Exercise 1

$X_\infty = \lim_{m\to+\infty} X_m$ a.s. So, fix $n \in \mathbb{N}$ and see

$$\mathbb{E}[X_\infty \mid \mathcal{F}_n] \underset{\uparrow}{\leq} \liminf_m \mathbb{E}[X_m \mid \mathcal{F}_n] \underset{\uparrow}{\leq} X_n \quad \text{a.s.}$$

Fatou's Lemma — supermartingale property for $m \geq n$

as desired.

Exercise 2

Let A_n = "at time n get a white ball"

B_n = "at time n get a black ball" $= A_n^c$

Then $Y_{n+1} = Y_n + \mathbb{1}_{A_n}$.

ⓐ Observe that $\mathbb{P}(A_n \mid \mathcal{F}_n) = X_n$

Then $\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = Y_n + \mathbb{E}[\mathbb{1}_{A_n} \mid \mathcal{F}_n] = Y_n + X_n$

ⓑ First $0 \leq X_n \leq 1$ is bounded, so $\{X_n\}_{n\geq 0}$ is u.i. and in $\mathcal{L}^1$. By the $\mathcal{L}^1$ Convergence theorem of martingales, it is enough to establish that $\{X_n\}_{n\geq 0}$ satisfies the martingale property.

Indeed, $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \frac{1}{n+3}\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \frac{1}{n+3}(Y_n + X_n)$

$$= \frac{1}{n+3}\left((n+2)\cdot X_n + X_n\right) = X_n \quad \square$$

ⓒ We first establish a Lemma

<u>Lemma:</u> $Y_n \to +\infty$ a.s.

Proof: Y_n is non decreasing. Label the first white ball in the bag and assume that this ball stays there whenever it is picked (in this way, we always keep track of one ball). We claim that this ball will be picked infinitely many times a.s, concluding the Lemma.

Let E_k be the event that this labelled ball is picked in turn k. Note that $\mathbb{P}(E_k) = \frac{1}{k+2}$, also note that the events $\{E_k\}_{k\geq 0}$ are independant. We wish to show that

$$\mathbb{P}(\limsup E_k) = \mathbb{P}\left(\bigcap_{n\geq 0} \bigcup_{k\geq n} E_k\right) = 1$$

Note that $\mathbb{P}\left(\bigcup_{k\geq n} E_k\right) = 1 - \mathbb{P}\left(\bigcap_{k\geq n} E_k^c\right) = 1 - \prod_{k\geq n} \mathbb{P}(E_k) = 1 - \prod_{k\geq n+2} \frac{1}{k}$

Thus, $\mathbb{P}\left(\bigcup_{k\geq n} E_k\right) = 1$ and $\bigcup_{k\geq n} E_k \searrow \bigcap_{n\geq 0} \bigcup_{k\geq 0} E_k$

so by monotone convergence theorem, $\mathbb{P}(\limsup E_k) = 1$ ∎

Let us now show that $Z_{n;r}$ is a martingale.

First, observe that $0 \leq Z_{n;r} \leq 1$, because $0 \leq Y_n + j \leq n+2+j$ for all $j = 0, \dots, r-1$.

Thus, $Z_{n;r} \in \mathcal{L}^1$. Let us use the notation $[a]^c = \frac{(a+c-1)!}{(a-1)!}$

Then $Z_{n;r} = \frac{[Y_n]^r}{[n+2]^r}$.

So $\mathbb{E}\left[Z_{n+1;r} \mid \mathcal{F}_n\right] = \mathbb{E}\left[\frac{[Y_n + 1\!\!1_{A_n}]^r}{[n+3]^r} \mid \mathcal{F}_n\right] =$

$$= \mathbb{E}\left[\frac{[Y_n+1]^r}{[n+3]^r} 1\!\!1_{A_n} + \frac{[Y_n]^r}{[n+3]^r} 1\!\!1_{B_n} \mid \mathcal{F}_n\right] =$$

$$= \frac{[Y_n+1]^{r-1}}{[n+3]^r} \mathbb{E}\left[(Y_n+r)\cdot 1\!\!1_{A_n} + Y_n 1\!\!1_{B_n} \mid \mathcal{F}_n\right] = \frac{[Y_n+1]^{r-1}}{[n+3]^r} \mathbb{E}\left[Y_n + r\, 1\!\!1_{A_n} \mid \mathcal{F}_n\right]$$

$$= \frac{[Y_n+1]^{r-1}}{[n+3]^r} \left(Y_n + r X_n\right) = \frac{[Y_n+1]^{r-1}}{[n+3]^r}\left(Y_n + \frac{r}{n+2} Y_n\right) = \frac{[Y_n]^r}{[n+2]^{r+1}}(n+2+r)$$

$$= \frac{[Y_n]^r}{[n+2]^r} \frac{n+\cancel{2+r}}{n+\cancel{2+r}} = Z_{n;r}$$

Thus, $Z_{n;r}$ is a martingale. Since it is bounded it is u.i., so by the $\mathcal{L}^1$ convergence theorem on martingales, there is a r.v. $U^{(r)}$ in $\mathcal{L}^1$ s.t. $Z_{n;r} \xrightarrow[n]{a.s.\ \mathcal{L}^1} U^{(r)}$.

