Exercise 9

(a) We wish to show that

(x). $$\mathbb{P}(M_{n+1}=s_{n+1} \mid M_n=s_n,\dots,M_1=s_1)=\mathbb{P}(M_{n+1}=s_{n+1}\mid M_n=s_n) \qquad \forall\, s_1,\dots,s_{n+1}\in S,\ n\geq 1$$

(y). $\mathbb{P}(M_{n+1}=i \mid M_n=j)$ does not depend on n.

Note that $$M_{n+1}=M_n\,\mathbb{1}[X_{n+1}\leq M_n]+X_{n+1}\,\mathbb{1}[X_{n+1}> M_n]=\begin{cases} M_n, & \text{if } M_n > X_{n+1}\\ X_{n+1}, & \text{if } M_n < X_{n+1}\end{cases}$$

Thus, if $s_{n+1}<s_n$, then both sides of (x) evaluate to 0.

if $s_{n+1}=s_n$, then

$$\mathbb{P}(M_{n+1}=s_{n+1} \mid M_n=s_n,\dots,M_1=s_1)=\mathbb{P}(M_{n+1}=M_n \mid M_n=s_n,\dots,M_1=s_1)=$$

$$=\mathbb{P}(X_{n+1}\leq s_n \mid M_n=s_n,\dots,M_1=s_1)\underset{X_{n+1}\perp\!\!\!\perp M_n,\dots,M_1}{=}\mathbb{P}(X_{n+1}\leq s_n)$$

and $$\mathbb{P}(M_{n+1}=s_{n+1}\mid M_n=s_n)=\mathbb{P}(M_{n+1}=M_n\mid M_n=s_n)=\mathbb{P}(X_{n+1}\leq s_n\mid M_n=s_n)$$

$$\underset{X_{n+1}\perp\!\!\!\perp M_n,\dots,M_1}{=}\mathbb{P}(X_{n+1}\leq s_n)=\mathbb{P}(X_1\leq s_n)$$

finally, if $s_{n+1}>s_n$,

$$\mathbb{P}(M_{n+1}=s_{n+1}\mid M_n=s_n,\dots,M_1=s_1)=\mathbb{P}(X_{n+1}=s_{n+1}\mid M_n=s_n,\dots,M_1=s_1)$$

$$\underset{\perp\!\!\!\perp}{=}\mathbb{P}(X_{n+1}=s_{n+1})=\mathbb{P}(X_{n+1}=s_{n+1}\mid M_n=s_n)$$ which concludes (x)

To establish (y), just note that $\mathbb{P}(X_{n+1}=s_{n+1})=\mathbb{P}(X_1=s_{n+1})$

Thus in both the cases $s_{n+1}>s_n$ and $s_{n+1}=s_n$ we get that

$\mathbb{P}(M_{n+1}=s_{n+1}\mid M_n=s_n)$ only depends on s_{n+1}, s_n and not on the time n □

Thus, $S = \mathbb{Z}_{\geq 0}$ and the transition matrix Q has

$$Q(b,a) = \mathbb{P}(M_{n+1} = a \mid M_n = b) = \begin{cases} 0 & , \text{ if } b > a \\ \mathbb{P}(X_1 \leq b) = \sum_{i=0}^{b} \binom{10}{i} 0.3^i 0.7^{10-i}, & \text{if } b = a \\ \mathbb{P}(X_1 = a) = \binom{10}{a} 0.3^a 0.7^{10-a}, & \text{if } b < a \end{cases}$$

Careful with the order ! ⚠

(b) $N_{n+1} = N_n + 1\!\!1[X_{n+1} = 0] = \begin{cases} N_n & , \text{ if } X_{n+1} > 0 \\ N_n + 1, & \text{if } X_{n+1} = 0 \end{cases}$

Thus, if $s_{n+1} \neq s_n$ and $s_{n+1} \neq s_n + 1$, both sides of ⊗ vanish.

