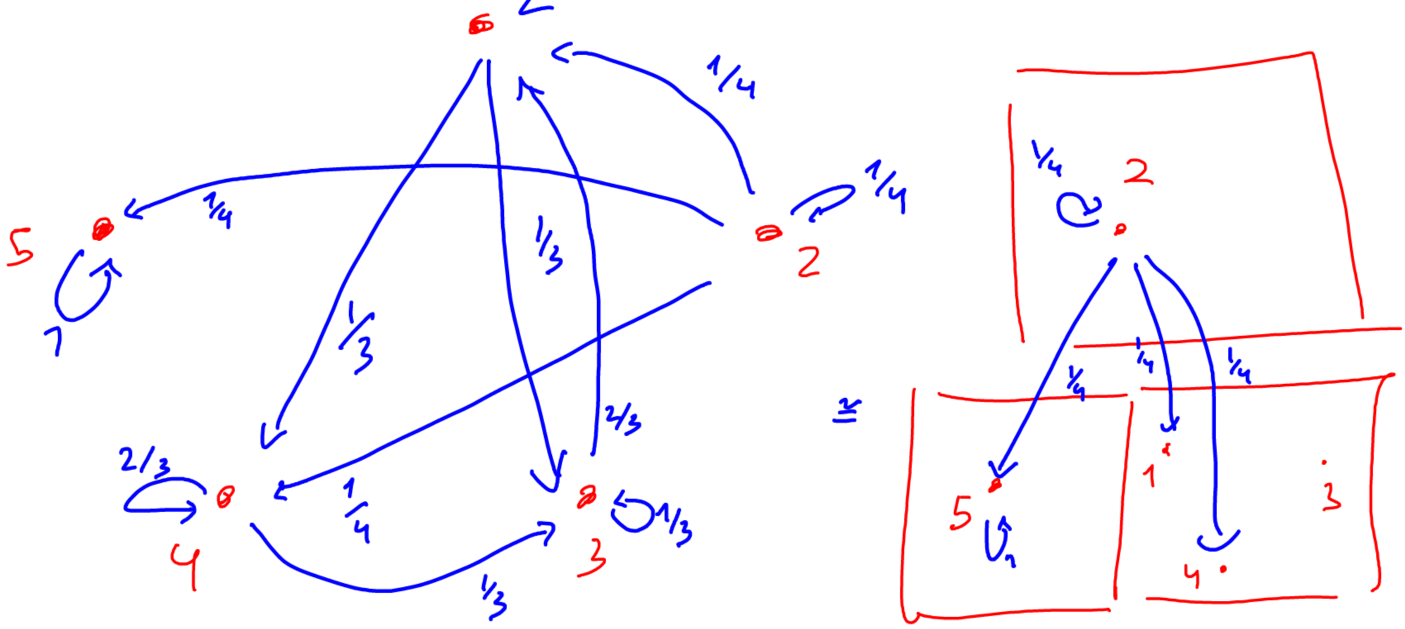
Exercise 1



$\{5\}$ and $\{1,4,3\}$ are closed irreducible sets. Because they are finite, all states within them are recurrent.

2 is transient because

$$\mathbb{P}_2\left[\exists\, n>0 : X_n = 2\right] \leq 1 - \mathbb{P}_2\left[X_1 = 5\right] = \frac{3}{4} < 1$$

this holds because $\{X_1 = 5\} = \{X_n = 5 \ \forall n \geq 1\} \subseteq \{\forall n>0 \ X_n \neq 2\}$

Then $\{1,2,3,4,5\} = \{2\} \cup \{1,3,4\} \cup \{5\}$ is the desired decomposition.

$$F^{(n)}_{2,5} = \mathbb{P}(X_n = 5, \ X_i \neq 5 \text{ for } i = 1,\dots,n-1 \mid X_0 = 2) =$$

$$= \frac{\mathbb{P}(X_n = 5, \ X_i \neq 5 \text{ for } i=1,\dots,n-1 \ X_0 = 2)}{\mathbb{P}(X_0 = 2)} =$$

