Exercise 1 (a) Since $(X_n)_{n\geq 1}$ is a martingale and $\varphi: t \mapsto t^2$ is convex, we have that $(X_n^2)_{n\geq 1}$ is a submartingale, that is

$$\mathbb{E}[X_{n+1}^2 \mid \mathcal{F}_n] \geq X_n^2 \quad a.s.$$

It follows that, by taking $\mathbb{E}$ on both sides

$$\forall n\geq 0 \quad \mathbb{E}[X_{n+1}^2] \geq \mathbb{E}[X_n^2]$$

Since $\mathbb{E}[X_n^2]$ is a monotone bounded (by $\sup_{n\geq 0} \mathbb{E}[X_n^2]$) sequence, it converges □

(b) If $p \geq q$, then $\mathbb{E}[X_p \mid \mathcal{F}_q] = X_q$, it follows that

$$\mathbb{E}[X_p \cdot X_q] = \mathbb{E}\big[\mathbb{E}[X_q \mid \mathcal{F}_q] \cdot X_q\big] \underset{cp}{=} \mathbb{E}[X_q^2]$$

(c) Because L^2 is a complete space, we only need to show that for any $\varepsilon > 0$ there exists N s.t. $m,n > N \Rightarrow \mathbb{E}[(X_n - X_m)^2] < \varepsilon$.

However, wlog $n \geq m$ gives $\mathbb{E}[(X_n - X_m)^2] = \mathbb{E}[X_n^2] - 2\,\mathbb{E}[X_n \cdot X_m] + \mathbb{E}[X_m^2]$

$$\underset{(b)}{=} \mathbb{E}[X_n^2] - 2\mathbb{E}[X_m^2] + \mathbb{E}[X_m^2] = \mathbb{E}[X_n^2] - \mathbb{E}[X_m^2]$$

Let $Q = \lim_{n\geq 0} \mathbb{E}[X_n^2]$ be the limit of $\mathbb{E}[X_n^2]$, that exists according to 1(a).

Then, by definition, there is some N s.t. $n > N \Rightarrow |\mathbb{E}[X_n^2] - Q| < \frac{\varepsilon}{2}$.

It follows that $m, n > N \Rightarrow \left|\mathbb{E}[X_n^2] - \mathbb{E}[X_m^2]\right| \leq \left|\mathbb{E}[X_n^2] - Q\right| + \left|Q - \mathbb{E}[X_m^2]\right| < 2 \cdot \varepsilon/2 = \varepsilon$

as desired □.

Exercise 2 (a) From ES3 Ex 4(b), $(Z_n)_{n\geq 0}$ is a martingale.
Hence, by Doob's maximal inequality, we have that

$$\mathbb{P}(Z_\infty^* \geq \alpha) \leq \frac{1}{\alpha}\mathbb{E}[|Z_0|]$$

But $Z_0 = 1$ a.s., and $Z_\infty^* := \sup_{k\geq 0} Z_k$, this gives

$$\mathbb{P}\left(\sup_{k\geq 0} Z_k \geq \alpha\right) \leq \alpha^{-1}$$

Because $t \mapsto \left(\frac{1-p}{p}\right)^t$ is increasing continuous function for $p\in(0, 0.5)$,

if we let $\alpha = \left(\frac{1-p}{p}\right)^k$ we get

$$\mathbb{P}\left(\sup_{k\geq 0} Z_k \geq \alpha\right) = \mathbb{P}\left(\sup_{n\geq 0}\left(\frac{1-p}{p}\right)^{X_n} \geq \left(\frac{1-p}{p}\right)^k\right) = \mathbb{P}\left(\sup_{n\geq 0} X_n \geq k\right)$$

It follows that

$$\mathbb{P}\left(\sup_{n\geq 0} X_n \geq k\right) \leq \alpha^{-1} = \left(\frac{p}{1-p}\right)^k \quad \triangleright$$

For the expectation: $\mathbb{E}\left[\sup_{n\geq 0} X_n\right] = \sum_{k=0}^{+\infty} k\cdot\mathbb{P}\left(\sup_{n\geq 0} X_n = k\right)$

$$= \sum_{k=0}^{+\infty}\sum_{j=1}^{k}\mathbb{P}\left(\sup_{n\geq 0} X_n = k\right) = \sum_{j=1}^{+\infty}\sum_{k=j}^{+\infty}\mathbb{P}\left(\sup_{n\geq 0} X_n = k\right)$$

$$= \sum_{j=1}^{+\infty}\mathbb{P}\left(\sup_{n\geq 0} X_n \geq j\right) \leq \sum_{j=1}^{+\infty}\left(\frac{p}{1-p}\right)^p = \frac{\frac{p}{1-p}}{1-\frac{p}{1-p}} = \frac{p}{1-2p} \quad \triangleright$$

(b) We have that $\frac{1}{n}X_n \to \mathbb{E}[Y_1] = 2p-1$ a.s. by the law of large numbers.

This means that there is some $A\subseteq\Omega$ such that
- $\mathbb{P}(A) = 1$
- $\omega\in A \Rightarrow \frac{1}{n}X_n(\omega)\to 2p-1 < 0$

Let $F_N = \{X_n \leq 0 \ \forall n\geq N\}$

We claim that $A \subseteq \bigcup_{N\geq 0} F_N \subseteq \{\sup_{n\geq 0} X_n < +\infty\}$. This concludes the proof since we get $\mathbb{P}\left(\sup_{n\geq 0} X_n < +\infty\right) \geq \mathbb{P}(A) = 1$

So take some $\omega \in A$. Then $\frac{1}{n} X_n(\omega) \to 2p-1$, so there is some N_ω s.t. $n \geq N_\omega \Rightarrow \left| \frac{1}{n} X_n(\omega) - (2p-1) \right| < 2p-1 \Rightarrow X_n(\omega) < 0$

Hence $\omega \in F_{N_\omega}$, so $\omega \in \bigcup_{N \geq 0} F_N$. It follows that $A \subseteq \bigcup_{N \geq 0} F_N$

On the other hand, if $\omega \in F_N$, $\sup_{n \geq 0} X_n(\omega) = \max\{ X_0(\omega), \dots, X_N(\omega), \sup_{n \geq N} X_n(\omega) \}$

$$\leq \max\{ X_0(\omega), \dots, X_N(\omega), 0 \} < +\infty \quad \square$$

Exercise 3 (a) Because $t \mapsto \exp(t)$ is convex, and $(M_n)_{n \geq 0}$ is a bounded martingale, we have that $X_n = e^{M_n}$ is a submartingale.

(b) Doob's decomposition theorem says that

$X_n = X_0 + N_n + A_n$ where $(N_n)_{n \geq 0}$ is martingale with $N_0 = 0$, A_n is predictable.

Then, because from (a) we have that X_n is submartingale, we have that $A_{n+1} \geq A_n$ a.s. It follows that

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[X_0] + \cancel{\mathbb{E}[N_{n+1}]} + \mathbb{E}[A_{n+1}]$$
$$\geq \mathbb{E}[X_0] + 0 + \mathbb{E}[A_n] = \mathbb{E}[X_n] \quad \square$$

(c) To show that this converges a.s., we need to show that $\sup_{n \geq 0} \mathbb{E}[X_n^+] < +\infty$. This is immediate because $M_n \leq A$ a.s.

so $X_n^+ \leq \exp(A)$ a.s.

It follows that $X_n \xrightarrow{a.s.} X_\infty$ to some r.v. X_∞ in L^1 $\square$.