The Birkoff von Neumann polytope

Volumes, triangulations, and magic squares
seminar diskrete mathematik

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The **Birkoff von Neumann** polytope

The **Birkoff von Neumann** polytope is the set of matrices that are *doubly stochastic*:

$$B_n = \{A \in \mathbb{R}_{\geq 0}^{n \times n} | \forall_{j=1,\dots,n}, \sum_i A_{i,j} = \sum_i A_{j,i} = 1\}.$$

Examples of matrices in this region:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Permutation matrices are particular examples of doubly-stochastic matrices!

Questions about B_n

Problem

What is the dimension of the ***Birkoff von Neumann*** *polytope?*

We have a *dimension preserving* injective map, by selecting the entries in the uppermost and leftmost minor:

$$\mathbb{R}^{n\times n} \to \mathbb{R}^{(n-1)\times(n-1)}; \quad B_n \mapsto A_n$$

$$\text{for example : } \frac{1}{5}\begin{pmatrix}1 & 2 & 2\\ 2 & 2 & 1\\ 2 & 1 & 2\end{pmatrix} \mapsto \frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}$$

$$A_n = \left\{ \sum_i A_{i,j} \leq 1, \sum_{i,j} A_{j,i} \leq 1, \sum_i A_{i,j} \geq 2n-2 \right\} .$$

Thus $\dim B_n = \dim A_n = (n-1) \times (n-1)$.

Questions about B_n

Theorem (Birkoff von Neumann theorem)

*The **Birkoff von Neumann** polytope is the convex hull of the permutation matrices of dimension n.*

Question: this is an exercise in the book? I have no idea how to establish this in a simple way...

Problem

*What is the $(n-1)^2$-volume of the **Birkoff von Neumann** polytope?*

Three ways of computing this volume: twice via the **Ehrhard polynomial** and once by finding a **triangulation**.

The polytope A_3

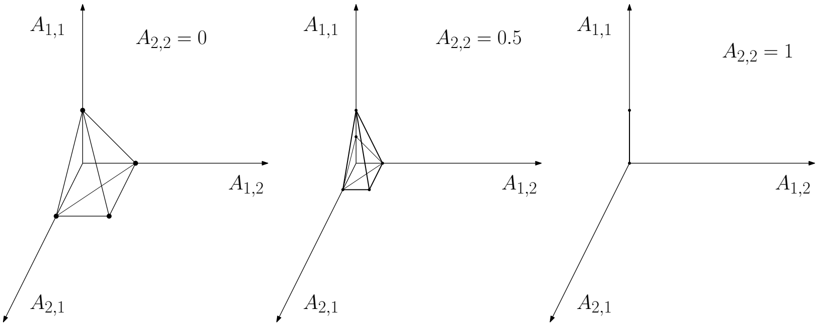


Figure: From left to right, we have $A_3 \cap \{x_{2,2} = 0\}$, $A_3 \cap \{x_{2,2} = 0.5\}$, $A_3 \cap \{x_{2,2} = 1\}$.

The Erhart polynomial

Let $P \subseteq \mathbb{R}^d$ be a polytope. The Ehrhart polynomial is defined as:

$$E_P(t) = |\mathbb{Z}^d \cap tP|, \ t \text{ non-negative integer}.$$

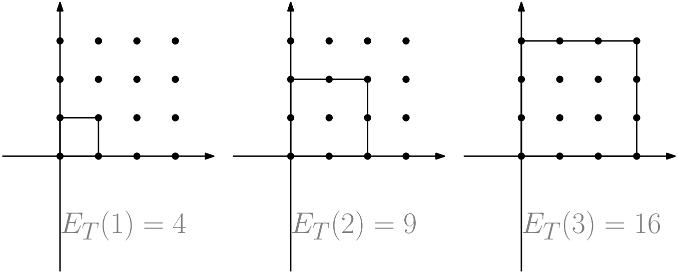


Figure: The value of the Ehrhart polynomial for a few values of t.
$E_T(t) = (t+1)^2 = t^2 + 2t + 1$

The Erhart polynomial

Theorem (Erharht theorem)

If $P = \operatorname{conv}\{v_1, \dots, v_n\}$ is a polytope of dimension d, where each v_i has integer coordinates, then $E_P(t)$ is a polynomial. Furthermore, it has degree d, and the leading coefficient is the volume of P.

