Exercise Sheet 5 – Solutions

Exercise 1 (a) We first observe that $X_0, \dots, X_n$ are all $\mathcal{F}_n$-m. because $X_j = k + \sum_{i=1}^{j} Y_i$ is a Borel-measurable function on the $\{Y_i\}_{i=1}^n$, so by Exercise 3 of ES1, X_j is $\mathcal{F}_n$-measurable.

To show that T is a stopping time, we only need to show that $\{T \le n\} \in \mathcal{F}_n$. Equivalently, we will show that $\{T \le n\}^c = \{T > n\} \in \mathcal{F}_n$.
In fact, $\{T > n\} = \bigcap_{i=0}^{n} \underbrace{\{X_i \ne 0 \text{ and } X_i \ne m\}}_{X_i^{-1}(\mathbb{Z} \setminus \{0,m\}) \in \mathcal{F}_n}$ $\in \mathcal{F}_n$ $\square$

$X_i^{-1}(\mathbb{Z} \setminus \{0,m\}) \in \mathcal{F}_n$ because each X_i is $\mathcal{F}_n$-measurable.

(b) Because $\{X_{n \wedge T}\}_{n \ge 0}$ is a bounded martingale, it converges a.s. to some r.v. X_∞, by the a.s. convergence theorem on submartingales.

(c) We start by showing that $T < +\infty$ a.s.
Let $E_n = \{Y_n = 1\}$ and $F_n = \bigcap_{j=1}^{m} E_{n+j}$.

Claim 1: If $\omega \in F_n$ for some $n \ge 1$ then $T(\omega) \le n + m < +\infty$

Proof: If $T \ge n$, then $X_n(\omega) \in \{1, \dots, m-1\}$.
Because $\omega \in F_n$, $X_{n+j}(\omega) = j + X_n(\omega)$ for $j = 1, \dots, m$

In particular, for $j = m - X_n(\omega)$ we have
$X_{n+j}(\omega) = m \Rightarrow T(\omega) \le n + j < n + m$
as desired $\square$

Claim 2: $\mathbb{P}\left(\bigcup_{n \ge 0} F_n\right) = 1$

Proof: Because $\{Y_i\}_{i \ge 1}$ are independent r.v., the events $F_0^c, F_m^c, F_{2m}^c, \dots$ are independent. Let $j \ge 1$ be an integer, then
$$\mathbb{P}\left(\bigcap_{n \ge 0} F_n^c\right) \le \mathbb{P}\left(\bigcap_{n=0}^{j} F_{n \cdot m}^c\right)$$

$$\mathbb{P}\left(\bigcap_{n=0}^{j} F_{n\cdot m}^{c}\right) \underset{\substack{\uparrow\\ \text{independence of } \{F_0^c, \dots, F_{j\cdot m}^c\}}}{=} \prod_{n=0}^{j} \mathbb{P}(F_{n\cdot m}) = \prod_{n=0}^{j} \mathbb{P}\left(Y_{n\cdot m+1} \neq 1 \text{ or } \dots \text{ or } Y_{n\cdot m+m} \neq 1\right)$$

$$= \prod_{n=0}^{j}\left(1-2^{-m}\right) = \left(1-2^{-m}\right)^{j}$$

By choosing j arbitrarily large, we get

$$\mathbb{P}\left(\bigcap_{n\geq 0} F_n^c\right) \leq 0, \quad \text{so} \quad \mathbb{P}\left(\bigcup_{n\geq 0} F_n\right) = 1 \quad \square$$

From claim 1 we have that $\bigcup_{n\geq 0} F_n \subseteq \{T < +\infty\}$.

From Claim 2 we conclude that $\mathbb{P}(T < +\infty) = 1$ ■

By the bounded optional stopping time, $\mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_0] = k$ for any $n \geq 0$. Hence $\lim_{n\to+\infty} \mathbb{E}[X_{n\wedge T}] = k$.

On the other hand, because $T < +\infty$ a.s., $X_{n\wedge T} \xrightarrow{a.s.} X_T$ and because $\{X_{n\wedge T}\}$ are bounded r.v., this is a convergence in $\mathcal{L}^1$ so

$$k = \mathbb{E}[X_{n\wedge T}] \longrightarrow \mathbb{E}[X_T] = 0\cdot\mathbb{P}(X_T = 0) + m\cdot\mathbb{P}(X_T = m)$$

So $k = m\cdot\mathbb{P}(X_T = m)$ and $\mathbb{P}(X_T = 0) = 1 - \mathbb{P}(X_T = m) = \frac{m-k}{m}$

<u>Obs</u>: The formula $\mathbb{E}[X_T] = 0\cdot\mathbb{P}(X_T = 0) + m\,\mathbb{P}(X_T = m)$ holds because we have that $T < +\infty$ a.s.

