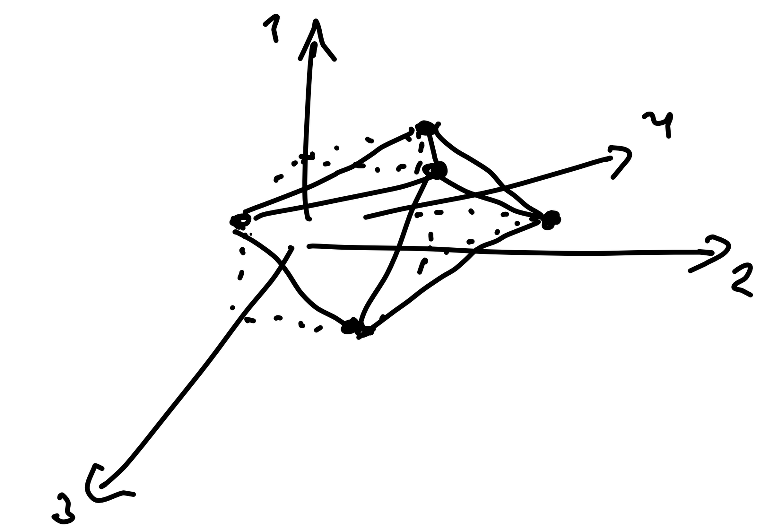
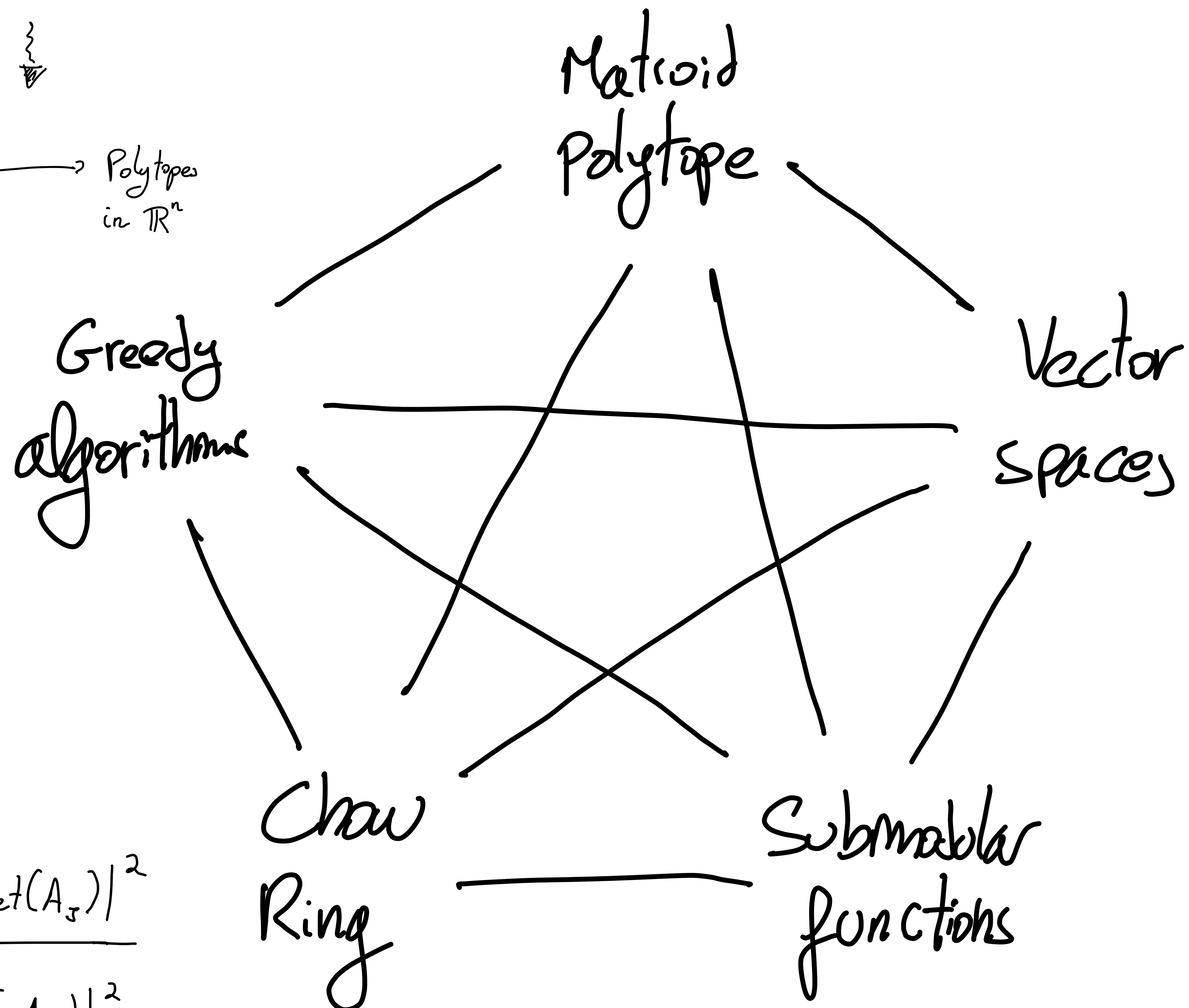
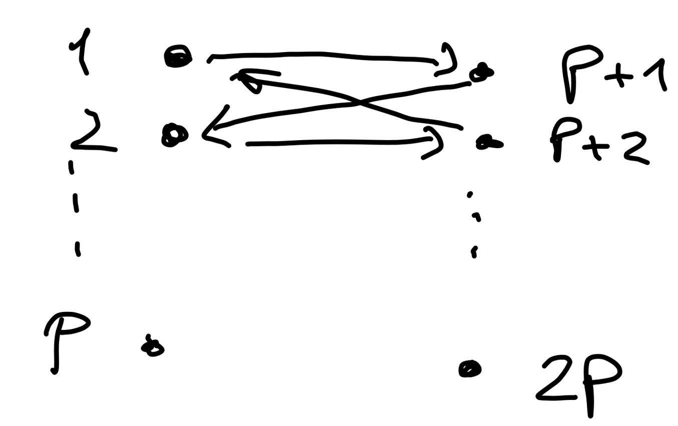
Matroids A short guide on how to invoke them

$$\begin{tikzcd} & Gr(r,\mathbb{C}^n) \arrow[dl] \arrow[d] \arrow[r, "\mu"'] & \mathbb{R}^n \\ \text{Matroids} & \text{Orbits of } Gr(r,\mathbb{C}^n) \arrow[l] \arrow[r] & \text{Polytopes in } \mathbb{R}^n \end{tikzcd}$$

$$(\alpha_1,\dots,\alpha_n)\cdot(v_1,\dots,v_n)=(\alpha_1 v_1,\dots,\alpha_n v_n)$$

