Exercise 1

X r.v. with $f_X(x)\begin{cases} 2x, & \text{if } x\in[0,1] \\ 0, & \text{o/w}\end{cases}$

$Y \sim \text{Unif}([0,X])$

Let $p_{(X,Y)}(x,y)$ be the density of the joint dist. (X,Y), and $q_{Y|X}(y,x)$ the density of $(Y \mid X=x) \sim \text{Unif}([0,x])$

Consider as well $q_{X|Y}(x,y)$, that is the density of $(X \mid Y=y)$

Let $f_X(x)$ and $f_Y(y)$ be the density function of the r.v. X, Y, resp.

$X \sim f_X(x)\,dx = 2x\,dx \quad x\in[0,1]$

$Y \sim f_Y(y)\,dy$

$(X,Y) \sim p_{(X,Y)}(x,y)\,dx\,dy$

$X \mid Y=y \sim q_{X|Y}(x,y)\,dx$

$Y \mid X=x \sim q_{Y|X}(y,x)\,dy$

Then
$$q_{Y|X}(y,x) = \begin{cases} 1/x, & \text{if } y\in[0,x] \\ 0, & \text{o/w}\end{cases}$$
$$= 1\!\!1_{\{0\le y\le x\}}\cdot\frac{1}{x}$$

It follows that $\mathbb{E}[Y \mid X] = H(X)$ where
$$H(x) = \int y\cdot q_{Y|X}(y,x)\,dy = \int_0^x y\,dy\ \frac{1}{x} = \frac{1}{2}x^2\cdot\frac{1}{x} = \frac{x}{2}$$

That is, $\boxed{\mathbb{E}[Y \mid X] = \frac{1}{2}X}$.

We also have that $f_X(x) = 1\!\!1[x\in[0,1]]\cdot 2x$

It follows that $p_{(X,Y)}(x,y) = f_X(x)\,q_{Y|X}(y,x) = 1\!\!1[0\le y\le x\le 1]\,2$

To recap:

$X \sim f_X(x)\,dx = 2x\,dx \quad x\in[0,1]$

$Y \sim f_Y(y)\,dy$

$(X,Y) \sim p(x,y)\,dx\,dy = 1\!\!1[0\le y\le x\le 1]\cdot 2\,dx\,dy$

$X \mid Y=y \sim q_{X|Y}(x,y)\,dx$

$Y \mid X=x \sim q_{Y|X}(y,x)\,dy = 1\!\!1[0\le y\le x]\cdot 1/x\,dy$

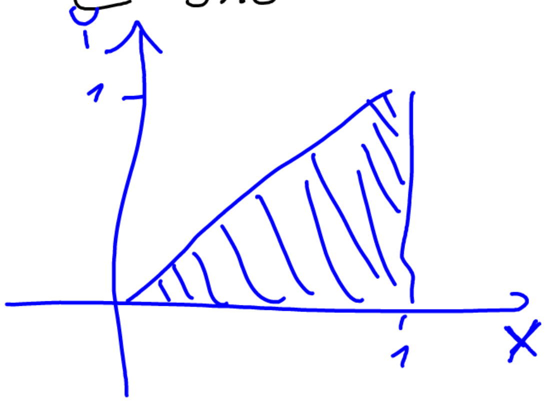
Finally, $f_Y(y) = \int p_{(X,Y)}(x,y)\,dx = \mathbb{1}[0 \le y \le 1] \cdot 2(1-y)$

For $y \in [0,1)$ $q_{X|Y}(x,y) = \dfrac{p_{(X,Y)}(x,y)}{f_Y(y)} = \dfrac{1}{1-y} \mathbb{1}[0 \le y \le x \le 1]$, that is $X|_{Y=y} \sim \mathrm{Unif}([y,1])$