Because $\{X_0 = 2, X_n = 5, X_i \neq 5 \text{ for } i = 1,\dots,n-1\}$

$= \{X_0 = X_1 = \dots = X_{n-1} = 2, \ X_n = 5\}$

$$= \frac{\mathbb{P}(X_n = 5, \ X_0 = X_1 = \dots = X_{n-1} = 2)}{\mathbb{P}(X_0 = 2)} =$$

$$= \left[\prod_{i=0}^{n-2} \mathbb{P}(X_{i+1} = 2 \mid X_i = 2)\right] \mathbb{P}(X_n = 5 \mid X_{n-1} = 2) = \left(\frac{1}{4}\right)^n$$

Thus, $\mathbb{P}(T=n) = F^{(n)}_{2,5} = (1/4)^n$ for $n \geq 1$

$\mathbb{P}(T=+\infty) = 1 - \sum_{n\geq 0} \mathbb{P}(T=n) = 1 - \frac{4^{-1}}{1-4^{-1}} = 2/3$ ∎

Exercise 2 2.1 We mimic the proof of Prop 14.2 (ii).

Let $x, y \in S$ be fixed, and define $B^n_r := \{X_n = y, X_{n-1} \neq x, \dots, X_{n-r+1} \neq x, X_{n-r} = x\}$

$A_n = \{X_n = y, X_0 = x\}$

Our goal is to show that

$$\sum_{r=0}^{n} Q^{(n-r)}_{x,x} \cdot L^{(r)}_{x,y} = Q^{(n)}_{x,y} \qquad *$$

the claim follows after multiplying $*$ by t^n and suming it for $n \geq 0$ (that this sum converges for $|t| < 1$ is because each term is < 1).

Then, $\mathrm{RHS}(*) = Q^{(n)}_{x,y} = \mathbb{P}(A_n \mid X_0 = x) = \dfrac{\mathbb{P}(A_n)}{\mathbb{P}(X_0 = x)}$

Now note that $A_n = \biguplus_{r=1}^{n} (B^n_r \cap A_n)$ so $\mathbb{P}(A_n) = \sum_{r=1}^{n} \mathbb{P}(B^n_r \cap A_n)$ and

$$Q^{(n)}_{x,y} = \sum_{r=1}^{n} \frac{\mathbb{P}(B^n_r \cap A_n)}{\mathbb{P}(X_0 = x)} = \sum_{r=1}^{n} \underbrace{\frac{\mathbb{P}(B^n_r, X_0 = x)}{\mathbb{P}(X_0 = x, X_{n-r} = x)}}_{A} \times \underbrace{\frac{\mathbb{P}(X_0 = x, X_{n-r} = x)}{\mathbb{P}(X_0 = x)}}_{B} \qquad *_2$$

$$A = \mathbb{P}(B^n_r \mid X_0 = x, X_{n-r} = x) \underset{\text{Markov Property}}{=} \mathbb{P}(B^n_r \mid X_{n-r} = x)$$

$$\underset{\text{time-homogeneous}}{=} \mathbb{P}_x(X_r = y, X_{r-1} \neq x, \dots, X_1 \neq x) = L^{(r)}_{x,y}$$

$$B = \mathbb{P}(X_{n-r} = x \mid X_0 = x) = Q^{(n-r)}_{x,x}$$

Thus $*_2$ becomes $Q^{(n)}_{x,y} = \sum_{r=1}^{n} L^{(r)}_{x,y} \, Q^{(n-r)}_{x,x}$.

Because $L^{(0)}_{x,y} = 0$, $*$ follows ∎

2.2. $Q_{z,y}(t) = Q_{y,y}(t) \cdot F_{x,y}(t)$ (Prop 14.2 ii)

$= Q_{z,z}(t) \cdot L_{x,y}(t)$ (Ex 2.1)

