Hopf algebras

Fix a field k

An <u>algebra</u> (over k) is a triple (A, μ, ι) where

- A is a vector space over k
- $\mu : A \otimes A \longrightarrow A$ is associative, that is

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] \arrow[dr, phantom, "\Omega"] & A\otimes A \arrow[d, "\mu"] \\ A\otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

- $\iota : k \to A$ is a unit wrt to μ, that is

$$\begin{tikzcd} k\otimes A \arrow[r, "\iota\otimes id"] \arrow[dr, leftrightarrow, "\simeq"] & A\otimes A \arrow[d, "\mu"] & A\otimes k \arrow[l, "id\otimes\iota"'] \arrow[dl, leftrightarrow, "\simeq"] \\ & A & \end{tikzcd}$$

commutes

Example : k

Example: $k[X]$, the ring of polynomials with coefficients in k, is an algebra under the usual multiplication, and $\iota : k \longrightarrow k[x]$, $1_k \longmapsto 1$

Rem: If A, B are algebras, $A\otimes B$ is also an algebra.

A <u>coalgebra</u> (over k) is a triple (C, Δ, ε) where

- C is a vector space over k
- $\Delta : C \to C\otimes C$ is coassociative, that is

$$\begin{tikzcd} C\otimes C\otimes C & C\otimes C \arrow[l, "id\otimes\Delta"'] \\ C\otimes C \arrow[u, "\Delta\otimes id"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

- $\varepsilon : C \to K$ is a counit wrt to Δ, that is

$$\begin{tikzcd} K\otimes C & C\otimes C \arrow[l, "id\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes id"] & C\otimes k \\ & C \arrow[ul, leftrightarrow, "\simeq"] \arrow[u] \arrow[ur, leftrightarrow, "\simeq"'] & \end{tikzcd}$$

commutes

Ex: k

Ex: $k[X]$ with $\Delta(X^n) = \sum_{k\geq 0} \binom{n}{k} X^k \otimes X^{n-k} = \sum_{(B_1,B_2)\models_S [n]} X^{|B_1|} \otimes X^{|B_2|}$

$\varepsilon(X^n) = \delta_{n,0}$ is a coalgebra.

Rem: If C, D are coalgebras, then $C\otimes D$ is also a coalgebra.

A <u>bialgebra</u> (over k) is a 5-tuple $(B, \mu, \iota, \Delta, \varepsilon)$ s.t.

① · (B, μ, ι) is an algebra and (B, Δ, ε) is a coalgebra.

② · Δ and ε are algebra morphisms.

③ · μ and ι are coalgebra morphisms.

④ · The following diagrams commute

$$\begin{tikzcd} k \arrow[r, "\simeq"] \arrow[d, "\iota"'] & k\otimes k \arrow[d, "\iota\otimes\iota"] \\ B \arrow[r, "\Delta"'] & B\otimes B \end{tikzcd} \qquad \begin{tikzcd} k \arrow[d, "\iota"'] \arrow[dr, "\simeq"] & \\ B \arrow[r, "\varepsilon"'] & k \end{tikzcd} \qquad \begin{tikzcd} B\otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"'] & k\otimes k \arrow[d, "\wr"] \\ B \arrow[r, "\varepsilon"'] & k \end{tikzcd}$$

$$\begin{tikzcd} B\otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "\mu"'] & B\otimes B\otimes B\otimes B \arrow[d, "\text{twist}", "\simeq"'] \\ & B\otimes B\otimes B\otimes B \arrow[d, "\mu\otimes\mu"] \\ B \arrow[r, "\Delta"'] & B\otimes B \end{tikzcd}$$

<u>Example</u>: $k[X]$ with the structures given above is a bialgebra

<u>Rem</u>: Assuming ①, the properties ②, ③, and ④ are equivalent.