It follows that $\mathbb{E}[X|Y] = H'(Y)$ where

$$H'(y) = \int x\, q_{X|Y}(x,y)\,dx = \int_y^1 x \frac{1}{1-y}\,dx = \left[\frac{1}{2}x^2\right]_{x=y}^{x=1} \frac{1}{1-y} =$$

$$= \frac{1}{2(1-y)}\left(1-y^2\right) = \frac{1+y}{2}$$



That is, $\boxed{\mathbb{E}[X|Y] = \dfrac{1+Y}{2}}$

Double check! $\mathbb{E}[X] = \int_0^1 x \cdot 2x\,dx = \left[\frac{2}{3}x^3\right]_{x=0}^{x=1} = 2/3$

$$\mathbb{E}[Y] = \int_0^1 y \cdot 2(1-y)\,dy = \left[-\frac{2}{3}y^3 + y^2\right]_{y=0}^{y=1} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\begin{cases} \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \frac{1}{2} + \frac{1}{2}\mathbb{E}[Y] \\ \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \frac{1}{2}\mathbb{E}[X] \end{cases} \iff \begin{cases} \mathbb{E}[X] = 2/3 \\ \mathbb{E}[Y] = 1/3 \end{cases} \quad \checkmark$$

Exercise 2

$X \sim \mathcal{N}(0, \sigma^2)$, $Z_i \sim \mathcal{N}(0, c_i^2)$ indep.

$Y_i := X + Z_i$

1 $(X, Y_1, \dots, Y_n)$ is a centered Gaussian vector. Find the covariance matrix.

Proof: A vector of r.v. is a Gaussian vector if any linear combination of its entries has a Gaussian distribution.

Indeed, let $W = \lambda X + \sum_{i=1}^{n} a_i Y_i$ be a linear combination of $X, Y_1, \dots, Y_n$, with $\lambda, a_i \in \mathbb{R}$ for $i = 1, \dots, n$.

Then, $W = \left(\lambda + \sum_{i=1}^{n} a_i\right) X + \sum_{i=1}^{n} a_i Z_i$ is also a Gaussian distribution because it is the linear combination of independent Gaussian distributions. This concludes that $(X, Y_1, \dots, Y_n)$ is a Gaussian vector. That this is centered it follows from $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y_i] = \mathbb{E}[X] + \mathbb{E}[Z_i] = 0 + 0$ for all $i = 1, \dots, n$.

Finally, to compute the covariance matrix,

- $\mathrm{Cov}[X, X] = \mathrm{Var}[X] = \sigma^2$
- $\mathrm{Cov}[X, Y_i] = \mathrm{Cov}[X, X] + \mathrm{Cov}[X, Z_i] \overset{Z_i \perp\!\!\!\perp X}{=} \sigma^2 + 0$ for all $i = 1, \dots, n$
- $\mathrm{Cov}[Y_i, Y_i] = \mathrm{Var}[X + Z_i] \underset{X \perp\!\!\!\perp Z_i}{=} \mathrm{Var}[X] + \mathrm{Var}[Z_i] = \sigma^2 + c_i^2$ for all $i = 1, \dots, n$

$(i \neq j)$

- $\mathrm{Cov}[Y_i, Y_j] = \mathrm{Cov}[X + Z_i, X + Z_j] = \mathrm{Var}[X] + \mathrm{Cov}[X, Z_j] + \mathrm{Cov}[Z_i, X] + \mathrm{Cov}[Z_i, Z_j] = \sigma^2$ $\quad (X \perp\!\!\!\perp Z_i \perp\!\!\!\perp Z_j \perp\!\!\!\perp X)$

  for $i = 1, \dots, n$; $j = 1, \dots, n$ with $i \neq j$

$$\begin{bmatrix} \sigma^2 & \sigma^2 & \cdots & \sigma^2 \\ \sigma^2 & \sigma^2 + c_1^2 & & \\ \vdots & & \sigma^2 + c_2^2 & \\ & & & \ddots \\ \sigma^2 & \sigma^2 & & \sigma^2 + c_n^2 \end{bmatrix}$$

Remark: Because X, Y_i are centered r.v., we have that

$$(*) \begin{cases} \mathrm{Cov}[X, Y_i] = \mathbb{E}[X \cdot Y_i] & i \geq 0 \\ \mathrm{Cov}[Y_i, Y_j] = \mathbb{E}[Y_i Y_j] & i, j \geq 0 \end{cases}$$