If $s_{n+1} = s_n$, $\mathbb{P}(N_{n+1} = s_{n+1} \mid N_n = s_n, \dots, N_1 = s_1) = \mathbb{P}(X_{n+1} > 0 \mid N_n = s_n, \dots, N_1 = s_1)$

$$\underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_{n+1} > 0) = 1 - 0.7^{10}$$

$$\mathbb{P}(N_{n+1} = s_{n+1} \mid N_n = s_n) = \mathbb{P}(X_{n+1} > 0 \mid N_n = s_n) \underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_{n+1} > 0) = 1 - 0.7^{10}$$

If $s_{n+1} = s_n + 1$, $\mathbb{P}(N_{n+1} = s_{n+1} \mid N_n = s_n, \dots, N_1 = s_1) = \mathbb{P}(X_{n+1} = 0 \mid N_n = s_n, \dots, N_1 = s_1) =$

$$\underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_{n+1} = 0) = 0.7^{10}$$

$$\mathbb{P}(N_{n+1} = s_{n+1} \mid N_n = s_n) = \mathbb{P}(X_{n+1} = 0 \mid N_n = s_n) \underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_{n+1} = 0) = 0.7^{10}$$

Thus ⊗ holds, and we have $S = \mathbb{Z}_{\geq 0}$, and the transition matrix Q has

$$Q(b,a) = \mathbb{P}(N_{n+1} = a \mid N_n = b) = \begin{cases} 0, & \text{if } a \notin \{b, b+1\} \\ 1 - 0.7^{10}, & \text{if } a = b \\ 0.7^{10}, & \text{if } a = b+1 \end{cases}$$

(d) We can write $Q_{n+1} = (Q_n + 1) \times 1\!\!1_{X_{n+1} = 0}$ for $n \geq 1$

$$Q_1 = 1\!\!1_{X_1 = 0}$$

So, for a seq. $s_1, \dots, s_n, s_{n+1}$, if $s_{i+1} \notin \{0, s_i + 1\}$ for all $i = 1, \dots, n-1$ then $\mathbb{P}(\vec{Q} = \vec{s}) = 0$, so we don't need to establish the Markov property.

If $s_{n+1} \notin \{0, s_n+1\}$ then both sides of are zero.

If $s_{n+1} = 0$,

$$\mathbb{P}(Q_{n+1} = s_{n+1} \mid \vec{Q} = \vec{s}) = \mathbb{P}(X_{n+1} = 0 \mid \vec{Q} = \vec{s}) \underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_{n+1} = 0)$$

$$\mathbb{P}(Q_{n+1} = s_{n+1} \mid Q_n = s_n) = \mathbb{P}(X_{n+1} = 0 \mid Q_n = s_n) \underset{\perp\!\!\!\perp}{=}$$

If $s_{n+1} = s_n + 1$

$$\mathbb{P}(Q_{n+1} = s_{n+1} \mid \vec{Q} = \vec{s}) = \mathbb{P}(X_{n+1} \neq 0 \mid \vec{Q} = \vec{s}) \underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_{n+1} \neq 0)$$

$$\mathbb{P}(Q_{n+1} = s_{n+1} \mid Q_n = s_n) = \mathbb{P}(X_{n+1} \neq 0 \mid Q_n = s_n) \underset{\perp\!\!\!\perp}{=}$$

It follows that this is a Markov process and that $S = \mathbb{Z}_{\geq 0}$.
Further, the transition matrix is

$$\mathbb{P}(Q_{n+1} = j \mid Q_n = i) = \begin{cases} 0, & \text{if } j \notin \{i+1, 0\} \\ \mathbb{P}(X_1 = 0), & \text{if } j = 0 \\ \mathbb{P}(X_1 \neq 0), & \text{if } j = i+1 \end{cases}$$