Exercise 2

$X_n = |x_n|$ and $\mathbb{E}[|x_n|] = 0\cdot\left(1-\frac{1}{n}\right) + n\cdot\frac{1}{n} = 1 < +\infty$.

$\{X_n\}_{n\geq 0}$ is u.i. only if the following holds

$$\lim_{c\to+\infty} \sup_{n\geq 0} \mathbb{E}\left[|X_n|\,\mathbb{1}[|X_n| > c]\right] = 0$$

Indeed, fix $c > 0$ wlog integer,

$$\sup_{n\geq 0} \mathbb{E}\left[|X_n|\,\mathbb{1}[|X_n| > c]\right] \leq \mathbb{E}\left[|X_{c+1}|\,\mathbb{1}[|X_{c+1}| > c]\right]$$

$$= \mathbb{E}\left[|X_{c+1}|\right] = 1$$

It follows that $\lim_{c\to+\infty} \sup_{n\geq 0} \mathbb{E}\left[|X_n| \mathbb{1}[|X_n|>c]\right] \geq 1$

So $\{X_n\}_{n\geq 0}$ is not u.i.

Exercise 3 (b) Let $X_n^{(\alpha)} = X_n \mathbb{1}[X_n < \alpha]$ and $X^{(\alpha)} = X \mathbb{1}[X<\alpha]$.

Then $X_n^{(\alpha)} \longrightarrow X^{(\alpha)}$ whenever $X \neq \alpha$, so $X_n^{(\alpha)} \to X^{(\alpha)}$ a.s. if $\mathbb{P}(X=\alpha)=0$.

Let $U_n = \{\alpha \in \mathbb{R}_0^+ \mid \mathbb{P}(X=\alpha) \geq \frac{1}{n}\}$ and $U = \bigcup_{n\geq 0} U_n = \{\alpha\in\mathbb{R}_0^+ \mid \mathbb{P}(X=\alpha)>0\}$

Note that $|U_n| \leq n$, so U is countable, and for $\alpha \in \mathbb{R}_0^+ \setminus U$

we have $X_n^{(\alpha)} \longrightarrow X^{(\alpha)}$ a.s. Since $X_n^{(\alpha)}, X^{(\alpha)} \leq \alpha$, by the DCT we have that $X_n^{(\alpha)} \longrightarrow X^{(\alpha)}$ in $\mathcal{L}^1$ as well, so

$$\mathbb{E}[X_n^{(\alpha)}] \xrightarrow[n]{} \mathbb{E}[X^{(\alpha)}] \quad \text{for } \alpha\in\mathbb{R}_0^+\setminus U \quad \square$$

(c) We wish to show that if $\mathbb{E}[X_n] \xrightarrow[n]{} \mathbb{E}[X]$ and X_n converges a.s. to X, then $\{X_n\}_{n\geq 0}$ is u.i.

Note that
$$X_n = X_n^{(\alpha)} + X_n \cdot \mathbb{1}[X_n \geq \alpha]$$
$$X = X^{(\alpha)} + X\cdot \mathbb{1}[X\geq\alpha]$$
so by taking expectation and using (b) we get that for $\alpha \in \mathbb{R}_0^+\setminus U$,

$$\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X] \Rightarrow \mathbb{E}\left[X_n \mathbb{1}[X_n\geq\alpha]\right] \to \mathbb{E}\left[X\mathbb{1}[X\geq\alpha]\right]$$

Fix $\varepsilon>0$, fix $\alpha\notin U$, and let N_α be such that $\forall n\geq N_\alpha$
$$\mathbb{E}[X_n \mathbb{1}[X_n\geq\alpha]] < \mathbb{E}[X\mathbb{1}[X>\alpha]] + \varepsilon$$

It follows that
$$\sup_{n\geq 0} \mathbb{E}[|X_n| \mathbb{1}[|X_n|\geq\alpha]] \underset{\substack{\uparrow \\ X_n\geq 0}}{=} \sup_{n\geq 0} \mathbb{E}\left[X_n \mathbb{1}[X_n\geq\alpha]\right]$$

$$\leq \max\left\{ \max_{n\in\{0,1,\dots,N-1\}} \mathbb{E}[X_n \mathbb{1}[X_n\geq\alpha]],\ \mathbb{E}[X\mathbb{1}[X\geq\alpha]]+\varepsilon \right\}$$