Example: if T is the unit square, then

$$\operatorname{vol} T = \operatorname{coeff}_{t^2} E_T(t) = \operatorname{coeff}_{t^2}(t+1)^2 = 1$$

Strategies for computing $E_{B_n}(t)$: via power series magic, and by interpolation/ computing the first few values.
Find a combinatorial interpretation of the value $E_{B_n}(t) = |\mathbb{Z}^{n\times n} \cap tB_n|$.

The Erhart polynomial - magic squares

A **magic square** (sometimes called **semi-magic square**) is an $n \times n$ matrix with coefficients in $\mathbb{Z}_{\geq 0}$, whose rows and columns sum up to t.

$$\text{For example, for } t = 5 : \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

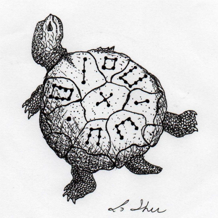


Figure: An example of a magic square dating back to the first century AC.

Triangulations of a polytope

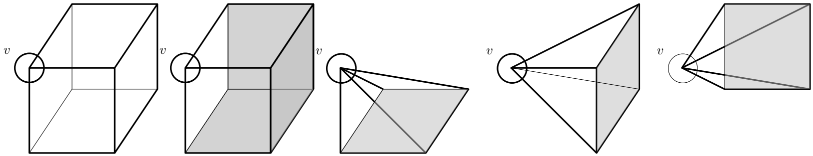


Figure: Computing the volume of the cube

Theorem (Magic and very rare theorem)

*Any simplex in the **Birkoff von Neumann** polytope has the same volume* $\frac{n^{n-1}}{(n-1)^2!}$. *Thus,*

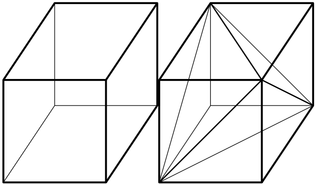
$$\operatorname{vol} B_n = \#\{ \text{ triangles in a triangulation of } B_n\} \frac{n^{n-1}}{(n-1)^2!} .$$

Why is it rare?

Theorem (Magic and very rare theorem)

Any simplex in the ***Birkoff von Neumann*** *polytope has the same volume.*

This does not happen in the cube! There are triangulations with six and with five simplices.



Outline of the talk

Power series galore

Introduce $2n$ variables $z_1, \dots, z_n, w_1, \dots, w_n$, and perform some power series magic.
For $n \times n$ matrix A, define

$$r_i(A) = \text{ sum of the entries in row } i$$

$$c_i(A) = \text{ sum of the entries in column } i$$

$$F(z_1, \dots, z_n, w_1, \dots, w_n) = \sum_{\substack{A \text{ non-negative matrix} \\ \text{integer}}} \prod_i z_i^{r_i(A)} w_i^{c_i(A)} .$$

Goal: find the coefficient at $z_1^t \cdots z_n^t w_1^t \cdots w_n^t$ algebraically, this **has** to be $E_{B_n}(t)$, the number of magic squares of row sum and column sum t.

Power series galore - case $n = 2$

$$F(z_1, z_2, w_1, w_2) = \sum_{\substack{a_{1,1}\geq 0\\ a_{1,2}\geq 0\\ a_{2,1}\geq 0\\ a_{2,2}\geq 0}} z_1^{a_{1,1}+a_{1,2}} z_2^{a_{2,1}+a_{2,2}} w_1^{a_{1,1}+a_{2,1}} w_2^{a_{1,2}+a_{2,2}}$$

$$= \sum_{\substack{a_{1,1}\geq 0\\ a_{1,2}\geq 0\\ a_{2,1}\geq 0\\ a_{2,2}\geq 0}} (z_1 w_1)^{a_{1,1}} (z_2 w_1)^{a_{2,1}} (z_1 w_2)^{a_{1,2}} (z_2 w_2)^{a_{2,2}} = \left(\sum_{a_{1,1}\geq 0} (z_1 w_1)^{a_{1,1}} \right)$$

$$\left(\sum_{a_{1,2}\geq 0} (z_1 w_2)^{a_{1,2}} \right) \left(\sum_{a_{2,1}\geq 0} (z_2 w_1)^{a_{2,1}} \right) \left(\sum_{a_{2,2}\geq 0} (z_2 w_2)^{a_{2,2}} \right) .$$