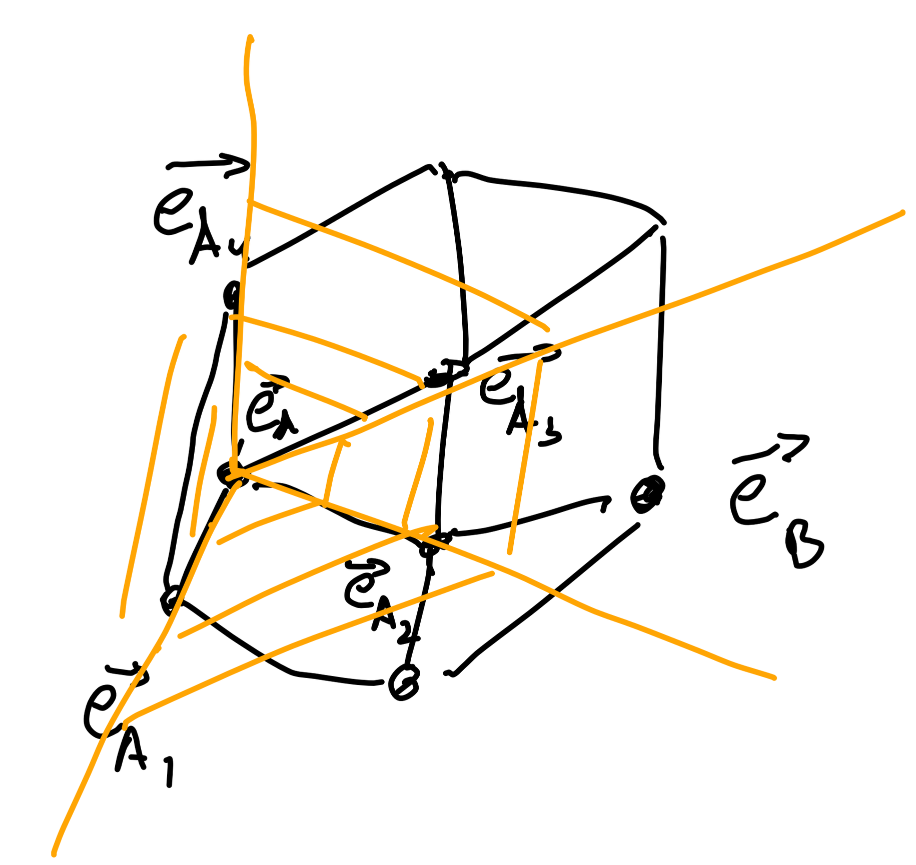






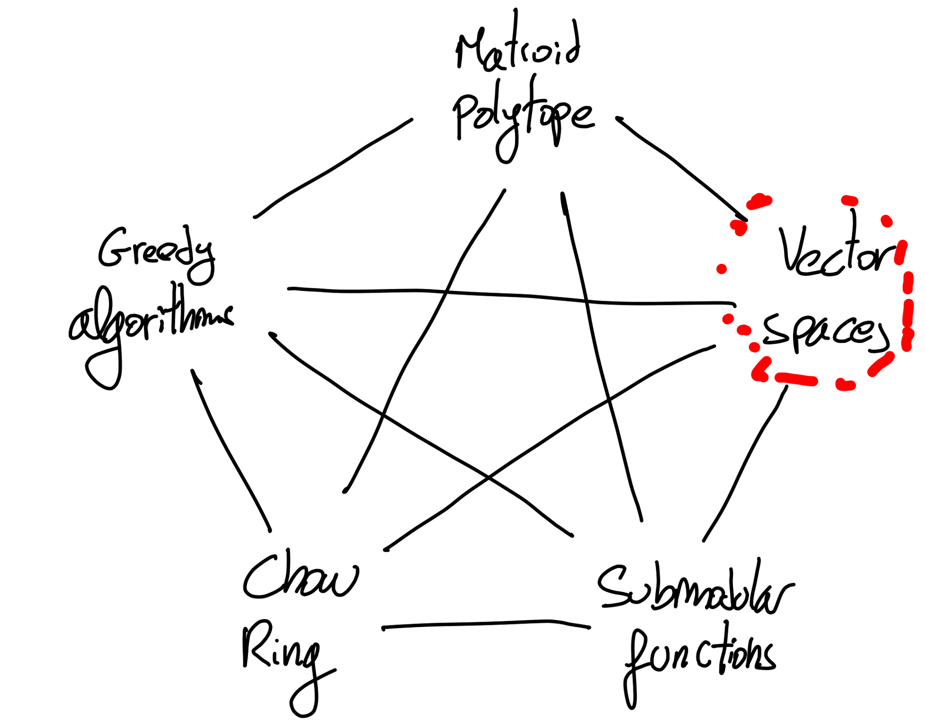
$$A=\begin{bmatrix} 1 & 0 & 1 & i \\ 0 & 2 & 1 & i \end{bmatrix}$$

$$\mu(U)_i = \frac{\sum_{\substack{J\in M[U] \\ i\in J}} |\det(A_J)|^2}{\sum_{J\in M[U]} |\det(A_J)|^2}$$



user.math.uzh.ch/penaguiao -> Talks

Our Conversation starts with a vector space $\mathbb{C}^n$, and a subspace $U \subseteq \mathbb{C}^n$ with dimension $r \leq n$.



Assume that $U = \text{rowspace}(A)$, where A is $r \times n$ matrix.

That is, $A = \begin{bmatrix} \text{———} \; r_1 \; \text{———} \\ \vdots \\ \text{———} \; r_r \; \text{———} \end{bmatrix}$ with $\text{span}\{r_1, \dots, r_n\} = U$.

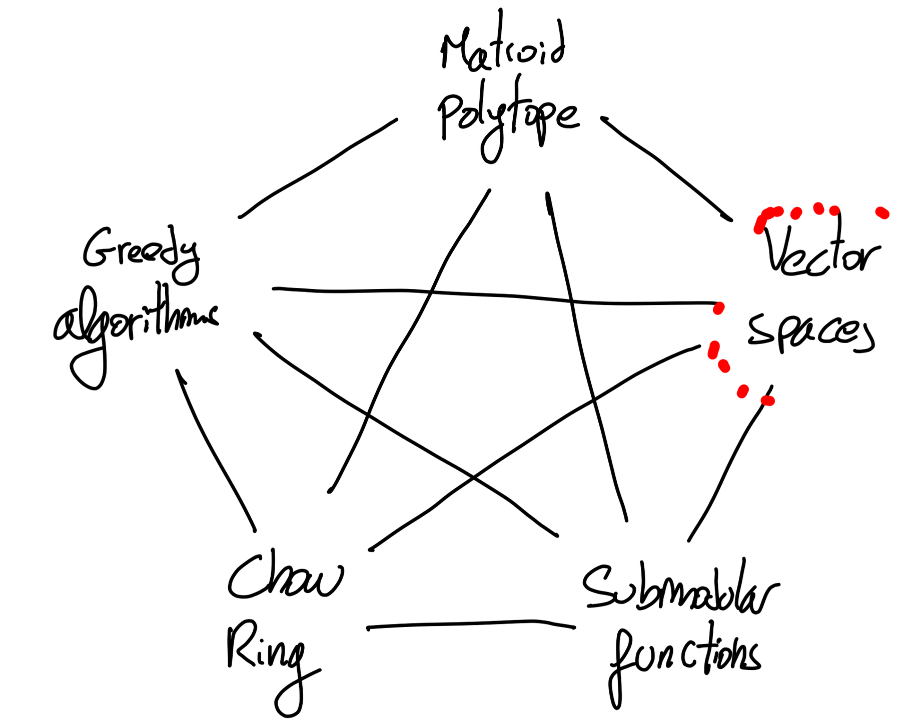
Def: A (representable) matroid is the collection of sets of columns that form non-zero minors.

Example If $A = \begin{bmatrix} 1 & 0 & 1 & i \\ 0 & 2 & 1 & i \end{bmatrix}$, then the matroid corresponding to $U = \text{rowspace}(A)$ is $\{12, 13, 14, 23, 24\} =: M[U]$

Question: Does this construction depend on A?

Answer: No! If $\text{rowspace}(A) = \text{rowspace}(B)$ for $r \times n$ matrices A, B, $\exists M$ $r \times r$-matrix non-singular s.t. $A = MB$.

The toric action Let $H = (\mathbb{C}^*)^n$ (this is a group!) act on $\mathbb{C}^n$ via



$$(\alpha_1, \dots, \alpha_n) \cdot (v_1, \dots, v_n) = (\alpha_1 v_1, \dots, \alpha_n v_n)$$

Grassmannian Define $Gr(r, \mathbb{C}^n) := \{k\text{-dimensional subspaces of } \mathbb{C}^n\}$

The toric action extends to $Gr(r, \mathbb{C}^n)$.

