Ex 1 & 2 after page 2

Exercise 3

(1) If f is a function that is differentiable in $x \in (0,1)$, let $\varepsilon > 0$ and find $\delta > 0$ s.t.

$$\begin{array}{c} |z-x| < \delta \\ z \neq x \end{array} \Rightarrow \left| \frac{f(z)-f(x)}{z-x} - f'(x) \right| < \varepsilon$$

it follows that for such z's we have

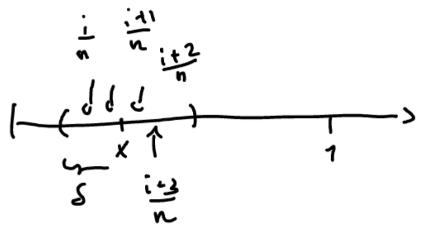
$$\left| f(z) - f(x) - f'(x)(z-x) \right| < \varepsilon \, |z-x|$$

Or

$$f(z) = f(x) + f'(x)(z-x) + g(z)$$

where $\quad |g(z)| < \varepsilon \, |z-x|$

Let M be such that $\delta \cdot M > 2$. In this way for any $n \geq M$ the integer $i \in \{0, 1, \dots, n-3\}$ s.t. $x \in \left[\frac{i+1}{n}, \frac{i+2}{n}\right]$ satisfies that $\left\{\frac{i}{n}, \frac{i+1}{n}, \frac{i+2}{n}, \frac{i+3}{n}\right\} \subseteq (x-\delta, x+\delta)$, because $\left|\frac{i}{n} - x\right| < \frac{2}{n} \leq \frac{2}{M} < \delta$.



Thus, we have that $\left| f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right| =$

$$= \left| f'(x) \frac{1}{n} + g\left(\frac{i}{n}\right) - g\left(\frac{i+1}{n}\right) \right| < |f'(x)| \frac{1}{n} + \left| g\left(\frac{i}{n}\right) \right| + \left| g\left(\frac{i+1}{n}\right) \right|$$

$$\leq |f'(x)| \frac{1}{n} + \varepsilon \left| \frac{i}{n} - x \right| + \varepsilon \left| \frac{i+1}{n} - x \right| < |f'(x)| \frac{1}{n} + 3\varepsilon \frac{1}{n}$$

$$= \frac{1}{n} \left(|f'(x)| + 3\varepsilon \right)$$

Thus, for $n \geq \left(|f'(x) + 3\varepsilon \right)^{10}$ and $n \geq M$, we have that

$$\left| f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right| < \frac{1}{n} \cdot n^{-0.1} = n^{-0.9}$$

In a similar way we get that $\left| f\left(\frac{i+1}{n}\right) - f\left(\frac{i+2}{n}\right) \right| < n^{-0.9}$ and that

$$\left|f\left(\tfrac{i+2}{n}\right)-f\left(\tfrac{i+3}{n}\right)\right| < n^{-0.9} \quad \text{for any} \quad n \geq \left\lceil \max\left(\left(|f'(x)+3\varepsilon\right)^{10}, M\right)\right\rceil \quad \square$$

② We know that $B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \sim N\left(0, \frac{1}{n}\right)$, thus

$$\mathbb{P}\left[\left|B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right| \leq n^{-0.9}\right] = \int_{-n^{-0.9}}^{n^{-0.9}} \frac{1}{\sqrt{2\pi\frac{1}{n}}} \exp\left(-t^2/_{1/n}\right) dt$$

$$\leq 2 \cdot n^{-0.9} \times \frac{1}{\sqrt{2\pi \, 1/n}} = \sqrt{\frac{2}{\pi}}\, n^{-0.9+0.5} = \sqrt{\frac{2}{\pi}}\, n^{-0.4} \quad \square$$

③ The variables $B_{\frac{i+1}{n}} - B_{\frac{i}{n}}$, $B_{\frac{i+2}{n}} - B_{\frac{i+1}{n}}$ and $B_{\frac{i+3}{n}} - B_{\frac{i+2}{n}}$ are independent. Therefore,

$$\mathbb{P}\left(E_i^{(n)}\right) = \mathbb{P}\left(\left|B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right| \leq n^{-0.9}\right) \cdot \mathbb{P}\left(\left|B_{\frac{i+2}{n}} - B_{\frac{i+1}{n}}\right| \leq n^{-0.9}\right)$$

$$\cdot\, \mathbb{P}\left(\left|B_{\frac{i+3}{n}} - B_{\frac{i+2}{n}}\right| \leq n^{-0.9}\right)$$

$$\overset{②}{\leq} \left(\sqrt{\frac{2}{\pi}}\, n^{-0.4}\right)^3 = n^{-1.2}\left(\sqrt{\frac{2}{\pi}}\right)^3 .$$

