Exercise 1

(a) Observe that $\mathbb{1}_A \in \mathcal{L}^1$ so $X_n := \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_n]$ is well defined and $X_n \in \mathcal{L}^1$. Also $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ from the tower law (iv)

On the other hand, $A \in \mathcal{G}_\infty \subseteq \mathcal{G}_{n+1} \perp\!\!\!\perp \mathcal{F}_n$, so $A \perp\!\!\!\perp \mathcal{F}_n$ and

$$X_n = \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A) \quad \text{a.s.}$$

It follows that $X_n \xrightarrow{a.s.} X_\infty := \mathbb{P}(A)$ and because the martingale is closed, we have that $X_\infty = \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_\infty] = \mathbb{1}_A$ from Theorem 10.2.

But $\mathbb{1}_A = \mathbb{P}(A)$ a.s. $\Rightarrow \mathbb{P}(A) \in \{0, 1\}$, as desired ☒

(b) Let $A_x = \{Z < x\}$. Because Z is $\mathcal{G}_\infty$-measurable, we have that $A_x \in \mathcal{G}_\infty$ $\forall x \in \mathbb{R}$. Let $x_0 = \inf_{x \in \mathbb{R}} \{\mathbb{P}(A_x) = 1\}$.

We claim that $Z = x_0$ a.s.

First, observe that $x_0 \in \mathbb{R}$ is finite, as

$$A_n \underset{n \to +\infty}{\nearrow} \bigcup_k \{Z < k\} = \Omega \quad \text{so} \quad \lim_{n \to +\infty} \mathbb{P}(A_n) = 1, \text{ hence from}$$

the 0-1 law, $\mathbb{P}(A_n) = 1$ for some $n \geq 0$

$$A_{-n} \underset{n \to +\infty}{\searrow} \bigcap_k \{Z < k\} = \emptyset \quad \text{so} \quad \lim_{n \to +\infty} \mathbb{P}(A_{-n}) = 0, \text{ hence from}$$

the 0-1 law, $\mathbb{P}(A_{-n}) = 0$ for some $n \leq 0$.

Now, if $x > x_0$, there is some $t \in [x_0, x)$ s.t. $\mathbb{P}(A_t) = 1$ by the properties of infimum and the definition of x_0.
So $\mathbb{P}(A_x) \geq \mathbb{P}(A_t) \Rightarrow \mathbb{P}(A_x) = 1$.

if $x < x_0$, by definition $\mathbb{P}(A_x) \neq 1$, by the 0-1 law we have $\mathbb{P}(A_x) = 0$. It follows that $\mathbb{P}(Z \neq x_0) = 0$ ☐

Exercise 2 (a) Because $Y_{n+1} \perp\!\!\!\perp \overbrace{\sigma(Y_1,\dots,Y_n)}^{\tilde{\mathcal{F}}_n}$, we have that

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[\underset{\uparrow \atop \mathcal{F}_n\text{-meas}}{X_n} \cdot Y_{n+1} | \mathcal{F}_n] = X_n \, \mathbb{E}[Y_{n+1}] = X_n$$

$$\mathbb{E}[|X_n|] \underset{\uparrow \atop X_n \geq 0}{=} \mathbb{E}[X_n] \underset{\{Y_i\} \text{ indep}}{=} \prod_{i=1}^{n} \mathbb{E}[Y_i] = 1$$

Also, $t \longmapsto -\sqrt{t}$ is convex function, so $\{-\sqrt{X_n}\}$ is a submartingale and $\{\sqrt{X_n}\}$ is a supermartingale.

(b) Because $\{Z_n\}_{n\geq 1}$ is a positive supermartingale, it converges a.s. to some r.v. Z. By Fatou's Lemma we have

$$\mathbb{E}[Z] = \mathbb{E}[\liminf_n Z_n] \leq \liminf_n \mathbb{E}[Z_n]$$

$$\underset{\uparrow \atop \{Y_i\} \text{ are indep}}{=} \liminf_n \prod_{i=1}^{n} \mathbb{E}[\sqrt{Y_i}] = 0$$

It follows that $\mathbb{E}[Z] \leq 0$, because $Z \geq 0$ we conclude that $Z = 0$ a.s.