A <u>Hopf algebra</u> is a 6-tuple $(H, \mu, \iota, \Delta, \varepsilon, S)$ such that

- $(H, \mu, \iota, \Delta, \varepsilon)$ is a bialgebra
- $S : H \to H$ is a linear map such that

$$\begin{tikzcd} & H\otimes H \arrow[r, "S\otimes id"] & H\otimes H \arrow[dr, "\mu"] & \\ H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & k \arrow[r, "\iota"] & & H \\ & H\otimes H \arrow[r, "id\otimes S"'] & H\otimes H \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes

Intuition: A Hopf algebra is to a bialgebra as a group is to a monoid

<u>Ex</u>: $k[X]$ is a Hopf algebra with $S : p(x) \longmapsto p(-x)$.

The character group of a bialgebra / The convolution algebra

Let A be an algebra and C a coalgebra.

If we have $f,g \in \mathrm{Hom}(C,A)$ then define $f*g$ as the composition

$$C \xrightarrow{\Delta} C\otimes C \xrightarrow{f\otimes g} A\otimes A \xrightarrow{\mu} B$$

This defines a monoid with a unit $\iota\circ\varepsilon$. (Also thought of as an algebra over k)

Indeed, if $b\in C$, $f*(\iota\circ\varepsilon)(b) = \sum_{(b)} f(b_1)\,\varepsilon(b_2)\cdot 1$

$$= f\Big(\sum_{(b)} b_1\cdot\varepsilon(b_2)\Big) = f(b).$$

Whenever B is a Hopf algebra, $id_B \in \mathrm{End}(B)$ is invertible and $S = id_B^{\langle -1\rangle}$

Furthermore, if $\alpha: H\to A$ is an algebra hom between an algebra A and a Hopf algebra, then $(\alpha\circ S)*\alpha = \iota\varepsilon = \alpha*(\alpha\circ S)$: let $b\in H$,

$$\alpha\circ S * \alpha\,(b) = \sum_{(b)} \alpha(S(b_1))\cdot\alpha(b_2) = \alpha\Big(\sum_{(b)} S(b_1)\,b_2\Big) = \alpha\big(\iota_H\circ\varepsilon(b)\big)$$
$$= \iota_A\circ\varepsilon(b)$$

The character group of a Hopf algebra

Let $Ch(H) = \mathrm{Alg}(H,k)$. Then this is a group!

The Takeuchi formula

Let H be a Hopf algebra, and suppose that for every $h\in H$, there is some N s.t. $n\geq N \Rightarrow \mu^{\circ n-1}\circ(id_H - \iota\varepsilon)^{\otimes n}\circ\Delta^{\circ n-1}(h) = 0$.

Then $$S = \sum_{k\geq 0} (-1)^k\,\mu^{\circ n-1}\circ(id_H-\iota\varepsilon)^{\otimes n}\circ\Delta^{\circ n-1}$$

$$\boxed{\begin{aligned}\mu^{\circ -1} &= \iota\\ \Delta^{\circ -1} &= \varepsilon\end{aligned}}$$

Proof: On $\mathrm{End}(H)$, we have the following formula

$$(\iota\circ\varepsilon + f)^{\langle -1\rangle} = \iota\circ\varepsilon - f + f^{*2} - f^{*3} + \cdots$$

whenever the RHS is a finite sum. For $f = id_H - \iota\circ\varepsilon$ this is precisely the assumption given, thus $id_H^{\langle -1\rangle} = S = \iota\circ\varepsilon - f + f^{*2} - \cdots$

Obs: If $H = \bigcup_{n\geq 0} H_n$ is a filtered connected Hopf algebra, that is

- $\mu(H_n \otimes H_m) \subseteq H_{n+m}$,
- $\Delta(H_n) \subseteq \bigcup_{k+j=n} H_k \otimes H_j$, · H_0 is 1-dim
- $\iota \circ \varepsilon|_{H_0} = id_{H_0}$;

Then for $b \in H_N$ we have that $n \geq N+1 \Rightarrow \mu^{\circ n-1} \circ (\iota\circ\varepsilon - id_H)^{\otimes n} \circ \Delta^{\circ n-1} = 0$

It follows that we can apply Takeuchi's formula for filtered Hopf algebras.

