Feasible regions meets pattern avoidance
The awaited 3rd part on feasible regions

University of Florida combinatorics seminar

Raúl Penaguião

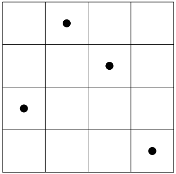
University of Zurich

24th November, 2020

Slides can be found in
`http://user.math.uzh.ch/penaguiao/`

Patterns in permutations

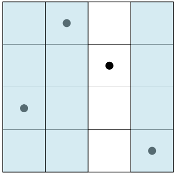
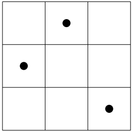
A permutation π of size n is an arrangement on an $n \times n$ table:

$\pi =$  $= 2431$

The set of permutations of size n : $\mathcal{S}_n$

The set of all permutations : $\mathcal{S}$

Select a set I of columns of the square configuration of π and define the **restriction** $\pi|_I$. This is a permutation.

$\pi|_{\{1,2,4\}}$  $=$  $= 231$

Number of occurrences

We can count occurrences!
For permutations π, σ, we define the pattern number:

$$\mathrm{occ}(\pi, \sigma) = \#\{\text{occurrences of } \pi \text{ in } \sigma\}\,.$$

In this way we have

$$\mathrm{occ}(12, 4132) = 2,\ \mathrm{occ}(312, 4132) = 2,\ \mathrm{occ}(12, 12345) = 10$$

$$\text{and} \quad \mathrm{occ}(312, 3675421) = 0$$

For a fixed integer k, what are the possible values of $\left(\mathrm{occ}(\pi, \sigma)\,|\sigma|^{-|\pi|}\right)_{\pi \in \mathcal{S}_k}$ when $|\sigma|$ is big?

Plotting these relationships

$$\widetilde{\mathrm{occ}}(\pi,\tau) = \frac{\mathrm{occ}(\pi,\tau)}{\binom{|\tau|}{|\pi|}}, \quad \widetilde{\mathrm{occ}}_k(\tau) = (\widetilde{\mathrm{occ}}(\pi,\tau))_{\pi\in\mathcal{S}_k} \in \mathbb{R}^{\mathcal{S}_k}.$$

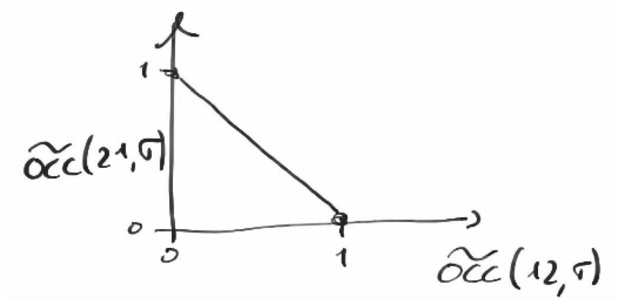


Figure: The interplay between proportion of occurrences of 12 and 21.

Feasible region - Classical patterns

For a fixed integer k, the corresponding feasible region (FReg) is defined as follows

$$F_k := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} \,|\, \exists\, \sigma^{(n)}, \widetilde{\text{occ}}_k(\sigma^{(n)}) \to \vec{v}\}\,.$$

Denote $F_{\leq k}$ for the FReg indexed by all permutations of size at most k, and more flexibly $F_{\mathcal{S}}$ for a set of permutations $\mathcal{S}$.

The FReg F_π, indexed by only one permutation, is an interval and is often studied in the context of *packing problems*.