This does not depend on the time n, so it is a Markov chain $\square$

(c) & (e) These are not Markov processes. This is so because

$$\mathbb{P}(P_3 = -5 \mid P_2 = 9, P_1 = 1) \neq \mathbb{P}(P_3 = -5 \mid P_2 = 9) \qquad (\dagger)$$

and

$$\mathbb{P}(S_4 = 2 \mid S_3 = 1, S_2 = 2, S_1 = 1) \neq \mathbb{P}(S_4 = 2 \mid S_3 = 1) \qquad (\dagger\dagger)$$

Indeed LHS of $(\dagger)$ is

$$\mathbb{P}(P_3 = -5 \mid P_2 = 9, P_1 = 1) = \mathbb{P}(P_3 = -5 \mid X_1 = 1, X_2 = 10)$$

$$= \mathbb{P}(X_3 = 5 \mid X_1 = 1, X_2 = 10) \underset{\perp\!\!\!\perp}{=} \mathbb{P}(X_3 = 5) = \binom{10}{5} 0.3^5 0.7^5$$

RHS of $(\dagger)$ is $\mathbb{P}(P_3 = -5 \mid P_2 = 9) = \mathbb{P}(P_3 = -5 \mid X_2 = 10, X_1 = 1 \text{ or } X_2 = 9, X_1 = 0)$

$$= \frac{\mathbb{P}(\{X_3 - X_2 = -5\} \cap \{X_2 = 10, X_1 = 1 \text{ or } X_2 = 9, X_1 = 0\})}{\mathbb{P}(X_2 = 10, X_1 = 1 \text{ or } X_2 = 9, X_1 = 0)} = \frac{\mathbb{P}(X_3 = 5, X_2 = 10, X_1 = 1) + \mathbb{P}(X_3 = 4, X_2 = 9, X_1 = 0)}{\mathbb{P}(X_2 = 10, X_1 = 1) + \mathbb{P}(X_2 = 9, X_1 = 0)}$$

$$= \frac{\binom{10}{5}\cancel{\binom{10}{10}}\cancel{\binom{10}{1}} 0.3^{5+10+1} 0.7^{5+0+9} + \binom{10}{4}\cancel{\binom{10}{9}}\cancel{\binom{10}{0}} 0.3^{4+9+0} 0.7^{6+1+10}}{\cancel{\binom{10}{10}}\cancel{\binom{10}{1}} 0.3^{10+1} 0.7^{0+9} + \cancel{\binom{10}{9}}\cancel{\binom{10}{0}} 0.3^{9+0} 0.7^{1+11}}$$

$\binom{10}{4} = \frac{5}{6}\binom{10}{5}$

$$= \binom{10}{5} \frac{0.3^{16} 0.7^{14} + 0.3^{13} 0.7^{17} \frac{5}{6}}{0.3^{11} 0.7^{9} + 0.3^{9} 0.7^{11}} = \binom{10}{5} \frac{0.3^3 + 0.7^3 \cdot \frac{5}{6}}{0.3^2 + 0.7^2} \cdot 0.3^4 \cdot 0.7^5$$

Thus,

$$\frac{\text{LHS of } (\dagger)}{\text{RHS of } (\dagger)} = \frac{\cancel{\binom{10}{5}} \cancel{0.3^5} \cancel{0.7^5} (0.3^2 + 0.7^2)}{\cancel{\binom{10}{5}} \cancel{0.3^4} \cancel{0.7^5} (0.3^3 + 0.7^3 \frac{5}{6})} = \frac{0.3 (0.3^2 + 0.7^2)}{0.3^3 + 0.7^3 \frac{5}{6}}$$

$$= \frac{0.3^3 + 0.7^2\, 0.3}{0.3^3 + 0.7^2 \frac{0.7 \times 5}{6}} \neq 1$$

Now to establish $(\dagger\dagger)$, we have

$$\text{LHS}(\dagger\dagger) = \mathbb{P}(S_4 = 2 \mid S_3 = 1, S_2 = 2, S_1 = 1) = \mathbb{P}(X_4 = X_3 \mid X_3 \neq X_2 = X_1) = \frac{\mathbb{P}(X_4 = X_3 \neq X_2 = X_1)}{\mathbb{P}(X_3 \neq X_2 = X_1)}$$