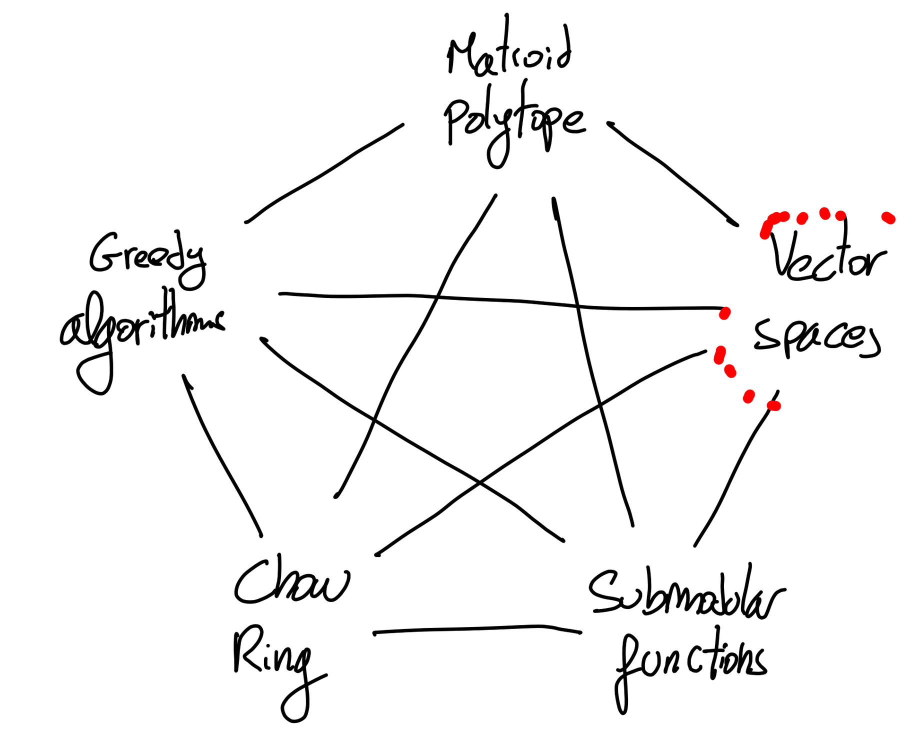
Example: $U = \text{rowspace}\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}\right)$ $\quad \vec{\alpha}\, U = \text{rowspace}\begin{pmatrix} 1 & 2i & -3 & -4i & 5 & 6i \\ 1 & i & -1 & -i & 1 & i \end{pmatrix}$

$n = 6 \qquad \vec{\alpha} = (1, i, -1, -i, 1, i)$

Question: How does the map $Gr(r, \mathbb{C}^n) \longrightarrow \text{Matroids}$ interact with this action?

Answer: Very well! Because $M[U] = M[\vec{\alpha}\, U] \ \forall \vec{\alpha} \in H$

$$\begin{tikzcd}
 & Gr(r, \mathbb{C}^n) \arrow[dl] \arrow[d] \arrow[r, "\mu"] & \mathbb{R}^n \arrow[d, squiggly] \\
\text{Matroids} & \text{Orbits of } Gr(r, \mathbb{C}^n) \arrow[l] \arrow[r] & \text{Polytopes in } \mathbb{R}^n
\end{tikzcd}$$

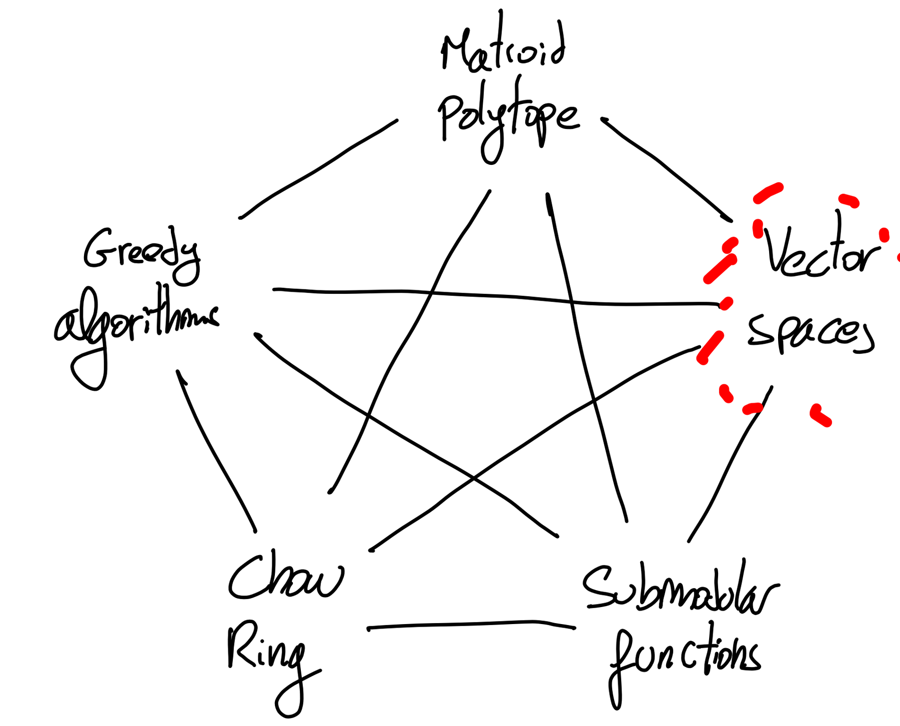


The moment map

Fix $U =$ rowspace(A), define $\mu(U)$ as

$$\mu(U)_i = \frac{\sum_{\substack{J \in M[U] \\ i \in J}} |\det(A_J)|^2}{\sum_{J \in M[U]} |\det(A_J)|^2} \in \mathbb{R}^n$$

$A_J \leadsto$ square submatrix corresponding to columns J.



Example: If $U = \text{rowspace}\begin{pmatrix}1 & 0 & 1 & i\\ 0 & 2 & 1 & i\end{pmatrix}$

then $M[U] = \{12, 13, 14, 23, 24\}$ and

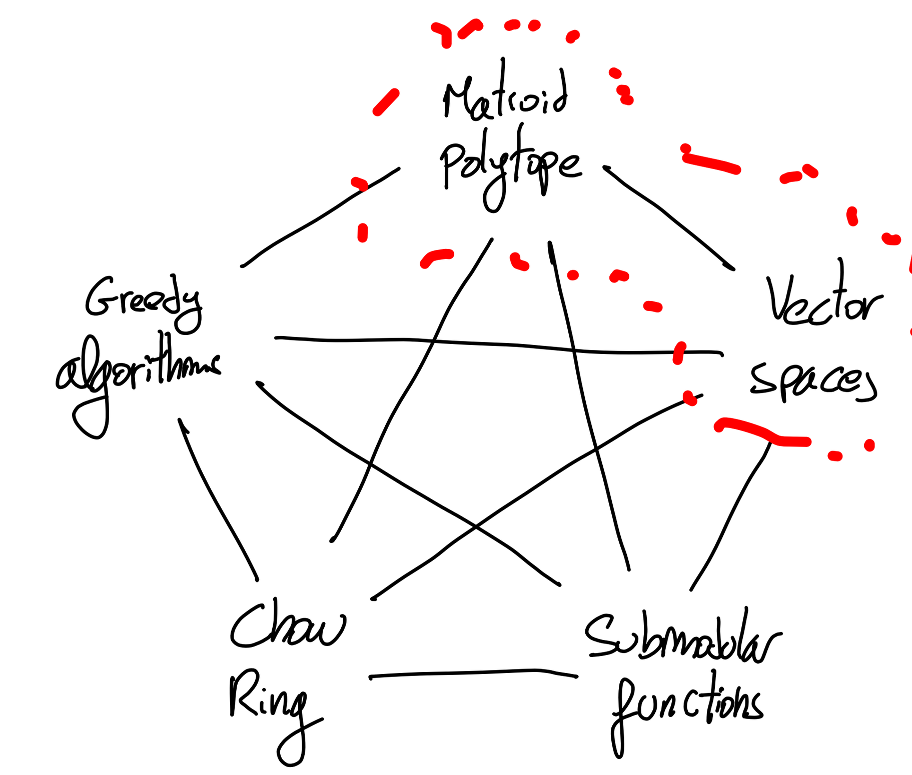
J	12	13	14	23	24
$\lvert\det(A_J)\rvert^2$	4	1	1	4	4

$$\boxed{\mu(U)_i = \frac{\sum_{J \in M[U],\, i \in J} \lvert\det(A_J)\rvert^2}{\sum_{J \in M[U]} \lvert\det(A_J)\rvert^2}}$$

$$\mu(U) = (6, 12, 5, 5)\frac{1}{14} \in \mathbb{R}^4$$

Observation 1 $\mu(U)$ does not depend on the choice of matrix A.

