Probability 2 - Solutions of ex sheet 2

Ex 1

$$\mathbb{E}[XY \mid \mathcal{B}] = \mathbb{E}[\mathbb{E}[XY \mid \mathcal{C}] \mid \mathcal{B}] =$$

$$\underset{Y \text{ is } \mathcal{C}\text{-measurable}}{=} \mathbb{E}[Y \, \mathbb{E}[X \mid \mathcal{C}] \mid \mathcal{B}]$$

$$\underset{X \perp\!\!\!\perp \mathcal{C}}{=} \mathbb{E}[Y \, \mathbb{E}[X] \mid \mathcal{B}]$$

$$= \mathbb{E}[X] \, \mathbb{E}[Y \mid \mathcal{B}]$$

Ex2 (a) Let $A = \{\mathbb{E}(Y \mid \mathcal{B}) = 0\}$. Then $A \in \mathcal{B}$ and by CP on $\mathbb{E}[Y \mid \mathcal{B}]$ we have

$$\mathbb{E}[\mathbb{E}[Y \mid \mathcal{B}] \cdot \mathbb{1}_A] = \mathbb{E}[Y \cdot \mathbb{1}_A]$$

But $\mathbb{E}[Y \mid \mathcal{B}] \cdot \mathbb{1}_A = 0$ a.s., so $\mathbb{E}[Y \cdot \mathbb{1}_A] = 0$.

Because $Y \cdot \mathbb{1}_A \geq 0$ a.s., we conclude that $Y \cdot \mathbb{1}_A = 0$ a.s.

It follows that $\{Y \neq 0\} \subseteq A \cup S$ for some S with $\mathbb{P}(S) = 0$.

(b) Define $A_n = \{ X \le n \}$, $A = \bigcup_n A_n$, $B = \{ Y = +\infty \}$

Note that $1\!\!1_{A_n \cap B} \nearrow 1\!\!1_{A \cap B}$ so by MCT we have

$$P(A_n \cap B) \xrightarrow[n]{} P(A \cap B) = P(Y = +\infty, \; X < +\infty)$$

However, $E[Y 1\!\!1_{A_n}] = E[X 1\!\!1_{A_n}] \le n$, by CP on $E[Y|B]$, so

$$P(B \cap A_n) = P(Y 1\!\!1_{A_n} = +\infty) = 0 \qquad \boxed{X 1\!\!1_{A_n} \le n \text{ a.s}}$$

We conclude that $P(\{Y = +\infty\} \setminus \{X = +\infty\}) = 0$,

as desired.

⚠ Atention we can use CP in $E(Y|B)$ with the function $1\!\!1_{A_n}$ because $A_n \in \mathcal{B}$!

Ex 3 $E[(X-Y)^2 | Y] =$

$$= E[X^2 | Y] - 2E[XY | Y] + E[Y^2 | Y]$$

$$= E[X^2 | Y] - 2Y E[X | Y] + Y^2 \qquad \leftarrow Y, Y^2 \text{ are } \sigma(Y)\text{-meas.}$$

$$= Y^2 - 2Y^2 + Y^2 = 0$$

It follows that $E\big[E[(X-Y)^2 | Y]\big] = E[(X-Y)^2]$ is zero. Since $(X-Y)^2 \ge 0$ a.s., we have that

$$(X-Y)^2 = 0 \text{ a. s.}$$

Hence $X = Y$ a. s., as desired.

Ex 4 Define $f(t) := \exp(st)$, where $s \in \mathbb{R} \setminus \{0\}$ is such that

$$\mathbb{E}\left(\exp(sX)\right) < +\infty,$$

note that s may be negative.

Then $\left(\frac{d}{dt}\right)^2 f = s^2 \exp(st) > 0$ for any $t \in \mathbb{R}$, so f is a convex function. Let $Y := \mathbb{E}(X \mid \mathcal{B})$.

From Jensen's inequality we have

$$f(Y) = f(\mathbb{E}(X \mid \mathcal{B})) \leq \mathbb{E}(f(X) \mid \mathcal{B}) \quad \text{a.s.}$$

Taking the expected value we get

$$\mathbb{E}(\exp(sY)) \leq \mathbb{E}\left(\mathbb{E}(f(X) \mid \mathcal{B})\right) = \mathbb{E}(X) = \mathbb{E}\left(\exp(sX)\right) < +\infty$$

So Y also has finite exponential moments $\square$