$$= \frac{\sum_{i \neq j} \mathbb{P}(X_3 = X_4 = i, X_1 = X_2 = j)}{\sum_{i \neq j} \mathbb{P}(X_3 = i, X_2 = X_1 = j)} \underset{\perp\!\!\!\perp \text{ and } X_i \sim \mathrm{Bin}(10, 0.3)}{=} \frac{\sum_{i=0}^{10} \sum_{j=0, j \neq i}^{10} \binom{10}{i}^2 \binom{10}{j}^2 0.3^{2i+2j} 0.7^{40-2i-2j}}{\sum_{i=0}^{10} \sum_{j=0, j \neq i}^{10} \binom{10}{i} \binom{10}{j}^2 0.3^{i+2j} 0.7^{30-i-2j}} \begin{matrix} =: c \\ =: b \end{matrix}$$

$$\text{RHS}(\dagger\dagger) = \mathbb{P}(S_4 = 2 \mid S_3 = 1) = \frac{\mathbb{P}(X_4 = X_3 \neq X_2)}{\mathbb{P}(X_3 \neq X_2)} = \frac{\sum_{i \neq j} \mathbb{P}(X_4 = X_3 = i, X_2 = j)}{\sum_{i \neq j} \mathbb{P}(X_3 = i, X_2 = j)}$$

$$= \frac{\sum_{i=0}^{10} \sum_{j=0, i \neq j}^{10} \binom{10}{i}^2 \binom{10}{j} 0.3^{i+j} 0.7^{20-2i-j}}{\sum_{i=0}^{10} \sum_{j=0, i \neq j}^{10} \binom{10}{i} \binom{10}{j} 0.3^{i+j} 0.7^{20-i-j}} \begin{matrix} = b \\ =: a \end{matrix}$$

In this way,

$\text{LHS}(\dagger\dagger) = c/b$

$\text{RHS}(\dagger\dagger) = b/a$

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In[ ]:= a = Sum[If[i == j, 0, Binomial[10, i] Binomial[10, j] (3/10)^(i+j) (7/10)^(20-i-j)],
      {i, 0, 10}, {j, 0, 10}];
    b =
      Sum[If[i == j, 0, Binomial[10, i]^2 Binomial[10, j] (3/10)^(2 i + j) (7/10)^(30 - 2 i - j)],
      {i, 0, 10}, {j, 0, 10}];
    c = Sum[If[i == j, 0, Binomial[10, i]^2 Binomial[10, j]^2
        (3/10)^(2 i + 2 j) (7/10)^(40 - 2 i - 2 j)], {i, 0, 10}, {j, 0, 10}];
    N[b/a]
    N[c/b]

Out[ ]= 0.186622

Out[ ]= 0.181764
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Computing w/ the help of a computer we get that these values are different $\square$

Exercise 2 We show that both processes are Markov processes but not Markov chains. Indeed, it is easy to see that the Markov property only needs to be establish whenever $s_{i+1} \in \{s_i+1, \dots, s_i+i+2\}$, $i<n$. Here

$$\mathbb{P}(S_{n+1} = s_{n+1} \mid \vec{S} = \vec{s}) = \mathbb{P}(X_{n+1} = s_{n+1} - s_n \mid \vec{S} = \vec{s}) = \mathbb{P}(X_{n+1} = s_{n+1} - s_n)$$

$$= \begin{cases} \frac{1}{n+1}, & \text{if } s_{n+1} \in \{s_n+1, \dots, s_n+n+2\} \\ 0, & \text{o/w} \end{cases}$$

$$\mathbb{P}(S_{n+1} = s_{n+1} \mid S_n = s_n) = \mathbb{P}(X_{n+1} = s_{n+1} - s_n \mid S_n = s_n)$$

$$= \mathbb{P}(X_{n+1} = s_{n+1} - s_n).$$

Thus, the transition matrix of $\{S_n\}_{n\geq 1}$ is

$$Q^n(i,j) = \mathbb{P}(S_{n+1} = j \mid S_n = i) = \begin{cases} 1/n+1, & \text{if } j-i \in \{1, \dots, n+1\} \\ 0, & \text{o/w} \end{cases}$$

This depends on n (for instance $Q^n(2,1) = 1/n+1$), so this is not a Markov chain.