$$\boxed{\mu(U)_i = \frac{\sum_{\substack{J\in M[U] \\ i\in J}} |\det(A_J)|^2}{\sum_{J\in M[U]} |\det(A_J)|^2}}$$



However, it is not action-invariant! (i.e. $\mu(U) \neq \mu(\vec{\alpha}\cdot U)$ in general)

Observation 2 $\mu(U) \in \{\sum x_i = r\}$, where $r = \dim U$, and $0 \leq \mu(U)_i \leq 1$. Thus $\mu(U) \in \Delta_r^n := \operatorname{Conv}\{e_J \mid J \subseteq [n], |J| = r\}$.

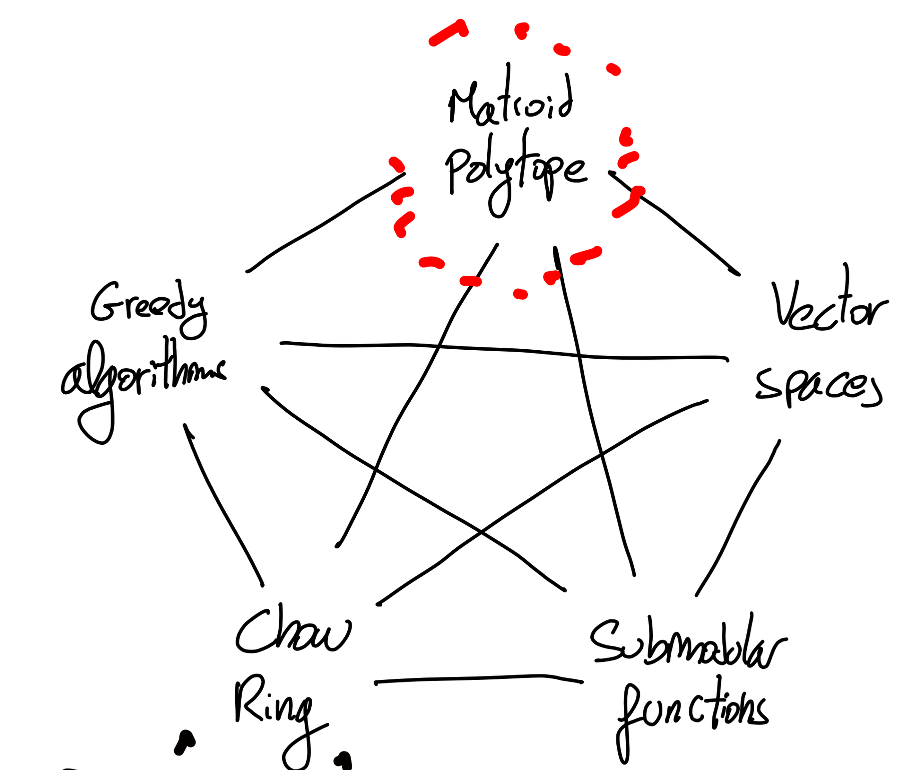
this is called the hypersimplex

Definition: The matroid polytope of U, $\Delta(U)$, is the image of the orbit of U through the moment map.

Zarisky closure, it's a long story.

$$\mu : \overline{H\cdot U} \longrightarrow \mathbb{R}^n.$$

Abstract matroid polytope is a polytope P s.t.



1. $V(P) \subseteq V(\Delta_n^r)$, for some r, n.
2. The edges of P are translates $\underbrace{\text{of } e_i - e_j}$, for some i, j.

If you are into root systems, you can plug in any such system here!

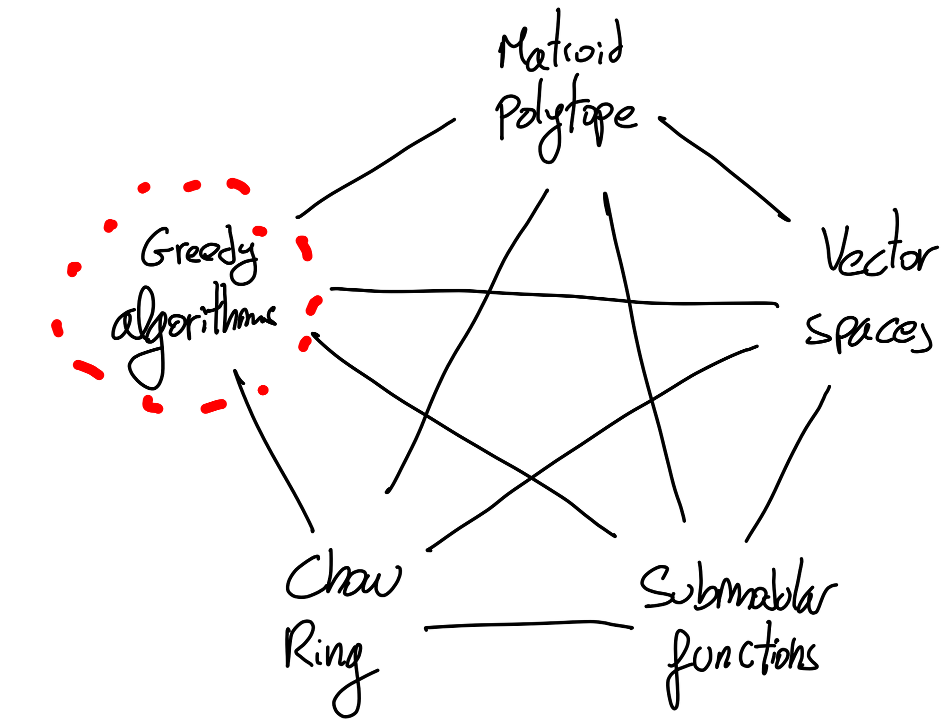
Theorem: The matroid polytope of a vector space is an abstract matroid.

The proof will be split into two parts

1. Vector spaces are "abstract matroids"
2. Any abstract matroid can be associated with an abstract matroid polytope.

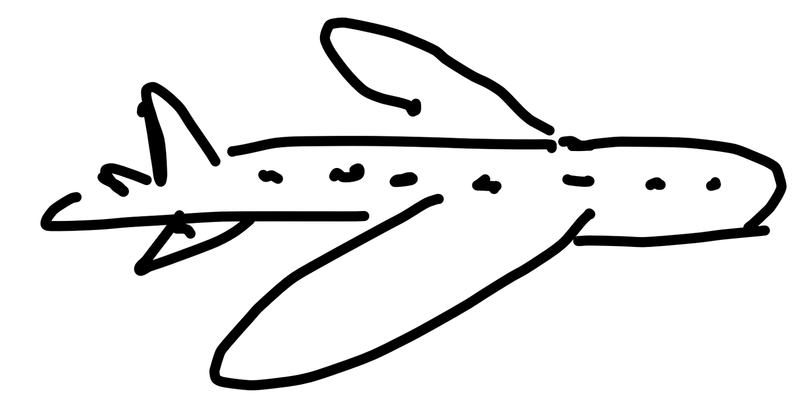
(Gelfand Goresky MacPherson Sergonova, 1987)