(2) From Proposition 4.3. and from (a), we have that $(Y_1, \dots, Y_n)$

$$\mathbb{E}[X \mid Y_1, \dots, Y_n] = \hat{X}_n = \sum_{i=1}^{n} \lambda_i^{(n)} Y_i \quad \text{for constants } \lambda_i^{(n)} \in \mathbb{R}.$$

Therefore, for any $m \leq n$

(1) $$\mathrm{Cov}[X, Y_m] \overset{(*)}{=} \mathbb{E}[X Y_m] \overset{(CP)}{=} \mathbb{E}[\hat{X}_n Y_m] \overset{(*)}{=} \sum_{i=1}^{n} \lambda_i^{(n)} \mathrm{Cov}[Y_i, Y_m]$$

For any $m \leq n$, because Y_m is $\sigma(Y_1, \dots, Y_n)$-measurable.

From (a) and (1), we have

(2) $$\sigma^2 = \left(\sum_{\substack{i=1\\ i\neq m}}^{n} \lambda_i^{(n)} \sigma^2\right) + \lambda_m^{(n)}\left(\sigma^2 + c_m^2\right) \quad \text{for } m=1,\dots,n$$

Let $a_n := \sum_{i=1}^{n} \lambda_i^{(n)} \sigma^2$. Eq. (2) becomes

(3) $$\sigma^2 - a_n = \lambda_m^{(n)} c_m^2 \quad \text{for } m = 1,\dots,n$$

Simplifying (3)

(4) $$\lambda_m^{(n)} = \frac{\sigma^2 - a_n}{c_m^2} = \frac{\sigma^2}{c_m^2}\left(1 - \sum_{i=1}^{n} \lambda_i^{(n)}\right) \quad \text{for } m=1,\dots,n.$$

Summing Eq (4) for $m=1,\dots,n$ and defining $b_n = \sum_{i=1}^{n} \lambda_i^{(n)}$,

(5) $$b_n = \sum_{m=1}^{n} \frac{\sigma^2}{c_m^2}\left(1-b_n\right)$$

Rearranging, gives us that $$b_n = \frac{\sum_{m=1}^{n} \frac{\sigma^2}{c_m^2}}{\underbrace{1+\sum_{m=1}^{n} \frac{\sigma^2}{c_m^2}}_{>0}} \quad (6)$$

Together with (4) we have $$\boxed{\lambda_m^{(n)} = \frac{\sigma^2}{c_m^2} \times \frac{1}{1+\sum_{i=1}^{n} \frac{\sigma^2}{c_i^2}}}$$

And this describes $\hat{X}_n$.

3 $$\mathbb{E}\left[(X-\hat{X}_n)^2\right] = \mathbb{E}\left[X^2\right] - 2\,\mathbb{E}\left[X\cdot \hat{X}_n\right] + \mathbb{E}\left[\hat{X}_n\cdot \hat{X}_n\right]$$

Because $$\mathbb{E}\left[X\cdot \hat{X}_n\right] \underset{\uparrow \text{ CP on } \mathbb{E}[X\mid\sigma(Y_1,\dots,Y_n)]}{=} \mathbb{E}\left[\mathbb{E}\left[X \mid \sigma(Y_1,\dots,Y_n)\right]\cdot \hat{X}_n\right] \underset{\text{Def of } \hat{X}_n.}{=} \mathbb{E}\left[\hat{X}_n\cdot\hat{X}_n\right]$$

So $$\mathbb{E}\left[(X-\hat{X}_n)^2\right] = \mathbb{E}\left[X^2\right] - \sum_{i=1}^{n} \mathbb{E}\left[X\cdot \lambda_i^{(n)}\cdot Y_i\right] =$$

$$= \mathrm{Var}[X] - \sum_{i=1}^{n} \lambda_i^{(n)} \mathrm{Cov}[X, Y_i] = \sigma^2 - \sum_{i=1}^{m} \lambda_i^{(n)}\sigma^2 = \sigma^2(1-b_n)$$