Then we have that, if $Q_{x,x}(t) = Q_{y,y}(t)$, because $Q_{x,x}(t) \neq 0$ we conclude

$$F_{x,y}(t) = L_{x,y}(t) \quad \text{for } t \in [0,1)$$

it follows that the coeficients coincide by taking derivatives

$$\frac{1}{r!}\left(\frac{d}{dt}\right)^r F_{x,y}(t)\Big|_{t=0} = F_{x,y}^{(r)}$$

$$\shortparallel$$

$$\frac{1}{r!}\left(\frac{d}{dt}\right)^r L_{x,y}(t)\Big|_{t=0} = L_{x,y}^{(r)}$$

$\square$

Exercise 3

3.1 $X_n = \sum_{i=1}^n Y_i = \left[\sum_{i=1}^n\left(\frac{1}{2}Y_i + \frac{1}{2}\right)\right] \times 2 - n = 2\left(\sum_{i=1}^n F_i\right) - n$

where $F_i := \frac{1}{2}Y_i + \frac{1}{2} \sim \text{Ber}(1/2)$ are i.i.d.

Thus $Z_n := \sum_{i=1}^n F_i \sim \text{Bin}(n, 1/2)$ and

$$Q_{0,0}^{2n} = \mathbb{P}(X_{2n} = 0 \mid X_0 = 0) = \mathbb{P}(2 Z_{2n} = 2n) = \binom{2n}{n} 0.5^n 0.5^{2n-n}$$

$$= \binom{2n}{n}\frac{1}{2^{2n}}$$

3.2 $Q_{0,0}^{2n} = \binom{2n}{n}\frac{1}{2^{2n}} = \frac{2n!}{(n!)^2}\frac{1}{2^{2n}} = \left(\frac{\cancel{2}n}{e}\right)^{2n} \cdot \left(\frac{e}{n}\right)^{2n} \cdot \frac{\sqrt{2\pi\, 2n}}{2\pi\, n} \frac{1}{\cancel{2^{2n}}} \cdot (1 + o(1))$

$$= \frac{1}{\sqrt{\pi n}}(1 + o(1))$$

Thus, the partial sums $\sum_{n=1}^{k} Q_{0,0}^{2n} = \sum_{n=1}^{k} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{\pi}} + o(1)\right)$

$$= \left[\int_1^{k+1} \frac{1}{\sqrt{x}}\,dx + O(1)\right] \times \left[\frac{1}{\sqrt{\pi}} + o(1)\right]$$

$$= \frac{1}{\sqrt{\pi}} \frac{\sqrt{k-1}-1}{2} + o(k^{1/2}) = \frac{\sqrt{k}}{2\sqrt{\pi}} + o(\sqrt{k})$$

It follows that $\sum_{n=1}^{+\infty} Q_{0,0}^{n} = \sum_{n=1}^{+\infty} Q_{0,0}^{2n} = +\infty$, so 0 is recurrent by Theorem 14.4 (i)

3.3 Because the process is given by independent Markov chains, we can compute

$$Q_{\vec{0},\vec{0}}^{2n} = \left(Q_{0,0}^{2n}\right)^{d} = \binom{2n}{n}^{d} \frac{1}{2^{2nd}} = \frac{1}{(\pi n)^{d/2}}\left(1+o(1)\right)$$

Then,

$$\sum_{n=1}^{k} Q_{\vec{0},\vec{0}}^{2n} = \sum_{n=1}^{k} \frac{1}{\sqrt{n}^{d}}\left(\frac{1}{\sqrt{\pi}^{d}} + o(1)\right) = \left(\int_{1}^{k-1} x^{-\frac{d}{2}}\,dx + O(1)\right)\left(\pi^{-\frac{d}{2}} + o(1)\right)$$

$$\overset{\text{for } d=2}{=} \left(\log(k-1) + O(1)\right)\left(\pi^{-1} + o(1)\right) \xrightarrow[k]{} +\infty$$

$$\overset{\text{for } d\geq 3}{=} \left(\frac{(k-1)^{-\frac{d-2}{2}} - 1}{-\frac{d-2}{2}} + O(1)\right)\left(\pi^{-d/2} + o(1)\right) \xrightarrow[k]{} \pi^{-d/2}\frac{2}{d-2} < +\infty$$

Thus, from Theorem 14.4 (i), $\vec{0}$ is recurrent for $d=2$, and transient for $d \geq 3$. □