Feasible region - Examples

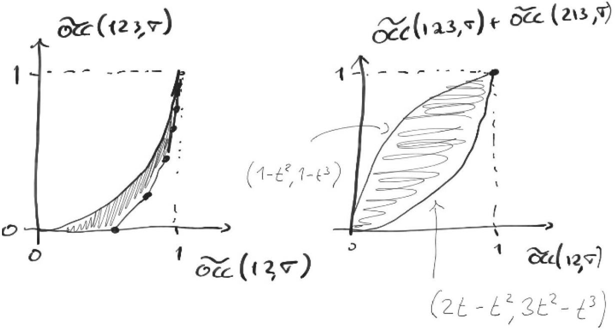
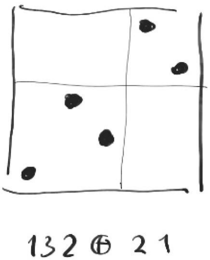


Figure: **Left:** The FReg comparing 12 and 123. **Right:** The FReg comparing patterns of 12 and patterns of 123 or 213 becomes a scalloped triangle. Feasible regions due to Kenyon, Kral, et.al. 2015.

Feasible region - The dimension problem

Given two permutations π of size n, and τ of size m, its **direct product** is $\pi \oplus \tau$ given as

$$\pi \oplus \tau = \pi(1) \cdots \pi(n)(\tau(1) + n) \cdots (\tau(m) + n) \,.$$



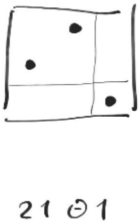


Figure: **Left:** The direct sum of the permutations 132 and 21. **Right:** The inverse sum of the permutations 12 and 1.

Feasible region - The dimension problem

A permutation σ is called **indecomposable** if it cannot be written as $\sigma = \pi \oplus \tau$ for π, τ non-trivial permutations.

Theorem (Glebov, Hoppen, et.al. 2017)

The dimension of the feasible region $F_{\leq k}$ is at least the number of indecomposable permutations of size k.

For instance, when $k = 3$, there are 3 permutations that are indecomposable. But the feasible region has dimension four.

Feasible region - Permutons

Permutons: A probability measure in the square $[0,1] \times [0,1]$ with *uniform marginals*.

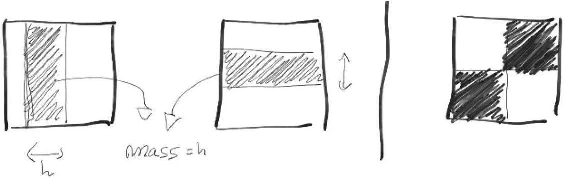


Figure: **Left:**A permuton has uniform marginals. **Right:** A permuton constructed from twice the Lebesgue measure on two squares.

Any permuton arises as the limit of permutations with unbounded size.

Feasible region - Algebra of patterns

Theorem (Vargas, 2014)

The pattern functions satisfy the following polynomial relations:

$$\mathrm{occ}(\pi,\cdot)\,\mathrm{occ}(\tau,\cdot) = \sum_{\sigma} \binom{\sigma}{\pi,\tau} \mathrm{occ}(\sigma,\cdot)\,.$$

*Furthermore, the **algebra** of pattern functions has a free basis indexed by **Lyndon permutations**.*

Consequence: we have an upper bound for $\dim F_{\leq k}$, given by the number of Lyndon permutations.

Conjecture (Borga, P., 2019)

This upper bound is tight.

Consecutive occurrences

We now consider only occurrences that form **an interval.** For instance, taking $\sigma = 2413$, there are two distinct consecutive restrictions of σ of size three, namely 231 and 312.

$$\text{c-occ}(\pi,\tau) = \#\{I \text{ interval s.t. } \tau|_I = \pi\}\,.$$

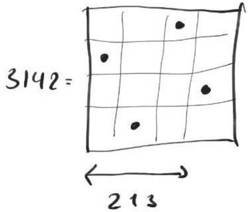


Figure: The permutation 3142, does not contain a consecutive occurrence of 231, but it does contain a consecutive occurrence of 213.