However, $$Z_{n;r} = \frac{[Y_n]^r}{[n+2]^r} = \frac{Y_n^{\ r}}{(n+2)^r} \cdot \frac{[Y_n]^r}{Y_n^r} \cdot \frac{(n+2)^r}{[n+2]^r}$$

Because $Y_n \xrightarrow{a.s.} +\infty$ we have that
$$\frac{[Y_n]^r}{Y_n^r} \cdot \frac{(n+2)^r}{[n+2]^r} \longrightarrow 1 \quad a.s.$$
Hence $$\frac{Z_{n;r}}{X_n^r} \longrightarrow 1 \quad a.s., \quad \text{thus } Z_{n;r} \to U^r \text{ a.s.}$$

It follows that $U^r = U^{(r)}$, as desired □

(d) $\mathbb{E}[Z_{0;r}] = \lim_{n\to+\infty} \mathbb{E}[Z_{n;r}] = \mathbb{E}[U^r]$

$$\Rightarrow \mathbb{E}[U^r] = \mathbb{E}\left[\frac{[Y_0]^r}{[2]^r}\right] = \mathbb{E}\left[\frac{[1]^r}{[2]^r}\right] = \frac{1}{r+1}$$

$$\mathbb{E}[V^r] = \int_0^1 t^r\, dt = \frac{1}{r+1} \quad \text{so} \quad \mathbb{E}[U^r] = \mathbb{E}[V^r] \quad \forall r \geq 1. \quad \square$$

Exercise 3 (a) For $n>1$,
$$\{T=-n\} = \{\underbrace{S_n \geq n}_{\in \mathcal{F}_{-n}}\} \cap \{\underbrace{S_{n+1} < n+1}_{\in \mathcal{F}_{-n}}\} \cap \cdots \cap \{\underbrace{S_N < N}_{\in \mathcal{F}_{-n}}\} \in \mathcal{F}_{-n}$$

For $n=1$, $$\{T=-1\} = \underbrace{\{S_1 \geq 1\}}_{\in \mathcal{F}_{-1}} \cup \bigcap_{k=1}^{N} \underbrace{\{S_k < k\}}_{\in \mathcal{F}_{-1}} \in \mathcal{F}_{-1}$$

It follows that T is a stopping time □

(b) We show that $\{X_{T\wedge n}\}_{n \leq -1}$ is a backward martingale.

In fact, $T \in \{1, \dots, N\}$ so $|X_{T\wedge n}| \leq \max_{i=1,\dots,N}\{|X_i|\} \in \mathcal{L}^1$.

Thus $X_{T\wedge n}$ is in $\mathcal{L}^1$. On the other hand, we have

$$X_{T\wedge(n+1)} = \left(\sum_{i=-N}^{n} X_i \, 1\!\!1_{T=i}\right) + X_{n+1} \, 1\!\!1_{T\geq n+1}$$

$$X_{T\wedge n} = \left(\sum_{i=-N}^{n} X_i \, 1\!\!1_{T=i}\right) + X_n \, 1\!\!1_{T\geq n+1}$$

It follows that $X_{T\wedge n}$ is $\mathcal{F}_n$ measurable. Also we have

$$\mathbb{E}[X_{T\wedge(n+1)} \mid \mathcal{F}_n] = \left(\sum_{i=-N}^{n} \mathbb{E}[X_i 1\!\!1_{T=i} \mid \mathcal{F}_n]\right) + \mathbb{E}[X_{n+1} 1\!\!1_{T\geq n+1} \mid \mathcal{F}_n]$$

$$= \left(\sum_{i=-N}^{n} X_i 1\!\!1_{T=i}\right) + \underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{=X_n} \cdot 1\!\!1_{T\geq n+1} = X_{T\wedge n}$$

because $1\!\!1_{T\geq n+1} = 1 - 1\!\!1_{T\leq n}$ is $\mathcal{F}_n$-measurable.

c) We start by showing that if $\omega \in A = \{S_k \geq k \text{ for some } k=1,\dots,N\}$ and $S_N(\omega) = b$, then $S_{-T} = -T$.

In fact, by definition, and because $\omega \in A$, $S_{-T} \geq -T$. On the other hand, because $S_N(\omega) = b < N$, $-T < N$ and by maximality we have that $S_{-T+1} < -T+1$. Now since S_n is non-decreasing we have $S_{-T+1} = -T = S_{-T}$, (*1) concluding the claim.

Now, if $\omega \notin A$ but $S_N(\omega) = b$, then $T = -1$ and $S_1 < 1$. Thus, $S_1 = 0$ and $\frac{S_{-T}}{-T} = 0$. (*2)

Putting (*1) and (*2) together, we get

$$1\!\!1_A \cdot 1\!\!1_{S_N=b} = \frac{S_{-T}}{-T} 1\!\!1_{S_N=b}. \qquad (*_s)$$

Taking expectations gives us $\mathbb{P}(A \cap S_N=b) = \mathbb{E}\left[\frac{S_{-T}}{-T} 1\!\!1_{S_N=b}\right] =$

$$= \mathbb{E}\left[\mathbb{E}\left(\frac{S_{-T}}{-T} 1\!\!1_{S_N=b} \mid \mathcal{F}_{-N}\right)\right] \underset{\substack{\uparrow \\ S_N \text{ is } \mathcal{F}_{-N}\text{-meas}}}{=} \mathbb{E}\left[\mathbb{E}\left[\frac{S_{-T}}{-T} \mid \mathcal{F}_{-N}\right] 1\!\!1_{S_N=b}\right] =$$

$$\overset{(3)}{=} \mathbb{E}\left[\frac{S_N}{N} \cdot \mathbb{1}_{S_N = b} \right] = \mathbb{E}\left[\frac{b}{N} \mathbb{1}_{S_N = b} \right] = \frac{b}{N} \mathbb{P}(S_N = b)$$

With the assumption that $\mathbb{P}(S_N = b) > 0$, we have that

$$\frac{\mathbb{P}(A \cap S_N = b)}{\mathbb{P}(S_N = b)} = \frac{b}{N} \qquad \square$$