Power series galore - major simplification

$$F(z_1, z_2, w_1, w_2) = \left(\sum_{a_{1,1} \geq 0} (z_1 w_1)^{a_{1,1}} \right) \left(\sum_{a_{1,2} \geq 0} (z_1 w_2)^{a_{1,2}} \right) \dots$$

$$= \frac{1}{(1 - z_1 w_1)(1 - z_2 w_1)(1 - z_1 w_2)(1 - z_2 w_2)}$$

This works in general for any n, so we have a product with n^2 terms:

$$F(\mathbf{z}, \mathbf{w}) = \sum_{\substack{A \text{ integer} \\ \text{non-negative matrix}}} \prod_i z_i^{r_i(A)} w_i^{c_i(A)} = \prod_{i,j=1,\dots,n} \frac{1}{1 - z_i w_j}.$$

Power series galore

$$E_{B_n}(t) = \mathrm{coeff}_{z_1^t \cdots w_n^t}(F) = \mathrm{coeff}_1 \left((z_1^t \cdots w_n^t)^{-1} \prod_{i,j=1,\ldots,n} \frac{1}{1 - z_i w_j} \right)$$

By splitting $\prod_{j=1,\ldots,n} \frac{1}{1 - z_i w_j} = \sum_{j=1,\ldots,n} \frac{A_j}{1 - z_i w_j}$ for some carefully chosen A_j (and a lot of "massaging") we get:

Proposition

$$E_{B_n}(t) = \mathrm{coeff}_1 \left((z_1^t \cdots z_n^t)^{-1} \sum_{k=1}^{n} \frac{z_k^{n-1+t}}{\prod_{i \neq k} z_k - z_i} \right)^n .$$

Power series galore - case $n = 2$

$$E_{B_2}(t) = \mathrm{coeff}_1 (z_1^t z_2^t)^{-1} \left(\frac{z_1^{t+1}}{z_1 - z_2} + \frac{z_2^{t+1}}{z_2 - z_1} \right)^2$$

$$= \mathrm{coeff}_1 (z_1^t z_2^t)^{-1} \left(\frac{z_1^{t+1} - z_2^{t+1}}{z_1 - z_2} \right)^2$$

Now assume that the variable t is an integer :

$$E_{B_2}(t) = \mathrm{coeff}_{z_1^t z_2^t} \left(\sum_{i=0}^{t} z_1^i z_2^{t-i} \right)^2 = \mathrm{coeff}_{z_1^t z_2^t} \sum_{a,b=0,\dots,t} z_1^{a+b} z_2^{2t-a-b}$$

$$= \sum_{\substack{a,b=0,\dots,t \\ a+b=t}} 1 = t + 1 \,.$$

Power series galore - case $n = 4$

Much in the same way, by using

$$E_{B_n}(t) = \mathrm{coeff}_1 \left((z_1^t \cdots z_n^t)^{-1} \sum_{k=1}^{n} \frac{z_k^{n-1+t}}{\prod_{i \neq k} z_k - z_i} \right)^n ,$$

one can compute by hand:

$$E_{B_4}(t) = \frac{11}{11340}t^9 + \frac{11}{630}t^8 + \frac{19}{135}t^7 + \frac{2}{3}t^6 + \frac{1109}{540}t^5 + \frac{43}{10}t^4 + \frac{35117}{5670}t^3 + \frac{379}{63}t^2 + \frac{65}{18}t + 1 .$$

Interpolating Ehrhart

If $E_{B_n}(t)$ is the number of magic squares of size n and row sum t, then we can simply try to compute all such squares. Can we?

Problem

How many magic squares are there of size n and row sum t?

Interpolating Ehrhart - some properties

Ehrhart theory says that:

$$E_{B_n}(-n-t) = (-1)^{n-1} E_{B_n}(t)\,,$$

$$E_{B_n}(0) = 1 = E_{B_n}(-n)(-1)^{n-1},\ E_{B_n}(-n+1) = \cdots = E_{B_n}(-1) = 0\,.$$

So in general it is enough to compute $E_{B_n}(t)$ for $t = 1, \ldots, \binom{n-1}{2}$, because $E_{B_n}(t)$ has degree $(n-1)^2$.