Penaguião's travel agency



Now offers trips to

Portugal	Beach, Nature, City
Brazil	Mountains, Beach, City
Switzerland	Ski, City, Mountains
Canada	City, Nature, Mountains



+ Norway
Ski, Nature, City

Magda: Ski > Mountains > Nature > Beach > City

Unique optimal solution: Switzerland

Switzerland

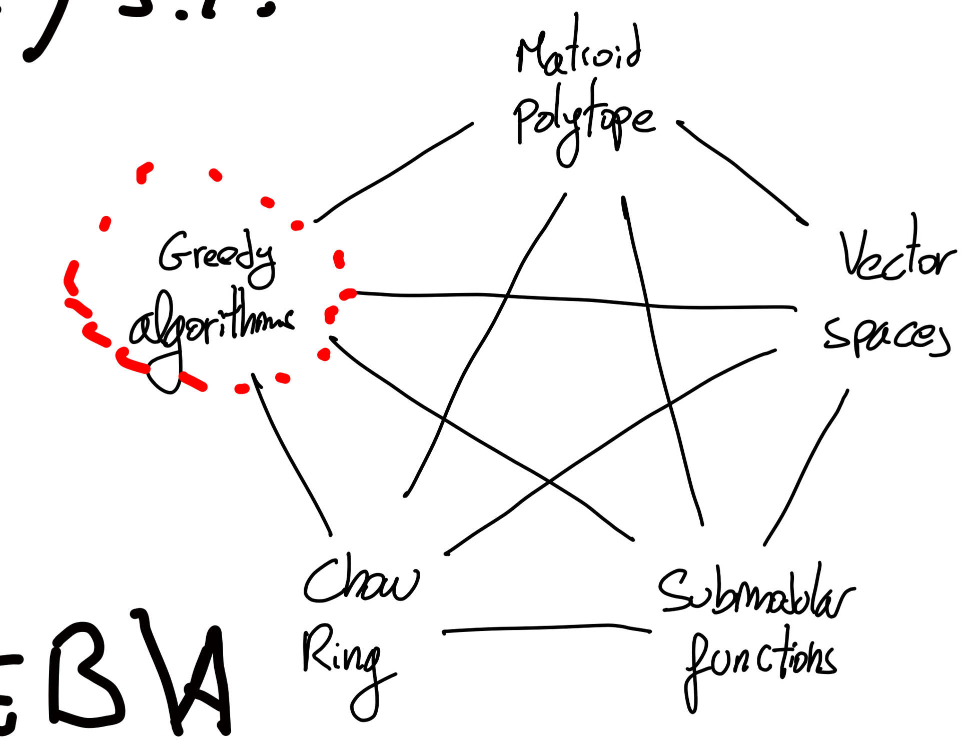
Peter: City > Ski > Nature > Beach > Mountains

No unique optimal solution

tie between Portugal & Switzerland

Norway

Bases axioms An abstract matroid is a collection $\mathcal{B} \subseteq \binom{I}{r}$ s.t.



1. $\mathcal{B} \neq \emptyset$
2. Basis exchange axiom if $A, B \in \mathcal{B}$, $\forall a \in A \setminus B \ \exists b \in B \setminus A$ s.t. $A \cup \{b\} \setminus \{a\} \in \mathcal{B}$.

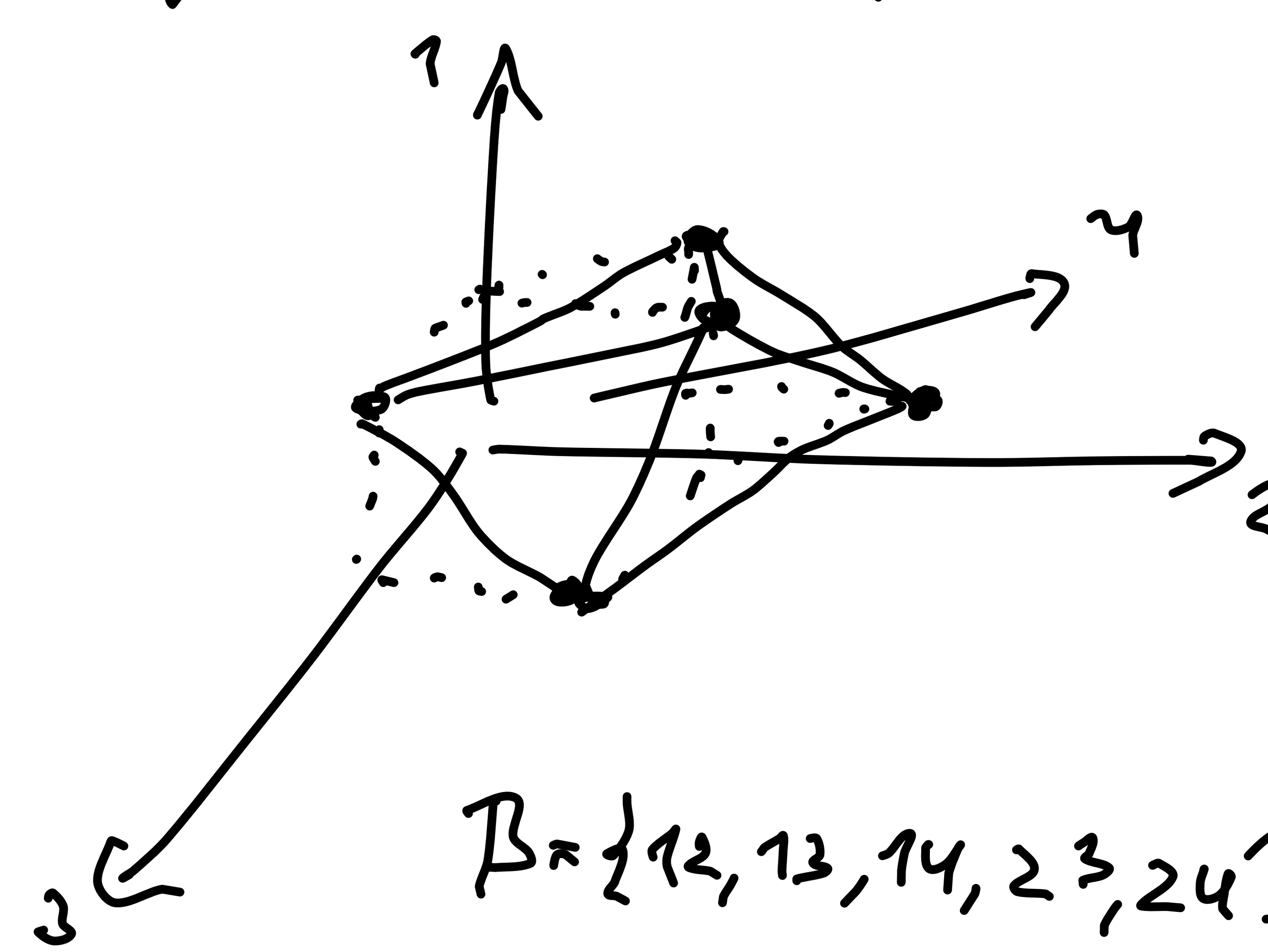
Examples: · The uniform matroid $\mathcal{U}^n_r \subseteq \binom{[n]}{r}$.

- The matroid of a vector space is an abstract matroid

The matroid polytope is $\Delta(M) := \mathrm{conv}\{ e_B \mid B \in \mathcal{B} \}$.

