# Another set composition Hopf algebra - and a commutative diagram with polytopes

A Hopf district productions seminar

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Slides can be found in
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# Symmetric functions

If $f : [n] \to \{1, 2, \dots\}$, write $x_f = x_{f(1)} \dots x_{f(n)}$, a symmetric function can be written as

$$\sum_f c_f x_f$$

where $c_f = c_g$ whenever $f = b \circ g$ for some order bijective map $b$ in the integers.
Basis indexed by **partitions**, endowed with a natural notion of product and *quasi-natural* notion of coproduct.
Context of symmetric function: chromatic invariants, representation theory (of the symmetric groups), and an algebraic study of integer partitions.

# Symmetric functions are cool - chromatic functions

Hopf algebra of graphs $\mathtt{Gr}$ maps to $Sym$ via the *chromatic symmetric function*, due to Stanley.

$$\Psi_{\mathtt{Gr}}(G) = \sum_f x_f \, ,$$

where the sum runs over *stable colourings* $f$ of the graph $G$.

## Conjecture (Tree conjecture)

*Any two non-isomorphic trees have distinct chromatic symmetric functions.*

# Quasi-symmetric functions

For a function $f : [n] \to \{1, 2, \dots\}$ define its **kernel** as the set partition

$$\ker f = f^{-1}(1)|f^{-1}(2)| \dots ,$$

where we further disregard empty sets.
In this way, quasi-symmetric functions can be written as

$$\sum_f c_f x_f ,$$

where $c_f$ are coefficients such that, for two different functions $f, g$ such that $\ker f = \ker g$, we have that $c_f = c_g$.
Basis indexed by **compositions**, endowed with a natural notion of product and *quasi-natural* notion of coproduct arises.
Quasi-symmetric functions $QSym$ used to study further chromatic invariants, representation theory (of the Hecke algebras), and an algebraic study of integer compositions.

# Quasi-symmetric functions are even cooler

Hopf algebra of matroids $\mathtt{Mt}$ maps to $QSym$ via *chromatic quasi-symmetric function* due to Billera-Jia-Reiner.

$$\Psi_{\mathtt{Mt}}(M) = \sum_f x_f \, ,$$

where the sum runs over $M$-*generic* colourings $f$, that is colourings that are maximized in exactly one basis of the matroid.

The fact that this is a quasi-symmetric function is non-trivial

## Theorem (Aguiar, Bergeron, Sottile 2000)

*Any combinatorial Hopf algebra* $\mathtt{H}$ *(HA + character) has a unique Hopf algebra morphism to* $QSym$ *denoted* $\Psi_{\mathtt{H}}$*.*

# Word symmetric functions in non-commutative variables

We now consider a family of non-commuting variables $\{\mathbf{a}_n\}_{n\geq 1}$.Word symmetric functions or symmetric functions in non-commutative variables can be written as

$$\sum_f c_f \mathbf{a}_f \ ,$$

where $c_f$ are coefficients such that, whenever $f = b \circ g$ for some bijection $b$, we have $c_f = c_g$.

(do not confuse with **non-commutative symmetric functions**, usually refers to the dual of $QSym$).

# Word quasi-symmetric functions

Word quasi-symmetric functions in non-commuting variables can be written as

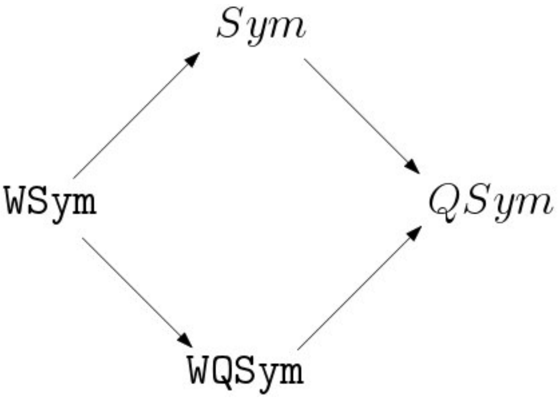
$$\sum_f c_f \mathbf{a}_f \,,$$

where $c_f$ are coefficients such that, whenever $\ker f = \ker g$, we have $c_f = c_g$.
Basis indexed by **set compositions**, endowed with a natural notion of product and *quasi-natural* notion of coproduct.

## Theorem (Aguiar and Mahajan 2010)

*There for any connected combinatorial Hopf monoid* $\mathtt{h}$ *there is a unique Hopf monoid morphism* $\boldsymbol{\Psi}_{\mathtt{h}} : \mathtt{h} \to \mathtt{WQSym}$.