Example To compute the antipode of $k[X]$, we use Takeuchi's formula.

$$S(x^n) = \sum_{k\geq 0} (-1)^k \mu^{\circ k-1} \circ (id_{k[x]} - \iota\circ\varepsilon)^{\otimes k} \circ \Delta^{\circ k-1}(x^n)$$

$$= \sum_{k\geq 0} (-1)^k \mu^{\circ k-1} \circ (id_{k[x]} - \iota\circ\varepsilon)^{\otimes k} \Big(\sum_{(\alpha_1,\dots,\alpha_k)\vDash_0 [n]} X^{|\alpha_1|} \otimes \cdots \otimes X^{|\alpha_k|} \Big)$$

$$= \sum_{k\geq 0} (-1)^k \mu^{\circ k-1} \Big(\sum_{(\alpha_1,\dots,\alpha_k)\vDash [n]} X^{|\alpha_1|} \otimes \cdots \otimes X^{|\alpha_k|} \Big) = X^n \underbrace{\sum_{k\geq 0} (-1)^k \left| \left\{ \begin{smallmatrix} \text{Set Comp of } [n] \\ \text{with size } k \end{smallmatrix} \right\} \right|}_{(*)}$$

Sign-reversing involutions (a proof from C.Benedetti, B. Sagan)

Define $\iota : \{\text{set Compositions of } n\} \longrightarrow \{\text{Compositions of } n\}$

$(\alpha_1,\dots,\alpha_k) \longmapsto$
- Let ℓ be smallest index such that either $|\alpha_\ell| \geq 2$ or $\alpha_\ell = \{a\}$ with $a < \min \alpha_{\ell+1}$.
- In the first case, define $\iota(\alpha_1,\dots,\alpha_k) = (\dots \alpha_{\ell-1}, \alpha_\ell \cup \alpha_{\ell+1}, \dots)$
- " " 2^{nd} ", $\iota(\alpha_1,\dots,\alpha_k) = (\dots, \alpha_{\ell-1}, \{\min \alpha_\ell\}, \alpha_\ell - \{\min\}, \dots)$
- If no such ℓ exists, $\iota(\alpha_1,\dots,\alpha_k) = (\alpha_1,\dots,\alpha_k)$

Rem: $\iota^2 = id$

Then the "contribution" of each α in $(*)$ is either canceled by $\iota(\alpha)$ or α is a fixed point of ι. (Only fixed point is $\alpha = (\{1\},\{2\},\dots,\{n\})$)

Thus $(*) = x^n \cdot (-1)^n$.

The permutation pattern Hopf algebra.

A permutation of size $n \geq 0$ is an ordering of the numbers $\{1, \dots, n\}$.

Example: 132, 1, 7142365, $\emptyset$ of size $3, 1, 7$ and 0 respectively.

We can consider "subpermutations" by dropping some numbers, and relabelling the remaining ones preserving the order

Example: $132 \leq 7142365$

We can count in this way how many times a small permutation π fits inside a big permutation τ: $p_\pi(\tau) = \#\{\text{patterns } \pi \text{ in } \tau\}$.

Example: $p_{132}(7142365) = 6$

<u>Cover number</u>: we define $\binom{\nu}{\pi_1, \pi_2} = \#\left\{\begin{array}{c}\text{Covers of } \nu \text{ with two} \\ \text{not-necessarily disjoint} \\ \text{subsequences of type } \pi_1 \text{ and } \pi_2\end{array}\right\}$

Example: $12, 21$ can cover 312 in two ways, and 4123 in three ways

$12 \to$

$21 \to$

312 $\quad 312$ $\quad \binom{312}{12, 21} = 2$ / 4123 $\quad 4123$ $\quad 4123$ $\quad \binom{4123}{12, 21} = 3$

<u>Product formula</u>:
$$p_{\pi_1}(\tau) \cdot p_{\pi_2}(\tau) = \sum_\nu p_\nu(\tau) \binom{\nu}{\pi_1, \pi_2} = \sum_{\substack{\nu \text{ a} \\ \text{quasi-shuffle} \\ \text{of } \pi_1, \pi_2}} p_\nu(\tau)$$

$\Rightarrow$ $\operatorname{span}\{p_\pi \mid \pi \text{ permutation}\} = \mathcal{A}(\text{per})$ is an algebra with the pointwise product.