Interpolating Ehrhart - magic squares $n = 3$

$$E_{B_3}(0) = 1 = E_{B_3}(-3)(-1)^2, \; E_{B_3}(-2) = E_{B_3}(-1) = 0$$

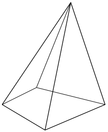
Only quartic polynomial that satisfies this and $E_{B_3}(1) = 6$ is

$$E_{B_3}(t) = (t+1)(t+2)\left(\frac{1}{8}t^2 + \frac{3}{8}t + \frac{1}{2}\right) .$$

Thus, $\operatorname{vol} B_3 = \frac{1}{8}$.
Challenge: Can you do the same for $n = 4$? Hint: $E_{B_4}(1) = 24$, $E_{B_4}(2) = 282$, $E_{B_4}(3) = 2008$.

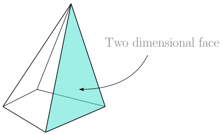
Faces of a polytope

What is a polytope, what are the faces of the polytope?



$\{x \in \mathbb{R}^n | Ax \leq b\}$

A face is determined by the set of vertices that it contains (but not the other way around)!



Faces of the Birkoff von Neumann polytope

Proposition

The faces of Birkoff von Neumann polytope are indexed by ***some*** *0-1 matrix U. This face corresponds to the set of doubly stochastic matrices*

$$F_U = \{A \in B_n | A_{i,j} > 0 \text{ only if } U_{i,j} = 1\}.$$

Example $n = 2$. $B_n = \operatorname{conv}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is a line segment. The one dimensional face corresponds to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Faces of the Birkoff von Neumann polytope - case $n = 3$

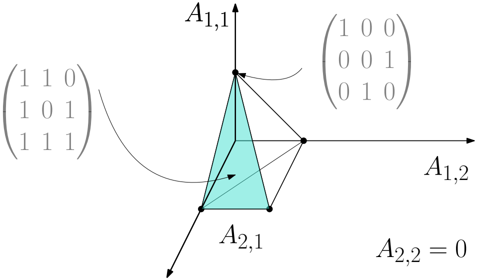
Example $n = 3$, $B_n = \mathrm{conv} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Some edges are indexed by the following matrices:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Faces of the **Birkoff von Neumann** polytope - case $n = 3$



Faces of the Birkoff von Neumann polytope - case $n = 3$

Some $0 - 1$ matrices do not correspond to a face, for instance:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

If a vertex A is in F_U, then $A_{1,1} = 0$.

Proposition

If $F_U \subseteq T_V$ then $U \leq V$.

Regular subdivisions of a polytope - and inductive procedure

Let P be a polytope.

- Pick a vertex v, and consider all **facets** that are **opposite** to v, $\{F_1, \dots, F_k\}$.
- Find a regular subdivision of each face $F_i = \bigcup_j T_{i,j}$ into simplices.
- Extend each simplex thus obtained $T_{i,j}$ to the piramid $T^o_{i,j} = \operatorname{conv} T_{i,j} \cup \{v\}$.

The resulting recursive algorithm gives us a triangulation of $P = \bigcup_{i,j} T^o_{i,j}$.

$$\operatorname{vol} P = \sum_{i,j} \operatorname{vol} T^o_{i,j} \, .$$

Regular subdivisions of a polytope - the **Birkoff von Neumann** polytope

- Start with a face F_U of B_n, say $U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.
- Select a vertex P of this face. A permutation matrix $P \leq U$.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Identify all facets of F_U that are opposed to P.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & {\color{red}1} & {\color{red}1} \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & {\color{red}1} & 1 \\ 1 & {\color{red}1} & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

Regular subdivisions of a polytope - Algorithm observations

- Method computes the volume of **all faces** of B_n.
- More efficient than the two methods described above, when $n \geq 6$.
- Already intractable when $n \geq 9$ (around 2^{n^2} faces).

n	2	3	4	5	6	7	8
Δ	1	3	352	4718075	$1.47 \cdot 10^{13}$	$1.78 \cdot 10^{22}$	$1.28 \cdot 10^{34}$

The relative volumes of the **Birkoff von Neumann** polytope.

Open problems

- The coefficients of the Erhard polynomial are all non-negative.
- Let F_n be the face of B_n corresponding to the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

Then $\operatorname{vol} F_n = \prod_{i=0}^{n-2} \frac{1}{i+1}\binom{2i}{i} = \prod_{i=0}^{n-2} C_i$.

Biblio

- Clara S. Chan and David P. Robbins (1999). On the volume of the polytope of douplby stochastic matrices. *Experiment. Math. 8, 291-300*.
- Matthias Beck and Sinai Robins (2015). Chapter 6: Computing the continuous Discretely. *Springer*.

Thank you