It follows that

$$0 \leq \mathbb{P}\left(\bigcup_{i=0}^{n-1} E_i^{(n)}\right) \leq \sum_{i=0}^{n-3} \mathbb{P}\left(E_i^{(n)}\right) \leq \sum_{i=0}^{n-3} n^{-1.2}\left(\sqrt{\frac{2}{\pi}}\right)^3 = \frac{n-2}{n^{1.2}}\left(\sqrt{\frac{2}{\pi}}\right)^3$$

Since $\lim_n \frac{n-2}{n^{1.2}}\left(\sqrt{\frac{2}{\pi}}\right)^3 = 0$, we conclude that

$$\lim_n \mathbb{P}\left(\bigcup_{i=0}^{n-1} E_i^{(n)}\right) = 0 \quad \square$$

④ $$\mathbb{P}\left(t \mapsto B_t \text{ differential somewhere}\right) \leq \mathbb{P}\left(\bigcup_{N\geq 0} \bigcap_{n\geq N} \bigcup_{i=0}^{n-3} E_i^{(n)}\right)$$

$$\leq \sum_{N\geq 0} \mathbb{P}\left(\bigcap_{n\geq N} \bigcup_{i=0}^{n-3} E_i^{(n)}\right). \text{ But } \bigcap_{n\geq N} \bigcup_{i=0}^{n-3} E_i^{(n)} \subseteq \bigcup_{i=0}^{m-3} E_i^{(m)}$$

for any $m \geq N$, thus $\mathbb{P}\left(\bigcap_{n\geq N} \bigcup_{i=0}^{n-3} E_i^{(n)}\right) \leq \mathbb{P}\left(\bigcup_{i=0}^{m-3} E_i^{(m)}\right) \longrightarrow 0$

It follows that $\mathbb{P}\left(t \mapsto B_t \text{ differential somewhere} \right) = 0$ ∎

Exercise 1

Take wlog $p \geq q$, then $B_p = B_q + (B_p - B_q)$.

By independent increments we have that $B_q \perp\!\!\!\perp B_p - B_q$, $B_q \sim \mathcal{N}(0, q)$

and so

$$\mathrm{Cov}(B_p, B_q) = \mathrm{Cov}(B_q, B_q) + \mathrm{Cov}(B_p - B_q, B_q) = q + 0 = q.$$

Exercise 2

ⓐ $X_t = B_t - tB_1 + t(y-x) + x$

$$= (1-t)B_t - t(B_1 - B_t) + t(y-x) + x$$

Since
$$(1-t)B_t \sim \mathcal{N}(0, (1-t)^2 t)$$
$$t(B_1 - B_t) \sim \mathcal{N}(0, t^2(1-t))$$
are independent, we have that

$$X_t \sim \mathcal{N}\left(t(y-x)+x,\ t(1-t) \right)$$

ⓑ Let $t_0 = 0$, $t_{n+1} = 1$ so that $0 = t_0 < t_1 < \dots < t_n < t_n = 1$.

Then set $Y_i := B_{t_{i+1}} - B_{t_i}$ for $i = 0, \dots, n$. Then

$$B_{t_i} = \sum_{j=0}^{i-1} Y_j \quad \text{for } i = ,\dots, n+1, \quad \text{and}$$

$$X_{t_i} = B_{t_i} - t_i B_1 + t_i(y-x) + x$$

$$= \left[\sum_{j=0}^{i-1} Y_j \right] - t_i \left[\sum_{j=0}^{n} Y_j \right] + t_i(y-x) + x$$

$$= \left[\sum_{j=0}^{i-1} (1-t_i) Y_j \right] - \left[\sum_{j=i}^{n} t_i Y_j \right] + t_i(y-x) + x$$

Thus, to show that $(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector, we simply observe that any linear combination $Z = \sum_{i=1}^{n} \lambda_i X_{t_i}$ satisfies

$$Z = \sum_{j=0}^{n} Y_j \left(-\sum_{i=1}^{j} \lambda_i t_i + \sum_{i=j+1}^{n} (1-t_i) \lambda_i \right) + \sum_{i=0}^{n} \lambda_i \left[t_i (y-x) + x \right]$$

So Z is the sum of independent normal r.v., thus Z is a normal r.v.

The covariance matrix is as follows: if $i < j$

$$\mathrm{Cov}(X_{t_i}, X_{t_j}) = \mathrm{Cov}\left(B_{t_i} - t_i B_1 + t_i (y-x) + x, B_{t_j} - t_j B_1 + t_j (y-x) + x \right)$$

$$= \mathrm{Cov}\left(B_{t_i} - t_i B_1, B_{t_j} - t_j B_1 \right) = \mathrm{Cov}(B_{t_i}, B_{t_j}) - t_i \mathrm{Cov}(B_1, B_{t_j})$$

$$- t_j \mathrm{Cov}(B_{t_i}, B_1) + t_i t_j \mathrm{Cov}(B_1, B_1)$$

$$= t_i - t_i t_j - t_i t_j + t_i t_j = t_i (1 - t_j) \quad .$$