Because $\sqrt{X_n} \to 0$ a.s., $X_n \to 0$ a.s. ($\sqrt{\cdot}$ is continuous)

Now suppose that $\{X_n\}_{n\geq 0}$ is u.i.. Then the convergence above is in L^1 and so $\mathbb{E}[X_n] \longrightarrow 0$

However, $\mathbb{E}[X_n] \underset{\{Y_i\} \text{ indep.}}{=} \prod_{k=1}^{n} \mathbb{E}[Y_k] = 1$ so $\mathbb{E}[X_n] \to 1$, a contradiction ↯

(c) $$\|Z_n - Z_m\|_2 = \mathbb{E}[Z_n^2 + Z_m^2 - 2Z_n Z_m] \overset{\text{wlog } n \leq m}{=}$$

$$= \mathbb{E}\left[X_n + X_m - 2X_n\sqrt{\frac{X_m}{X_n}}\right] = 2 - 2\,\underbrace{\mathbb{E}[X_n]}_{=1} \cdot \prod_{k=n+1}^{m} \mathbb{E}[\sqrt{Y_k}]$$

$$= 2\left(1 - \prod_{k=n+1}^{m} \mathbb{E}[\sqrt{Y_k}]\right).$$ Fix $\varepsilon > 0$ and suppose wlog $\varepsilon < 2$

Because $t \mapsto -\sqrt{t}$ is a convex function, by Jensen's ineq.

$$-1 = -\sqrt{\mathbb{E}[Y_i]} \leq \mathbb{E}[-\sqrt{Y_i}] \quad (*)$$

So $\mathbb{E}[\sqrt{Y_i}] \leq 1$. Let $M = \lim_{N\to+\infty} \prod_{i=0}^{N} \mathbb{E}[\sqrt{Y_i}] = \prod_{i=0}^{\infty} \mathbb{E}[\sqrt{Y_i}]$

Because of $(*)$, M is the limit of a decreasing sequence.

So there is some N s.t. $\prod_{i=0}^{N} \mathbb{E}[\sqrt{Y_i}] < M / 1-\frac{1}{2}\varepsilon$ $\left(\text{Note } \frac{M}{1-\frac{1}{2}\varepsilon} > M\right)$

It follows that $\prod_{i=N+1}^{\infty} \mathbb{E}[\sqrt{Y_i}] = \dfrac{M}{\prod_{i=0}^{N} \mathbb{E}[\sqrt{Y_i}]} > 1-\frac{1}{2}\varepsilon$

Then, for $m, n > N$, $m \geq n$ wlog, we have

$$\| Z_m - Z_n \|_2 = 2\left[1 - \prod_{i=n+1}^{m} \mathbb{E}[\sqrt{Y_i}]\right] \underset{\mathbb{E}[\sqrt{Y_i}] \leq 1}{\leq} 2\left[1 - \prod_{i=N}^{\infty} \mathbb{E}[\sqrt{Y_i}]\right]$$

$$< 2\left[1 - (1-\tfrac{1}{2}\varepsilon)\right] = \varepsilon$$ So $\{Z_n\}_{n\geq 0}$ is Cauchy in L^2.

Now observe that $\| X_n - X_m \|_1 = \mathbb{E}[|X_n - X_m|] = \mathbb{E}[|Z_n^2 - Z_m^2|]$

$$= \mathbb{E}[|(Z_n - Z_m)\cdot(Z_n + Z_m)|] \underset{\text{Cauchy-Schwarz inequality.}}{\leq} \underbrace{\mathbb{E}[|Z_n - Z_m|^2]^{\frac{1}{2}}}_{\|Z_n - Z_m\|_2} \cdot \mathbb{E}[|Z_n + Z_m|^2]^{\frac{1}{2}}$$

But $\mathbb{E}[|Z_m + Z_n|^2]^{1/2} \leq \|Z_m\|_2 + \|Z_n\|_2$

$\leq 1 + 1 = 2$ So $\|X_m - X_n\|_1 \leq \|Z_m - Z_n\|_2 \cdot 2$.

It follows that $\{X_n\}_{n\geq 0}$ is Cauchy in L^1 so it converges in L^1 to some r.v. X. It follows from Prop A2 that $\{X_n\}_{n\geq 0}$ is u.i., so from theorem 10.2, this is a closed martingale $\square$

Exercise 3 ⓐ First, let $A_n = \{X_{n+1} = \frac{1}{2}X_n\}$, $B_n = \{X_{n+1} = \frac{1+X_n}{2}\}$.

Observe that $P[A_n \cup B_n] = \mathbb{E}[\mathbb{1}_{A_n} + \mathbb{1}_{B_n}] = \mathbb{E}[\mathbb{E}[\mathbb{1}_{A_n} | \mathcal{F}_n] + \mathbb{E}[\mathbb{1}_{B_n} | \mathcal{F}_n]]$

$= \mathbb{E}[1 - X_n + X_n] = 1$.