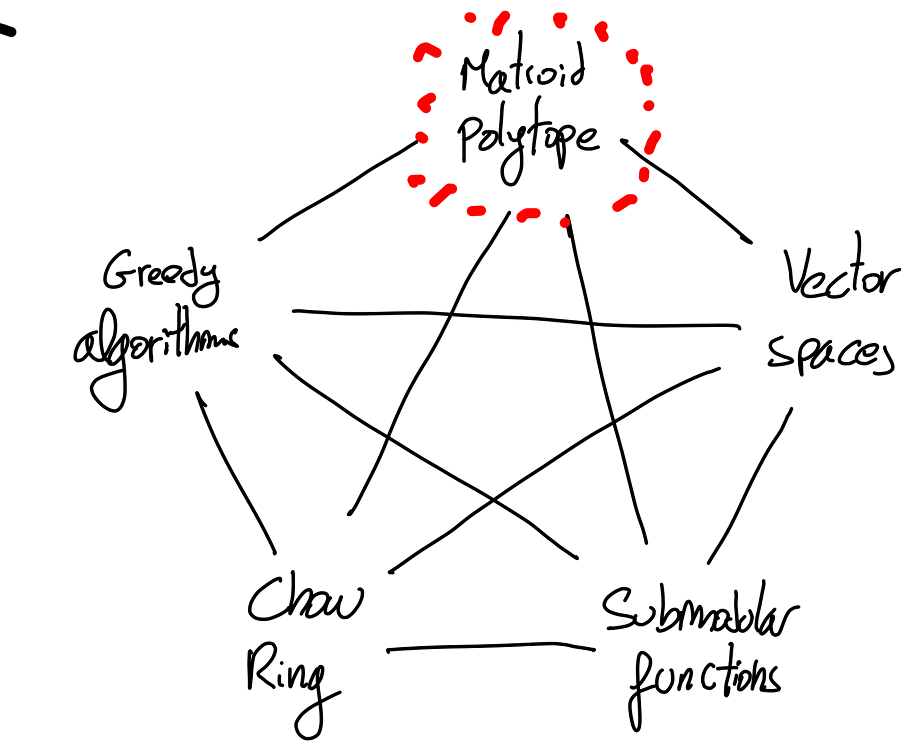
(Hard) Observation:

$$\Delta(M[U]) = \Delta(U) \left(= \mu\left(\overline{H \cdot U}\right)\right)$$



$\mathcal{B} = \{12, 13, 14, 23, 24\}$

Theorem (Gelfand, Goresky, MacPherson, Serganova, 1987) A polytope P is an abstract matroid polytope iff there is an abstract matroid M s.t. $\Delta(M) = P$.



Proof: ($\Leftarrow$) $\Delta(M) \subseteq \Delta_r^n$ is clear, take an edge of the polytope connecting $\vec{e}_{B_1}$ and $\vec{e}_{B_2}$. Wlog these vectors look like

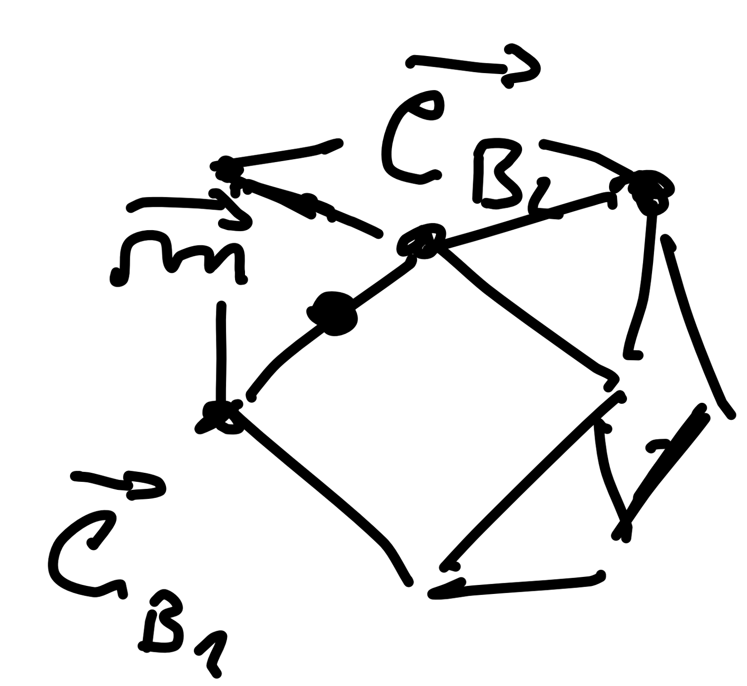
$$\vec{e}_{B_1} = (\overbrace{1, \dots, 1}^{p}, \overbrace{0, \dots, 0}^{p}, *, \dots, *)$$

$$\vec{e}_{B_2} = (0, \dots, 0, 1, \dots, 1, *, \dots, *)$$

same

$$\vec{m} = \frac{1}{2}\left(\vec{e}_{B_1} + \vec{e}_{B_2}\right)$$

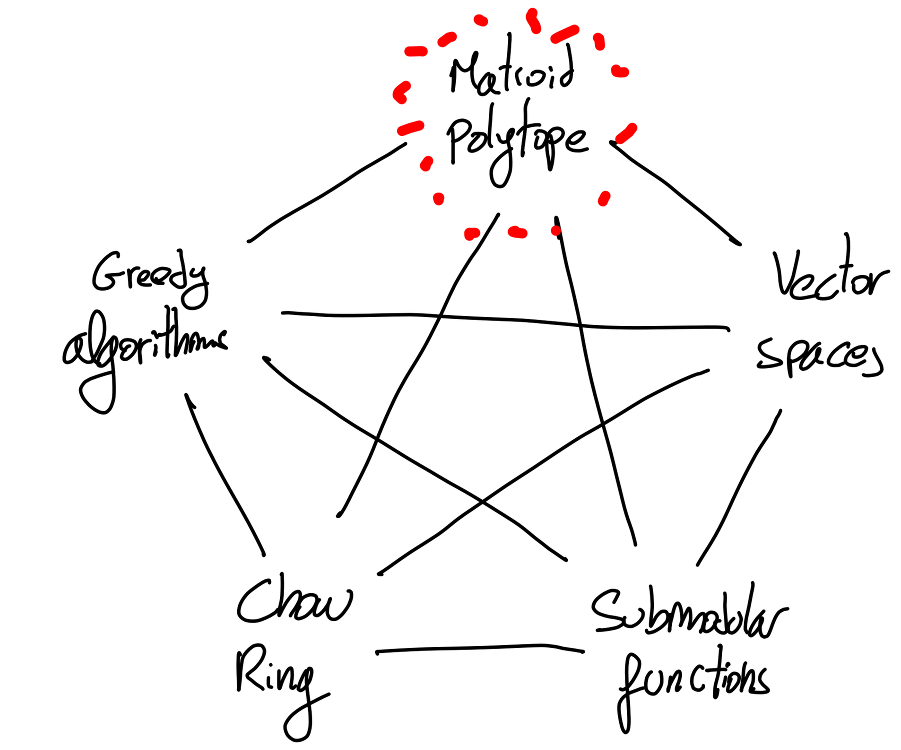
Goal: show that $\vec{m}$ arises as a convex combination of other vertices.