Consecutive occurrences

The number $\text{c-occ}(\pi, \tau)$ varies between 0 and $|\tau| - |\pi| + 1$. So we define

$$\widetilde{\text{c-occ}}(\pi, \tau) = \frac{\text{c-occ}(\pi, \tau)}{|\tau|}, \ \widetilde{\text{c-occ}}_k(\tau) = (\widetilde{\text{c-occ}}(\pi, \tau))_{\pi \in \mathcal{S}_k} \in \mathbb{R} \ .$$

The “permuton” version for consecutive occurrences are called *shift-invariant random orders of* $\mathbb{Z}$, due to Borga(2018).

FRegIve

$$\mathcal{F}_k := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} | \exists \sigma^{(n)}, \widetilde{\text{c-occ}}_k(\sigma^{(n)}) \to \vec{v}\} \subseteq \mathbb{R}^{\mathcal{S}_k} .$$

This is still a closed region. Is it convex?
Claim: The feasible region is convex. Proof: use $\oplus$.

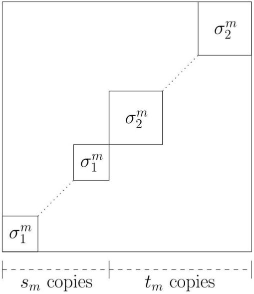


Figure: The construction of a particular permutation to establish convexity.

FRegIve

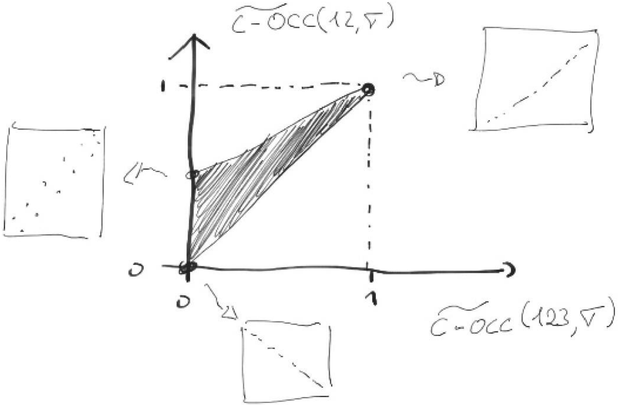


Figure: The FRegIve comparing patterns 12 and 123.

The overlap graph

Consider the case $k = 3$ and the permutation $\sigma = 2714365$.

$$2714365 \mapsto 231 - 312 - 132 - 213 - 132\,.$$

We can construct a graph from this:

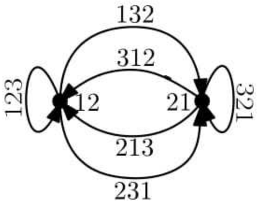


Figure: The overlap graph for $k = 3$

$\{$ permutations $\} \to \{$ paths in $\mathcal{O}v(k)\}$, is this map invertible?

The overlap graph - inverting a path

Lemma

Any path comes from a permutation.

Consider the path

$$\omega = 2413 \rightarrow 4123 \rightarrow 1342 \rightarrow 2413\,.$$

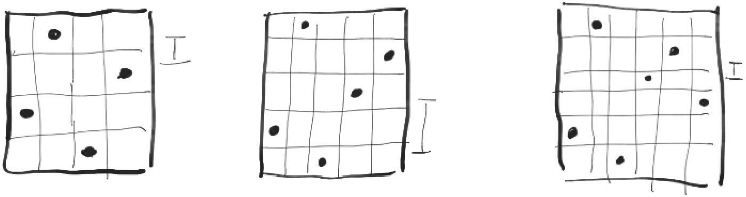


Figure: The construction of the path ω.

Cycle polytopes

Simple cycles in $\mathcal{O}v(k)$ correspond to a sequence of walks that give us a sequence of permutations of increasing size.

What are these points? $\vec{e}_{\mathcal{C}} = \left(\frac{\mathbb{1}[e \in \mathcal{C}]}{|\mathcal{C}|} \right)_{e \in E(\mathcal{O}v(k))}$.

For a graph G, define $P(G) = \text{conv}\{\vec{e}_{\mathcal{C}} |$ for $\mathcal{C}$ a simple cycle of $G\}$.