For $\{X_n\}_{n\geq 0}$, the Markov property is easily established

$$\mathbb{P}(X_{n+1} = s_{n+1} \mid \vec{X} = \vec{x}) = \mathbb{P}(X_{n+1} = s_{n+1}) = \mathbb{P}(X_{n+1} = s_{n+1} \mid X_n = s_n)$$

And the transition matrix is, then, $Q^n(i,j) = \mathbb{P}(X_{n+1} = j) = \begin{cases} 1/n+1, & \text{if } j \in \{1, \dots, n+1\} \\ 0, & \text{o/w} \end{cases}$

which is time dependent, so this is not a Markov chain.

Exercise 3 We wish to show the Markov property on $\{X_{n\wedge T}\}_{n\geq 1}$.

Fix n and $s_1, \dots, s_{n+1}$.

Case 1: there is no i s.t. $s_i \in E$, $i \leq n$. (Note: s_{n+1} may be in E)

Then $$\mathbb{P}(X_{n+1\wedge T} = s_{n+1} \mid X_{i\wedge T} = s_i \;\; i=1,\dots,n) = \frac{\mathbb{P}(X_{n+1\wedge T} = s_{n+1},\; X_{i\wedge T} = s_i \;\; i=1,\dots,n)}{\mathbb{P}(X_{i\wedge T} = s_i \;\; i=1,\dots,n)}$$

$$= \frac{\mathbb{P}(X_{n+1} = s_{n+1}, X_i = s_i \ i=1,\dots,n)}{\mathbb{P}(X_i = s_i \ i=1,\dots,n)} = \mathbb{P}(X_{n+1} = s_{n+1} \mid X_i = s_i \ i=1,\dots,n) =$$

because we have that $\{X_{n\wedge T} = s_n\} \subseteq \{T \geq n+1\}$, since $s_n \notin E$

Rem: this is only true in case 1!

because $\{X_i\}_{i\geq 1}$ is a Markov Chain

$$= \mathbb{P}(X_{n+1} = s_{n+1} \mid X_n = s_n) =$$

On the other hand, because $\{X_{n\wedge T} = s_n\} \subseteq \{T \geq n+1\}$,

$$\mathbb{P}(X_{n+1\wedge T} = s_{n+1} \mid X_{n\wedge T} = s_n) = \frac{\mathbb{P}(X_{n+1\wedge T} = s_{n+1}, X_{n\wedge T} = s_n)}{\mathbb{P}(X_{n\wedge T} = s_n)} = \frac{\mathbb{P}(X_{n+1} = s_{n+1}, X_n = s_n)}{\mathbb{P}(X_n = s_n)}$$

$$= \mathbb{P}(X_{n+1} = s_{n+1} \mid X_n = s_n)$$

Case 2: There is some i s.t. $s_i \in E$, for $i \leq n$. Let I be the smallest $i \leq n$ s.t. $S_I \in E$. Then $\{X_{T\wedge i} = s_i$ for $i=1,\dots,n\} \subseteq \{T = I\}$ and thus

$$\mathbb{P}(X_{T\wedge i} = s_i \text{ for } i=1,\dots,n) = 0 \text{ whenever } s_n = s_{n-1} = \dots = s_I \text{ is false.}$$

So to establish the Markov property we need only to consider the case where $s_n = s_{n-1} = \dots = s_{I+1} = s_I$. There, we have two cases