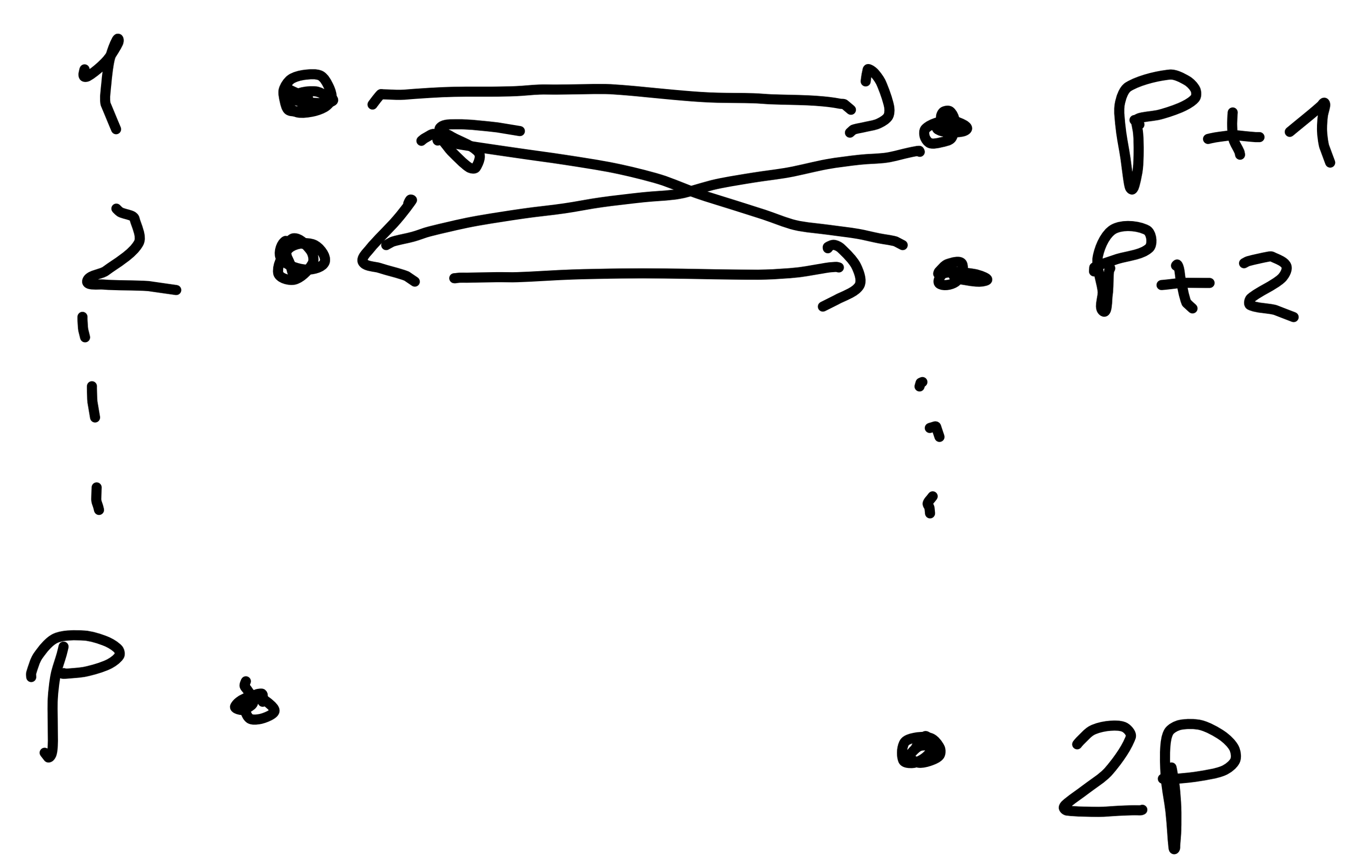


$$\vec{e}_{B_1} = (\overbrace{1, \dots, 1}^{p}, 0, \dots, 0, *, \dots, *)$$

same

$$\vec{e}_{B_2} = (0, \dots, 0, 1, \dots, 1, *, \dots, *)$$





if $i \le p$, $j > p$ $\quad i \to j$ if $A \cup \{j\} \setminus \{i\} \in \mathcal{B}$

if $i > p$, $j \le p$ $\quad i \to j$ if $B \cup \{j\} \setminus \{i\} \in \mathcal{B}$

To a cycle it corresponds an expression of the form

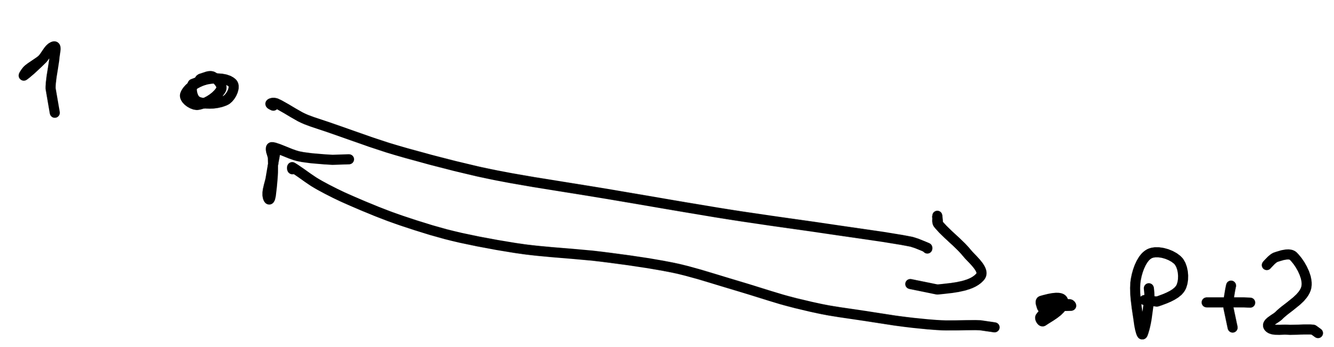
$$\frac{1}{2k} \sum \vec{e}_C = \vec{m}$$