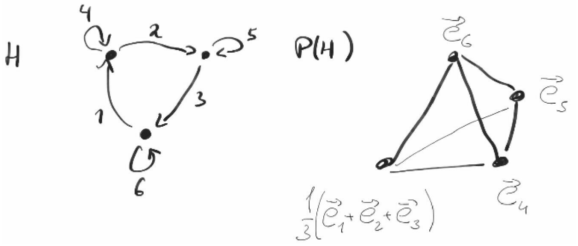


Figure: The cycle polytope of a graph H.

FRegIve is a cycle polytope

Theorem (Borga, P., 2019)

$$P(\mathcal{O}v(k)) = \mathcal{F}_k\,.$$

In particular, $\mathcal{F}$ is a polytope with dimension $k! - (k-1)!$.

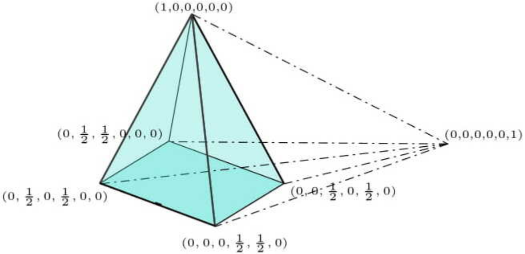


Figure: The feasible region of $k = 3$.

Avoiding set patterns - ReFRegIve

Let's introduce pattern avoidance in this problem! Fix a finite set $\mathcal{P}$ of patterns, and let $\mathrm{Av}(\mathcal{P})$ be the set of permutations that avoid all patterns of the set $\mathcal{P}$:

$$\mathrm{Av}(\mathcal{P}) = \{\tau \in \mathcal{S} \,|\, \forall\, \pi \in \mathcal{P},\, \mathrm{occ}(\pi, \tau) = 0\}\,,$$

and let $\mathrm{Av}_k(\mathcal{P})$ be $\mathrm{Av}(\mathcal{P}) \cap \mathcal{S}_k$.

A set of the form $\mathrm{Av}(\mathcal{P}) \subseteq \mathcal{S}$ is called a **permutation class.**

Permutations classes are a world to be investigated!

Permutation classes

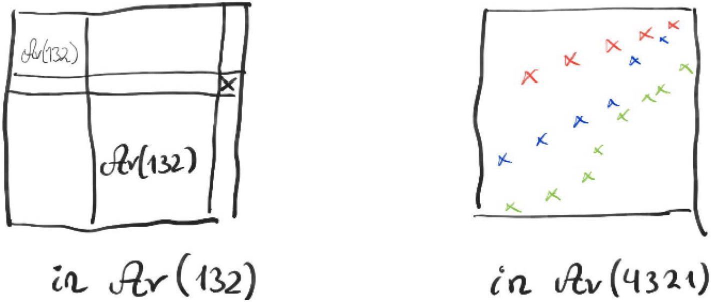


Figure: **Left:** the permutation class $\mathrm{Av}(132)$ is characterized by in inductive construction. **Right:** the permutation classes $\mathrm{Av}(n \cdots 1)$ are characterized by $n-1$ increasing monochromatic subsequences.

The feasible region (ReFRegIve) is:

$$\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})} := \{\vec{v} \in \mathbb{R}^{\mathrm{Av}_k(\mathcal{P})} \,|\, \exists\, \sigma^{(n)} \in \mathrm{Av}(\mathcal{P}) \text{ with } \widetilde{\text{c-occ}}(\sigma^{(n)}) \to \vec{v}\}\,.$$

Does anyone read these titles?

ReFRegIve is still a closed set. **Convexity on ReFRegIve**?
The argument above is not valid in general anymore.
Example: if $\mathcal{P} = \{132, 312, 231, 213\}$, then $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is a set with only two points.