Because each X_n and X is integrable, we have that

$$\lim_{\alpha \to +\infty} \mathbb{E}[X_n 1\!\!1[X_n \geq \alpha]] = 0$$

$$\lim_{\alpha \to +\infty} \mathbb{E}[X \, 1\!\!1[X \geq \alpha]] = 0$$

It follows that $\lim_{\substack{\alpha \to +\infty \\ \alpha \notin U}} \sup_{n \geq 0} \mathbb{E}[|X_n| 1\!\!1[|X_n| \geq \alpha]] \leq$

$$\leq \lim_{\alpha \to +\infty} \max\left\{ \max_{n \in \{0,1,\dots,N-1\}} \mathbb{E}[X_n 1\!\!1[X_n \geq \alpha]], \ \mathbb{E}[X 1\!\!1[X \geq \alpha]] + \varepsilon \right\} = \varepsilon$$

Because the seq. $\sup_{n \geq 0} \mathbb{E}[|X_n| 1\!\!1[|X_n| \geq \alpha]]$ is decreasing in α, it suffices to take the limit outside of U, so $\lim_{\alpha \to +\infty} \sup_{n \geq 0} \mathbb{E}[|X_n| 1\!\!1[|X_n| \geq \alpha]] \leq \varepsilon$.

Since ε is arbitrary, it follows that $\{X_n\}_{n \geq 0}$ is u.i. $\square$

(a) We use the fact that $\{X\} \cup \{X_n\}_{n \geq 0}$ is u.i. to choose C s.t. $\mathbb{E}[|Y| 1\!\!1[|Y| \geq \alpha]] < \frac{1}{4}\varepsilon \quad \forall Y \in \{X\} \cup \{X_n\}_{n \geq 0} \ \forall \alpha \geq C$.

From (b), we have that $\mathbb{E}[X_n^{(\alpha)}] \to \mathbb{E}[X^{(\alpha)}]$ for $\alpha \in \mathbb{R}_0^+ \setminus U$, so fix $\alpha \geq C$, $\alpha \notin U$ and let N s.t. $n \geq N \Rightarrow |\mathbb{E}[X_n^{(\alpha)}] - \mathbb{E}[X^{(\alpha)}]| < \frac{1}{2}\varepsilon$

From
$$X_n = X_n^{(\alpha)} + X_n \cdot 1\!\!1[X_n \geq \alpha]$$
$$X = X^{(\alpha)} + X \cdot 1\!\!1[X \geq \alpha]$$
we get for $n \geq N$

$$|\mathbb{E}[X_n] - \mathbb{E}[X]| \leq |\mathbb{E}[X_n^{(\alpha)}] - \mathbb{E}[X^{(\alpha)}]| + |\mathbb{E}[X_n \cdot 1\!\!1[X_n \geq \alpha]]| + |\mathbb{E}[X 1\!\!1[X \geq \alpha]]|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon.$$ Since $\varepsilon > 0$ is arbitrary, we show that $\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X]$.

Exercise 4 $\mathbb{E}[|X_n|] = \frac{1}{n} \cdot n + 0 \cdot \left(1 - \frac{2}{n}\right) + \frac{1}{n}|-n| = \frac{2}{n} n = 2$

$\mathbb{E}[X_n] = \frac{1}{n} n + 0\left(1 - \frac{2}{n}\right) + \frac{1}{n}(-n) = 0$, hence $\mathbb{E}[X_n] \to \mathbb{E}[V]$

$U(\omega) \in (0,1) \Rightarrow \{X_n(\omega)\}_{n \geq 0}$ is eventually zero $\left(\text{for } n \geq \frac{1}{U(\omega)}, \frac{1}{1-U(\omega)}\right)$

$\Rightarrow \{X_n(\omega)\}_{n \geq 0}$ converges to $V(\omega) = 0$

Since $\mathbb{P}(U \in (0,1)) = 1$, $_n \longrightarrow V$ a.s.

However,

$$|X_n| \mathbb{1}[|X_n| \geq c] = \begin{cases} 0 & \text{a.s., if } n < c \\ |X_n| & \text{a.s., if } n \geq c \end{cases}$$

So $\mathbb{E}\big[|X_n| \mathbb{1}[|X_n| \geq c]\big] = \begin{cases} 0, & \text{if } n < c \\ 2, & \text{if } n \geq c \end{cases}$

It follows that $\lim_{c \to +\infty} \sup_{n \geq 1} \mathbb{E}\big[|X_n| \mathbb{1}[|X_n| \geq c]\big] \underset{i = \lfloor c+1 \rfloor}{\geq} \lim_{c \to +\infty} \mathbb{E}\big[|X_i| \mathbb{1}[|X_i| \geq c]\big]$

$= 2$. So $\{X_n\}_{n \geq 1}$ is not u.i. □