# The big picture



# The combinatorial object

Symmetric set composition: a **symmetric** set composition $\pi$ of $n$ is a set composition of $\{-n, -n+1, \ldots, n-1, n\}$ such that

$$rev(\pi) = (-1) * \pi$$

Example: $\{-2\}|\{1\}|\{0\}|\{-1\}|\{2\}, \{2-1\}|\{0\}|\{-21\}$ and $\{1\}|\{-202\}|\{-1\}$.
Obs: always has an odd number of parts, zero is in the centre.
Let $\mathrm{forg}(\pi)$ be the set composition of $[n]$ resulting from dropping all non-positive integers from $\pi$.
Example: $\mathrm{forg}(\{1\}|\{-202\}|\{-1\}) = 1|2$

# The Hopf algebra

On non-commutative variables $\{\mathbf{a}_n\}_{n\in\mathbb{Z}}$, for a symmetric set composition $\pi$, we define the type B word quasi-symmetric functions as a sum

$$\sum_f c_f \mathbf{a}_f \, ,$$

where the sum runs over all **odd** functions $f : \{-n, \dots, n\} \to \mathbb{Z}$ and $c_f$ are coefficients such that, whenever $\ker f = \ker g$, we have $c_f = c_g$.

Further define the basis elements $\mathbb{N}_\pi = \sum_{\ker f = \pi} \mathbf{a}_f$.

Example: for $\pi = -1|2|0|-2|1$ we have that

$$\mathbb{N}_\pi = \mathbf{a}_1\mathbf{a}_{-2}\mathbf{a}_0\mathbf{a}_2\mathbf{a}_{-1} + \mathbf{a}_1\mathbf{a}_{-3}\mathbf{a}_0\mathbf{a}_3\mathbf{a}_{-1} + \mathbf{a}_2\mathbf{a}_{-3}\mathbf{a}_0\mathbf{a}_3\mathbf{a}_{-2} + \dots \, .$$

The projection $\mathbb{M}_\pi \mapsto \mathbb{M}_{\mathrm{forg}(\pi)}$ is a Hopf algebra morphism.

# Let's add polytopes - Generalized permutahedra

Why do we care about $\mathtt{WQSym}$ and $\mathtt{BWQSym}$?

$$\mathtt{Per}_n = \mathrm{conv}\{S_n(1,\ldots,n)\}$$

and set compositions of $n$ correspond to faces of $\mathtt{Per}_n$.
Consider the set $\mathtt{GPer}$ of polytopes arising from deformations of $\mathtt{Per}_n$.It's a Hopf algebra (Aguiar, Ardila 2017)
Thus arrises a map

$$\mathtt{GPer} \to \mathtt{WQSym}$$

$$\mathfrak{q} \mapsto \sum_{\pi} \mathbb{M}_\pi \chi[\text{The face of } \mathfrak{q} \text{ correponding to } \pi \text{ is a point}] .$$

This map generalizes the chromatic symmetric function, and many other chromatic invariants.

# Let's add polytopes - Type B

Let $D_n$ be the reflection group of $\mathbb{R}^n$ generated by the reflections accross the hyperplanes $x_i = x_j$ and $x_j = 0$.
The type B permutahedron arises as

$$\texttt{BPer}_n = \mathrm{conv}\{D_n(1, \dots, n)\}$$

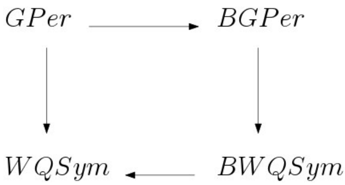
and symmetric set compositions of $n$ correspond to faces of $\texttt{BPer}_n$.
Consider the set $\texttt{GBPer}$ of polytopes arising from deformations of $\texttt{BPer}_n$.
Thus arrises a map

$$\Psi^B_{\texttt{BGPer}} : \texttt{BGPer} \to \texttt{BWQSym}$$

$$\mathfrak{q} \mapsto \sum_{\pi} \mathbb{N}_{\pi} \chi[\text{The face of } \mathfrak{q} \text{ correponding to } \pi \text{ is a point}] .$$

# The diagram

The following diagram commutes:




Problem: BWQSym is not even a Hopf algebra!

# Future work

- Is there some algebraic structure on BGPer that gives this diagram some meaning?
- Chromatic questions in the new invariant?
- Does $\mathtt{BWQSym}$ have some universal property, similarly to $\mathtt{WQSym}$?

# The end