It follows that $X_{n+1} \in \{\frac{1}{2}X_n, \frac{1+X_n}{2}\}$ a.s., or

$$X_{n+1} = 1\!\!1[A_n]\,\frac{1}{2}X_n + 1\!\!1[B_n]\,\frac{1+X_n}{2} \qquad a.s.$$

<u>Claim 1:</u> $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ a.s.

Proof: $$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \underset{\substack{\uparrow \\ X_n \text{ is } \mathcal{F}_n\text{-m}}}{=} \mathbb{E}[1\!\!1_{A_n} \mid \mathcal{F}_n]\,\frac{1}{2}X_n + \mathbb{E}[1\!\!1_{B_n} \mid \mathcal{F}_n]\,\frac{1+X_n}{2}$$

$$= \frac{1}{2}X_n(1-X_n) + \frac{1}{2}X_n(1+X_n) = X_n \qquad \square$$

<u>Claim 2:</u> $0 \le X_n \le 1$ a.s.

Proof: Induction on n. This is the case for $n=0$, as $a \in [0,1]$.

Now, for $X_n \in [0,1]$, then $\frac{1}{2}X_n,\ \frac{1+X_n}{2} \in [0,1]$. Since
$X_{n+1} \in \{\frac{1}{2}X_n, \frac{1+X_n}{2}\}$ a.s., $X_{n+1} \in [0,1]$ a.s. $\square$

It follows that $X_n \in \mathcal{L}^1$, so it is a bounded martingale.
Because it is bounded, it converges a.s. and in $\mathcal{L}^p$ for $p < +\infty$ from theorem 10.5, to some X_∞.

It follows that $\mathbb{E}[X_k] \xrightarrow[k]{} \mathbb{E}[X_\infty]$

But $\mathbb{E}[X_k] = \mathbb{E}[X_0] = a \quad \forall k \ge 0$, so $\mathbb{E}[X_\infty]$.

(b) $$\mathbb{E}[(X_{n+1} - X_n)^2] = \mathbb{E}[(X_{n+1} - X_n)^2 \cdot (1\!\!1_{A_n} + 1\!\!1_{B_n})] =$$

$$= \mathbb{E}[(X_{n+1} - X_n)^2 1\!\!1_{A_n}] + \mathbb{E}[(X_{n+1} - X_n)^2 1\!\!1_{B_n}] =$$

$$= \mathbb{E}[(\tfrac{1}{2}X_n - X_n)^2 1\!\!1_{A_n}] + \mathbb{E}[(\tfrac{1+X_n}{2} - X_n)^2 1\!\!1_{B_n}] =$$

(CP)
$$= \mathbb{E}[\tfrac{1}{4}X_n^2\,\mathbb{E}[1\!\!1_A \mid \mathcal{F}_n]] + \mathbb{E}[\tfrac{1}{4}(1-X_n)^2\,\mathbb{E}[1\!\!1_{B_n} \mid \mathcal{F}_n]] =$$

$$= \mathbb{E}[\tfrac{1}{4}X_n^2(1-X_n)] + \mathbb{E}[\tfrac{1}{4}(1-X_n)^2 X_n] = \mathbb{E}[\tfrac{1}{4}X_n(1-X_n)(\underbrace{X_n + 1 - X_n}_{=1})]$$

$$= \frac{1}{4}\mathbb{E}[X_n(1-X_n)] \qquad \square$$

(c) $\mathbb{E}[X_n^2] \to \mathbb{E}[X_\infty^2]$ because $X_n \to X_\infty$ in L^2.

So $\mathbb{E}[X_n(1-X_n)] \to \mathbb{E}[X_\infty(1-X_\infty)]$

On the other hand, $\mathbb{E}[X_n(1-X_n)] = 4 \cdot \mathbb{E}[(X_{n+1}-X_n)^2] \to 0$ because $(X_n)_{n\geq 0}$ converges in L^2.

It follows that $\mathbb{E}[X_\infty(1-X_\infty)] = 0$. $(**)$

However, $X_n \xrightarrow{a.s} X_\infty$, and $X_n \in [0,1]$ a.s from part (a). It follows that $X_\infty \in [0,1]$ a.s., so that $X_\infty(1-X_\infty) \geq 0$ a.s.

Together with $(**)$, this gives us that $X_\infty(1-X_\infty) = 0$ a.s.

So $X_\infty = 0$ or $X_\infty = 1$ a.s.

Because $\mathbb{E}[X_\infty] = a$, $\mathbb{P}(X_\infty = 1) = a$

$\mathbb{P}(X_\infty = 0) = 1-a$ $\square$