<u>Coproduct on $\mathcal{A}(\text{per})$</u> Given two permutations $\begin{array}{l}\pi = \pi_1 \cdots \pi_n \\ \tau = \tau_1 \cdots \tau_m\end{array}$, define

$\pi \oplus \tau = \pi_1 \cdots \pi_n (\tau_1 + n) \cdots (\tau_m + n)$ of size $n+m$. This is associative

Then define the coproduct $\Delta p_\pi = \displaystyle\sum_{\pi = \pi_1 \oplus \pi_2} p_{\pi_1} \otimes p_{\pi_2}$

<u>Obs</u>: Any permutation π has a <u>unique</u> factorization $\pi = \pi_1 \oplus \cdots \oplus \pi_j$ into $\oplus$-indecomposable permutations. Thus, we have in this case

$$\Delta^{\circ k-1}(p_\pi) = \sum_{(\alpha_1, \dots, \alpha_k) \models^{\circ} [j]} p_{\pi_{\alpha_1}} \otimes \cdots \otimes p_{\pi_{\alpha_k}}$$

interval set composition

Where $\pi_{\alpha_i} = \pi_a \oplus \pi_{a+1} \oplus \cdots \oplus \pi_{b-1} \oplus \pi_b$ whenever $\alpha_i = \{a, a+1, \dots, b-1, b\}$

The Takeuchi formula on $\mathcal{H}(Per)$

Because $\mathcal{H}(Per)$ is a filtered connected Hopf algebra, Takeuchi's formula holds. Thus, we have that if $\pi = \pi_1 \oplus \cdots \oplus \pi_j$ is the $\oplus$-factorization

$$S(P_\pi) = \sum_{k \geq 0} (-1)^k \mu^{\circ k-1} \circ (id_H - u \circ \varepsilon)^{\otimes k} \circ \Delta^{\circ k-1} (P_\pi)$$

$$= \sum_{k \geq 0} (-1)^k \sum_{\substack{(\alpha_1, \dots, \alpha_k) \vDash [j] \\ \text{interval set composition}}} P_{\pi_{\alpha_1}} \cdots P_{\pi_{\alpha_k}}$$

$$= \sum_{k \geq 0} (-1)^k \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_k) \vDash [j] \\ \text{interval set composition}}} \sum_{\tau} \sum_{\substack{(A_1, \dots, A_j) \text{ sub seqs} \\ \text{of } \tau \text{ of type } \pi_1, \dots, \pi_j \text{, resp.} \\ \text{that are } \alpha\text{-interlaced}}} P_\tau .$$

α-interlaced means that this is in fact a cover contributing to the product $P_{\pi_{\alpha_1}} \cdots P_{\pi_{\alpha_k}}$

$$= \sum_{\tau} P_\tau \sum_{\substack{(A_1, \dots, A_j) \text{ sub seqs} \\ \text{of } \tau \text{ of type } \pi_1, \dots, \pi_j \text{, resp.}}} \sum_{\substack{\alpha \vDash [j] \text{ interval set composition} \\ (A_1, \dots, A_j) \text{ is } \alpha\text{-interlaced}}} (-1)^{\ell(\alpha)} \qquad (**)$$

<u>Claim:</u> $\displaystyle\sum_{\substack{\alpha \vDash [j] \text{ interval set composition} \\ (A_1, \dots, A_j) \text{ is } \alpha\text{-interlaced}}} (-1)^{\ell(\alpha)}$ $= (-1)^j$, whenever $(A_1, \dots, A_j)$ is not α-interlaced for any α, except if $\alpha = (\{1\}, \dots, \{j\})$.

$= 0$, otherwise.

<u>Proof</u>: The first part is clear, since the sum on the right has only one term. For the remaining, define $I_\tau^\pi = I_\tau^\pi(A_1, \dots, A_j) = \{ \alpha \vDash [j] \text{ interval set compositions} \mid (A_1, \dots, A_j) \text{ is } \alpha\text{-interlaced} \}$

<u>Prop 1</u>: I_τ^π is an ideal of the poset of set compositions.