Using (5) we get $\|X - X_n\|_2^2 = \sigma^2 \dfrac{1}{\sum_{i=1} \frac{\sigma^2}{c_i^2}} = \dfrac{1}{\sum_{i=1}^n c_i^{-2}}$

We conclude that $\|X - X_n\|_2 \to 0$ iff $\sum_{i=1}^{\infty} c_i^{-2} = \infty$ $\square$

Exercise 3

To show that T is a stopping time, we need only to show that $\{T \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Fix n.

$$\{T \leq n\} = \biguplus_{i=0}^{n} \{T = i\} = \biguplus_{i=0} \{X_i > 0, X_{i-1} \leq 0, \dots, X_0 \leq 0\}$$

disjoint union

$$= \bigcup_{i=0}^{n} \underbrace{\{X_i > 0\}}_{\in \mathcal{F}_i \subseteq \mathcal{F}_n} \in \mathcal{F}_n \quad \square$$

Exercise 4

1. To show that $\{X_n\}_{n \geq 0}$ is a martingale, we only need to show that
   - X_n is $\mathcal{F}_n$-measurable
   - $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ a.s. for any $n \in \mathbb{N}$

Because $Y_0, \dots, Y_n$ are $\mathcal{F}_n$-measurable, and $f: \mathbb{R}^{n+1} \to \mathbb{R}$, $(y_i) \mapsto \sum_{i=0}^n y_i$ is Borel-measurable, then $X_n = f(Y_0, \dots, Y_n)$ is $\mathcal{F}_n$-measurable.

Because $Y_{n+1} \perp\!\!\!\perp \sigma(Y_1, \dots, Y_n)$, from Proposition 4.1. we have

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \left(\sum_{i=0}^n Y_i\right) + \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] \\
&= X_n + \mathbb{E}[Y_{n+1}] \\
&= X_n + 1 \cdot p + (-1)(1-p) \underset{p=0.5}{=} X_n
\end{aligned}$$

2 Because $f: \mathbb{R} \to \mathbb{R}$ is Borel-measurable, and
$$t \mapsto \left(\frac{1-p}{p}\right)^t$$

X_n is $\mathcal{F}_n$-measurable (as seen in 1, independently of $p=0.5$) then $Z_n = f(X_n)$ is $\mathcal{F}_n$-measurable.

Because $Y_{n+1} \perp\!\!\!\perp \sigma(Y_0, \dots, Y_n)$, and X_n is $\mathcal{F}_n$-measurable,

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_n}\left(\frac{1-p}{p}\right)^{Y_{n+1}} \Big| \mathcal{F}_n\right]$$
$$= \left(\frac{1-p}{p}\right)^{X_n} \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right]$$
$$= Z_n \cdot \left[\frac{1-p}{p} \cdot p + \left(\frac{1-p}{p}\right)^{-1}(1-p)\right] = Z_n \qquad \square$$

Exercise 5 1 T is $\mathcal{F}_T$-measurable if $\{T \le m\} \in \mathcal{F}_T$ for all $m \ge 0$.

Indeed, for any $n \ge 0$,

$$\{T \le m\} \cap \{T \le n\} = \{T \le \min\{n, m\}\} \in \mathcal{F}_n$$

because T is a stopping time. So $\{T \le m\} \in \mathcal{F}_T$ for all $m \ge 0$

2 Take $A \in \mathcal{F}_S$. We wish to show that $A \cap \{T \le n\} \in \mathcal{F}_n$ for any $n \ge 0$. Fix $n \ge 0$.

Because $A \in \mathcal{F}_S$, we have that $A \cap \{S \le n\} \in \mathcal{F}_n$

Because T is a stopping time, we have that $\{T \le n\} \in \mathcal{F}_n$

It follows that $A \cap \{S \le n\} \cap \{T \le n\} \in \mathcal{F}_n$
$$\overset{S \le T}{\Rightarrow} A \cap \{T \le n\} \in \mathcal{F}_n$$

Because n is generic, this concludes the proof $\square$