size of the cycle

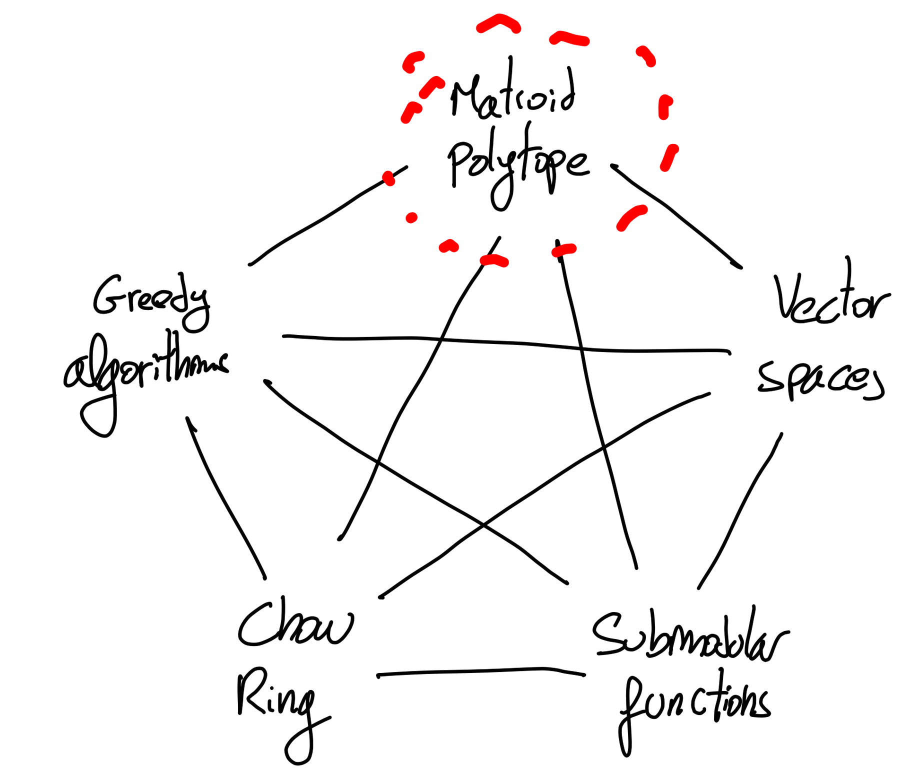
$C = A \cup \{j\} \setminus \{i\}$

or $C = B \cup \{j\} \setminus \{i\}$

Example

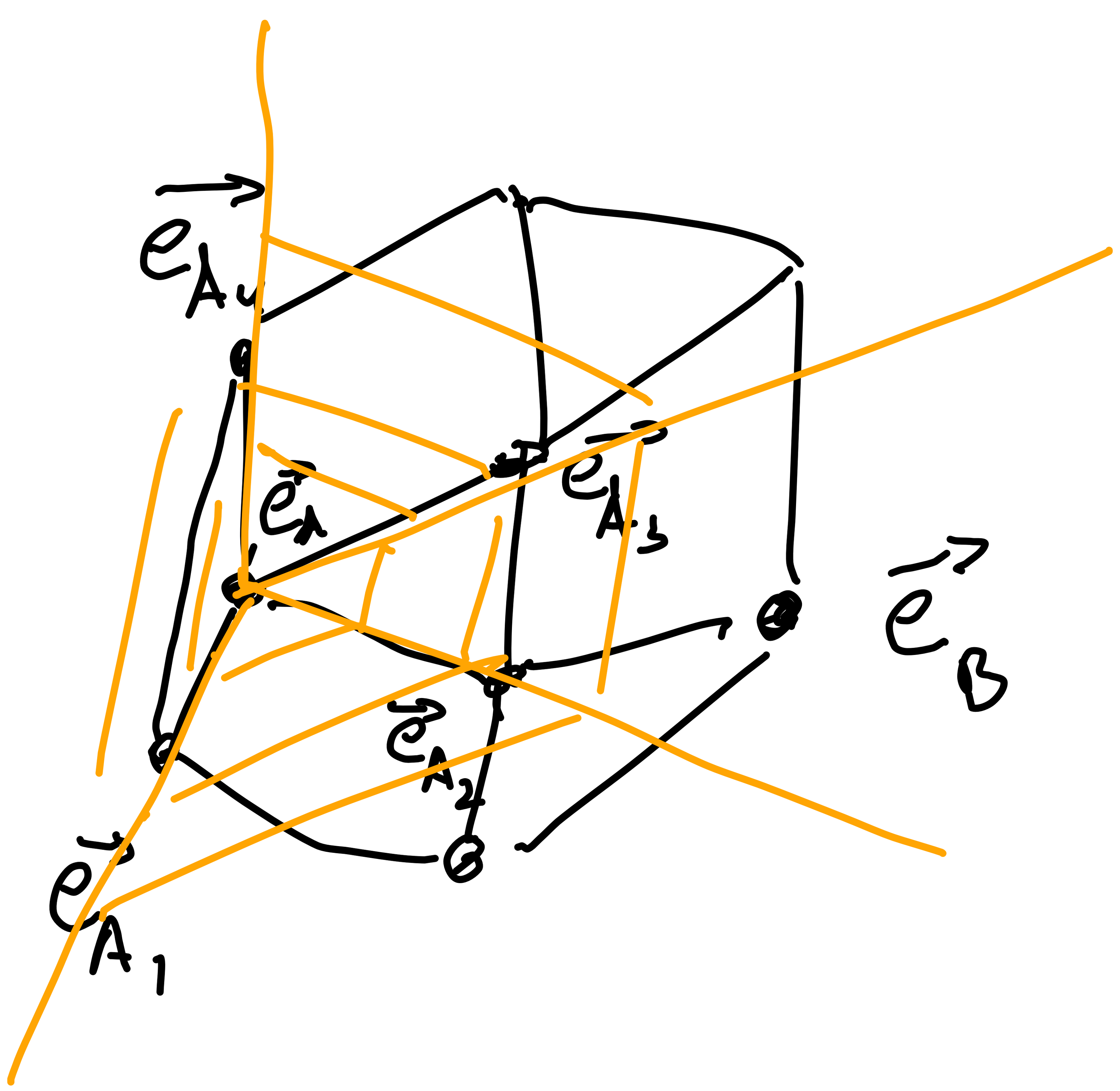


then $$\frac{1}{2}\left(\vec{e}_{A \cup \{p+2\} \setminus \{1\}} + \vec{e}_{B \cup \{1\} \setminus \{p+2\}}\right) = \vec{m}$$



<u>Proof</u>: $(\rightarrow)$ If P is an abstract matroid polytope, we have $V(P) \subseteq V(\Delta_r^n) = \{ e_J \mid |J|=r,\ J \subseteq [n] \}$

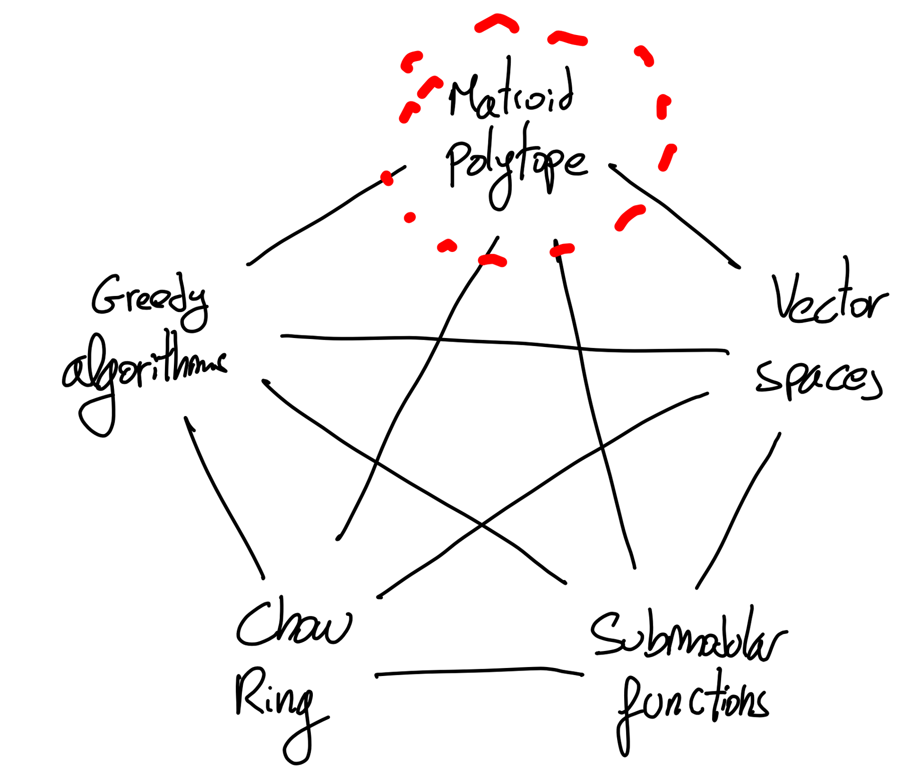
Let $\mathcal{B} = \{ J \mid e_J \in V(P) \} \neq \emptyset$; we need to prove the basis exchange property: $\forall A, B \in \mathcal{B}$, $a \in A \setminus B$, $\exists b \in B \setminus A$ s.t. $A \cup \{b\} \setminus \{a\}$.



$$\vec{e}_{A_i} = \vec{e}_A + (\vec{e}_{j_i} - \vec{e}_{k_i}) \qquad i = 1, \dots, s$$

$$\vec{e}_B \in P \subseteq \text{revolution cone at } \vec{e}_A = \vec{e}_A + \sum_{i=1}^{s} \mathbb{R}_{\geq 0} (\vec{e}_{A_i} - \vec{e}_A)$$

$$\vec{e}_B = \vec{e}_A + \sum_{i=1}^{s} \alpha_i (\vec{e}_{j_i} - \vec{e}_{k_i})$$



$$\vec{e}_B = \vec{e}_A + \sum_{i=1}^{s} \alpha_i (\vec{e}_{j_i} - \vec{e}_{k_i}), \quad \alpha_i \geq 0.$$

$$\vec{e}_{A_i} = \vec{e}_A + \vec{e}_{j_i} - \vec{e}_{k_i} \Rightarrow j_i \notin A \text{ and } k_i \in A$$

If $\alpha_i > 0$, $j_i \in B$ and $k_i \notin B$.

Now $(\vec{e}_B - \vec{e}_A)_a = -1$ so $\sum_{i=1}^{s} \alpha_i (\vec{e}_{j_i} - \vec{e}_{k_i})_a = -1$

There is some i s.t. $\alpha_i \neq 0$ and $k_i = a$. $b = j_i$ is a good choice!

Conclusion: $\mathcal{B}$ forms the basis set of a matroid.

That's all for now