Proposition

If the patterns of $\mathcal{P}$ are all $\oplus$ indecomposable, or are all $\ominus$ indecomposable, then $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is convex.

The overlap graph - general cases

A permutation avoiding $\mathcal{P}$ still corresponds to a path in the overlap graph. This path patterns in $\mathcal{P}$, so it does not populate the whole overlap graph. Thus, $P(\mathcal{O}v(k)|_{\mathrm{Av}(\mathcal{P})}) \subseteq \mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$.
Conjecture: this inclusion is almost an equality.

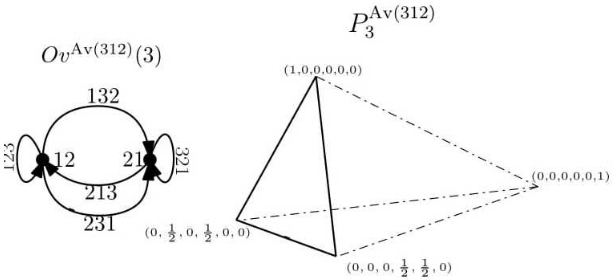


Figure: **Left:** The restricted overlap graph for $\mathcal{P} = \{132\}$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{132\}$.

Example of path inversion - 132

On the case 132, can we always invert such paths? Example:

$$\omega = 123 \to 231 \to 321 \to 213\,.$$

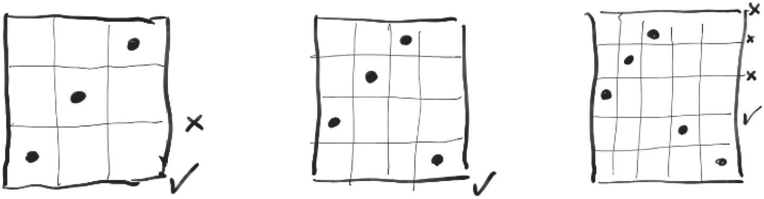


Figure: The construction of a permutation corresponding to the path ω.

The upshot - 132

$$\mathcal{F}_k^{\mathrm{Av}(132)} = P(\mathcal{O}v(k)|_{\mathrm{Av}_k(132)}) \text{ and } \dim \mathcal{F}_k^{\mathrm{Av}(132)} = C_k - C_{k-1}\,.$$

Example: the cycle $231 \to 213$ corresponds to a sequence of permutations $\sigma^{(n)}$ avoiding 132 that has

$$\widetilde{\text{c-occ}}_k(\sigma^{(n)}) \to \frac{1}{2}(\vec{e}_{231} + \vec{e}_{213})\,.$$

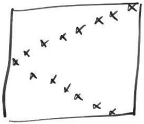


Figure: A big permutation that corresponds to the cycle $231 \to 213$.

The feasible region - 132

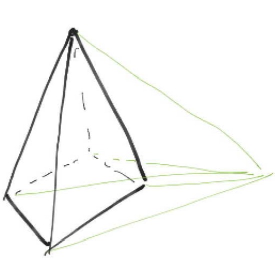


Figure: **Left:** The unrestricted feasible region for $k = 3$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{132\}$.

The overlap graph - 321

On the case 321, can we always invert such paths? Example:

$$\omega = 312 \to 123 \to 231\,.$$

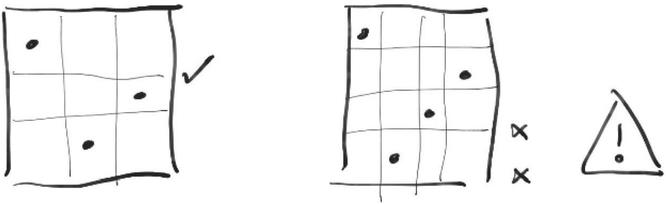


Figure: The construction of a permutation corresponding to the path ω.

We run into a big problem here: there are no permutations that map to the path.