Prop 2: Let $G = ([j], E)$ be the graph defined as follows. We say that $i_1 \longleftrightarrow i_2$ whenever $i_1 < i_2$ and

- The positions of the subseq. A_{i_1} are all before the positions of A_{i_2}
- The values of the subseq. A_{i_1} are all before the values of A_{i_2}

Then $\alpha = (\alpha_1, \dots, \alpha_k) \in I_\tau^\pi$ iff each α_i is a clique in G.

Prop 3: The cliques in G that are intervals form a poset. This poset is the vee prod. of Boolean posets $B_{A_1} \vee \cdots \vee B_{A_k}$, where A_i are the maximal interval cliques of G. As a consequence, the poset I_τ^π is isomorphic to the Boolean poset $\prod_{i=1}^{k} B_{|A_i|-1} \cong B_n$, where $n = j - k$.

It follows that

$$\sum_{\substack{\alpha \models [j] \text{ interval set composition} \\ (A_1, \dots, A_j) \text{ is } \alpha\text{-interlaced}}} (-1)^{\ell(\alpha)} = \sum_{\alpha \in I_\tau^\pi(A_1, \dots, A_j)} (-1)^{\ell(\alpha)} =$$

$$= \sum_{A \subseteq [j-k]} (-1)^{|A|} = 0 \quad \text{whenever } k > j.$$

$\square$

Consequence: We have a cancelation-free formula for the antipode of $\mathcal{A}(\mathrm{Per})$ from $(\ast\ast)$ as $\pi = \pi_1 \oplus \cdots \oplus \pi_j$ gives us

$$S(P_\pi) = (-1)^j \sum_{\substack{\tau \text{ non-interlacing} \\ \text{quasi-shuffle of} \\ \text{of } \pi_1, \dots, \pi_j}} P_\tau$$

A non-interlacing quasi-shuffle of $\pi_1, \dots, \pi_k$ is a perm τ and subseqs $(A_1, \dots, A_k)$ s.t. $\tau|_{A_i} = \pi_i$ and for $i = 1, \dots, k-1$ we have either

- max pos $A_i \geq$ min pos A_{i+1}
- max val $A_i \geq$ min val A_{i+1}

Example: Take $\pi = 1 \oplus 21$. We can describe the non-interlacing quasi-shuffles of $1, 21$: these are always permutations of size at most three and at least two.

no quasi-shuffles

interlacing

interlaced in position but not in value!

$21 \to$

$1 \to$

21 ✓ 21 ✓ ~~12~~ ~~123~~ ~~132~~ 231 ✓ 231 ✓ 213 ✓ 312 ✓ 312 ✓ 321 ✓ 321 ✓ 321 ✓

Thus, $S(P_{132}) = 2P_{21} + 2P_{231} + P_{213} + 2P_{312} + 3P_{321}$

On the other hand, we can observe that this is the expected value, as

$$\Delta P_{132} = P_{\emptyset} \otimes P_{132} + P_1 \otimes P_{21} + P_{132} \otimes P_{\emptyset} \qquad \text{so}$$

$$0 = \underbrace{S(P_{\emptyset})}_{=1} \cdot P_{132} + \underbrace{S(P_1)}_{=-P_1} \cdot P_{21} + S(P_{132}) \cdot \underbrace{P_{\emptyset}}_{=1}$$

$$\Rightarrow S(P_{132}) = -P_{132} + P_1 P_{21}$$

thus, the only quasi-shuffle that does not contribute to $S(P_{132})$ is precisely the interlaced one.

Obs: Note that $\mathfrak{H}(Per)$ is commutative, therefore $S^2 = id$.

This means that there is massive cancelation to get

$$S\left(\sum_{\substack{\sigma \text{ non-interlacing} \\ q\text{-s of } \pi_1, \dots, \pi_k}} P_\sigma \right) = P_{\pi_1 \otimes \cdots \otimes \pi_k}$$