Case 2.1: $s_{n+1} = s_n$

Then $\mathbb{P}(X_{n+1\wedge T} = s_{n+1} \mid X_{i\wedge T} = s_i \text{ for } i=1,\dots,n) =$

$$= \frac{\mathbb{P}(X_{n+1\wedge T} = s_{n+1}, X_{i\wedge T} = s_i \text{ for } i=1,\dots,n)}{\mathbb{P}(X_{i\wedge T} = s_i \text{ for } i=1,\dots,n)} = \frac{\mathbb{P}(X_i = s_i \text{ for } i \leq I)}{\mathbb{P}(X_i = s_i \text{ for } i \leq I)} = 1$$

Case 2.2: $s_{n+1} \neq s_n$. Then

$$\mathbb{P}(X_{n+1\wedge T} = s_{n+1} \mid X_{i\wedge T} = s_i \text{ for } i=1,\dots,n) =$$

$$= \frac{\mathbb{P}(X_{n+1\wedge T} = s_{n+1}, X_{i\wedge T} = s_i \text{ for } i=1,\dots,n)}{\mathbb{P}(X_{i\wedge T} = s_i \text{ for } i=1,\dots,n)} = \frac{0}{\mathbb{P}(X_i = s_i \text{ for } i \leq I)} = 0 \quad \square$$

Exercise 4 Wlog, $i > j$. There are five cases to consider

(i) If $2 \mid i, j$, then $X_i \perp\!\!\!\perp X_j$ by def.

(ii) If $i - j \geq 3$, then $i-1 > j+1$, thus

X_i is $\sigma(X_{2n}, \text{ for } 2n \geq i-1)$ - measurable

X_j is $\sigma(X_{2n}, \text{ for } 2n < i-1)$ - measurable

so X_i, X_j are indep

(iii) If $i = j+2$, both odd, then $X_j = X_{i-3} \cdot X_{i-1}$; $X_i = X_{i-1} X_{i+1}$.

Recall that $X \perp\!\!\!\perp Y$ if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$ ⊛
for any Borel-measurable sets A, B.

We will show directly that $X_i \perp\!\!\!\perp X_j$.

Because $X_i, X_j \in \{1, -1\}$ it suffices to study $A, B \subseteq \{1, -1\}$. Note that whenever A or B are $\emptyset$ or $\{1,-1\}$, the claim is trivial. The remaining probabilities can be directly computed to give $\frac{1}{4}$. That is for any $\varepsilon, \delta \in \{1, -1\}$ we have

$$*_1 \qquad \mathbb{P}(X_i = \varepsilon, X_j = \delta) = \frac{1}{4} = \mathbb{P}(X_i = \varepsilon) \cdot \mathbb{P}(X_j = \delta)$$

(iv) If $i = j+1$, with $i = 2n$, we can also show directly that $X_i \perp\!\!\!\perp X_j$ by way of ⊛. Specifically $*_1$ holds

(v) If $i = j+1$ with $j = 2n$, we can also show directly that $X_i \perp\!\!\!\perp X_j$ by way of ⊛. Specifically $*_1$ holds

(b) From (a), $\mathbb{P}(X_{n+1} = \varepsilon \mid X_n = \delta) = \mathbb{P}(X_{n+1} = \varepsilon) = \frac{1}{2}$ $\quad *_2$

On the other hand, because $X_{2n+1} = X_{2n} X_{2n+2}$, we have that

$$X_{2n} \cdot X_{2n+1} \cdot X_{2n+2} = X_{2n}^2 \cdot X_{2n+2}^2 = 1. \qquad *_3$$

In particular, the value of X_{2n}, X_{2n+1} determines X_{2n+2}.

So this is not a Markov chain. Specifically, the Markov property is not fulfiled for $n=0$, $s_0 = s_1 = 1$, $s_2 = -1$ as

$$\mathbb{P}(X_2 = -1 \mid X_1 = 1, X_0 = 1) = 1 \quad \text{from } *_3$$

$$\mathbb{P}(X_2 = -1 \mid X_1 = 1) = 1/2 \quad \text{from } *_2$$

$\square$