The coloured overlap graph - 321

Let's add colours to the path, in such a way that each color is a monotone sequence:

$$\omega = 312 \to 123 \to 231\,.$$

We can see that we cannot colour it using only two colours, so this path is not "transversible" while avoiding 321.
On the other hand, a valid sequence would be, for instance

$$\omega = 312 - 123 - 123 - 132\,.$$

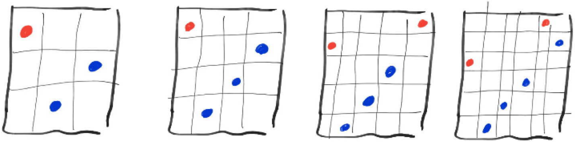


Figure: The construction of a permutation corresponding to the

The coloured overlap graph - 321

Let's add colours to the overlap graph itself and call it $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(k)$

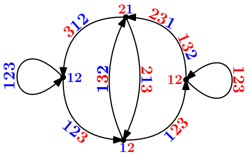


Figure: The overlap graph for $k = 3$ adapted to $\mathcal{P} = \{321\}$, where now we include all possible colouring of each edge.

RITMO colouring of a permutation:

$$\sigma = 1327456 \mapsto 1327456 \mapsto 1327456\,.$$

This gives us a map
$\{$permutations avoiding $321\} \to \{$walks on $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$ $\}$

The path lemma - monotone avoiding

Lemma

For any path ω, we can prepend a path ω' with constant size so that $\omega' \star \omega$ is invertible.

Let $\Pi : \mathbb{R}^{\mathcal{C}\,\mathrm{Av}_k(321)} \to \mathbb{R}^{\mathrm{Av}_k(321)}$ be the projection that forgets the colour of the edges. Then

$$\mathcal{F}_k^{\mathrm{Av}(321)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(k)))\,.$$

$$\dim \mathcal{F}_k^{\mathrm{Av}(321)} = |\,\mathrm{Av}_k(321)| - |\,\mathrm{Av}_{k-1}(321)| = C_k - C_{k-1}\,.$$

The restricted feasible region - 321

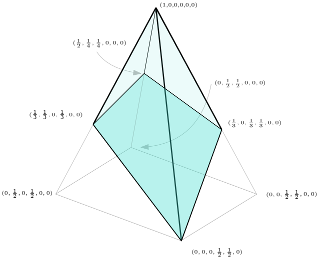


Figure: **Left:** $P(\mathcal{O}v(3))$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{321\}$, overlaid with $P(\mathcal{O}v(3)|_{\mathrm{Av}_3(321)})$.

The coloured overlap graph - $n\cdots 1$

Let's add more colours to the permutations of the graph, and use the RITMO colouring of a permutation:

$$\sigma = 5467231 \mapsto 5467231 \mapsto 5467231 \mapsto 5467231\,.$$

This gives us a map

$$\{\text{permutations}\} \to \{\text{walks on } \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(n\cdots 1)}(k)\ \}\,.$$

Theorem (Borga, P. 2020)

$$\mathcal{F}_k^{\mathrm{Av}(n\cdots 1)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(n\cdots 1)}(k)))\,,$$

$$\dim \mathcal{F}_k^{\mathrm{Av}(n\cdots 1)} = |\,\mathrm{Av}_k(n\cdots 1)| - |\,\mathrm{Av}_{k-1}(n\cdots 1)|\,.$$

Future work

- Other permutation classes are also given by generating trees. We believe that any such permutation class will have a direct description of the feasible region, and that we can totally describe all the extremal points.
- The dimension conjecture: if $\mathcal{P}$ has only one pattern, then

$$\dim \mathcal{F}_k^{\mathrm{Av}_k(\mathcal{P})} = |\,\mathrm{Av}_k(\mathcal{P})| - |\,\mathrm{Av}_{k-1}(\mathcal{P})|\,.$$

- The other dimension conjecture on the classical FReg.
- What is the volume of all these regions? Related with triangulations.

The end