Exercise 1

$$Q = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \qquad P_Q = -\lambda\left(\tfrac{1}{2}-\lambda\right) - \tfrac{1}{2}$$
$$= \lambda^2 - \tfrac{1}{2}\lambda - \tfrac{1}{2} = (\lambda - 1)(\lambda + 1/2)$$

Note that $Q\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ and $Q\begin{pmatrix}2\\-1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}2\\-1\end{pmatrix}$ so

$$Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

(1) Eighenvalues: 1, $-1/2$, left eighenvectors

$$(\mu_1, \mu_2)\,Q = (\mu_1, \mu_2) \Rightarrow \begin{cases} \frac{1}{2}\mu_2 = \mu_1 \\ \frac{1}{2}\mu_2 + \mu_1 = \mu_2 \end{cases} \Rightarrow (\mu_1, \mu_2) = \lambda\cdot(1, 2)$$

for some $\lambda \in \mathbb{C}$

$$(\mu_1, \mu_2)\,Q = -\frac{1}{2}(\mu_1, \mu_2) \Rightarrow \left\{ \underline{\frac{1}{2}\mu_2 = -\frac{1}{2}\mu_1} \right. \Rightarrow (\mu_1, \mu_2) = \lambda\cdot(1, -1)$$

Thus, $(\mu_1, \mu_2) = \left(\frac{1}{3}, \frac{2}{3}\right)$ is a stat. prob. measure.

(2)

$$Q^n = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1/2)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1} \longrightarrow \begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}\frac{1}{-3} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\frac{1}{3}$$

$$= \frac{1}{3}\left[\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & (-1/2)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2\cdot\left(\frac{-1}{2}\right)^n & -2\left(\frac{-1}{2}\right)^n \\ -(-1/2)^n & (-1/2)^n \end{pmatrix}\right]$$

$$= \begin{pmatrix} \dfrac{1+2(-1/2)^n}{3} & \dfrac{2-2(-1/2)^n}{3} \\[2ex] \dfrac{1-(-1/2)^n}{3} & \dfrac{2+(-1/2)^n}{3} \end{pmatrix}$$

Thus, $\mathbb{P}(X_n = 2 \mid X_0 = 1) = Q^{(n)}_{1,2} = \dfrac{2-2(-1/2)^n}{3}$

with $n \to +\infty$, $\lim_n Q^{(n)}_{1,2} = 2/3 = \mu_2$

③ $\mathbb{P}(X_{n+1}=1,\ X_n=2) = \mathbb{P}(X_n=2)\cdot \mathbb{P}(X_{n+1}=1 \mid X_n=2)$
$= \mathbb{P}(X_n=2)\cdot Q_{2,1} = \frac{1}{2}\mathbb{P}(X_n=2)$

Thus, $\lim_n \mathbb{P}(X_{n+1}=1,\ X_n=2) = \frac{1}{2}\cdot \mu_2 = \frac{1}{3}$

Similarly, $\mathbb{P}(X_{n+1}=2,\ X_n=1) = \mathbb{P}(X_n=1)\cdot \mathbb{P}(X_{n+1}=2 \mid X_n=1)$
$= \mathbb{P}(X_n=1)\cdot Q_{1,2} = 0$

Thus, $\lim_n \mathbb{P}(X_{n+1}=2,\ X_n=1) = 0.$

Exercise 2

① Because the chain is irreducible, it has a well defined period and is the period of any of it's elements.

The following claim will show that any $n \in A_x$ is even, thus $2 \mid \gcd(A_x)$.

Claim: Take $x \in S^i$, $y \in S^j$, and assume $Q^n_{x,y} > 0$.

If n is odd, then $i \neq j$

If n is even, then $i = j$

Proof: We use induction on n. For $n=0$, $Q^n_{x,y} > 0$ implies $x=y$, so because $S_1 \cap S_2 = \emptyset$, $i=j$ holds.

For the induction step, note that $Q^{n+1}_{x,y} = \sum_{z\in S} Q^n_{x,z}\cdot Q_{z,y} > 0 \Rightarrow \exists z \in S$ s.t. $Q^n_{x,z} > 0$, $Q_{z,y} > 0$.

Say that $z \in S_k$. Then $k \neq j$ from $Q_{z,y}$. By induction hip. we have that $i \neq k$ if n is odd ($i=k$ if n is even), thus $i=j$ if $n+1$ is even ($i \neq j$ if $n+1$ is odd) □

② We find directly an eighenvector $\vec{v} = \vec{1}_{S_1} - \vec{1}_{S_2}$, where $\vec{1}_{S_i}$ is the vector indexed by S that has $(\vec{1}_{S_i})_z = \begin{cases} 1, & \text{if } z \in S_i \\ 0, & \text{o/w} \end{cases}$

Now if $y \in S_1$, $(Q\vec{v})_y = \sum_{x\in S_1} Q_{x,y} - \sum_{x \in S_2} Q_{x,y} = 0 - \sum_{x\in S_2} Q_{x,y} = -1$

if $y \in S_2$, $(Q\vec{v})_y = \sum_{x\in S_1} Q_{x,y} - \sum_{x \in S_2} Q_{x,y} = \sum_{x\in S_1} Q_{x,y} - 0 = 1$

Thus $Q\vec{v} = \vec{1}_{S_2} - \vec{1}_{S_1} = -\vec{v}$. □

Exercise 3

① We compare the r.v. $\tilde{Z}_1$ in the MC $\left((X_{n+T_k})_{n\geq 0},\ \delta_x\right)$

and $\quad Z_1 \quad$ in $\quad ((X_n)_{n\geq 0}, \delta_x)$.

Because T_k is a stopping time, we have that $\tilde{Z}_1 \sim Z_1$ and $\tilde{Z}_1 \perp\!\!\!\perp (X_n)_{n=0}^{T_k}$. In particular, $\tilde{Z}_1 \perp\!\!\!\perp (Z_1, \dots, Z_{k-1})$.

On the other hand $\tilde{Z}_1 = Z_k$ by definition, so $\{Z_i\}_{i\geq 0}$ are indep. r.v.

(2) Recall $\quad \nu^{(x)}(y) = \sum_{i=1}^{+\infty} \mathbb{P}(X_i = y, T_x \leq i)$

$$= \mathbb{E}_x\left[\sum_{i=1}^{T_x} 1\!\!1[X_i = y]\right] = \mathbb{E}_x\left[\sum_{i=0}^{T_x-1} 1\!\!1[X_i = y]\right]$$

In fact, $\nu^{(x)}(S) = \mathbb{E}_x[T_x] = \frac{1}{\mu(x)}$. Assume $f \geq 0$

$$Z_0 = \sum_{i=0}^{T_1-1} f(X_i) = \sum_{i=0}^{T_1-1} \sum_{y\in S} 1\!\!1[X_i = y] f(y) = \sum_{y\in S}\left(\sum_{i=0}^{T_1-1} 1\!\!1[X_i = y]\right) f(y)$$

Thus $\quad \mathbb{E}_x[Z_0] = \mathbb{E}_x\left[\sum_{i=0}^{T_1-1} f(X_i)\right] = \mathbb{E}_x\left[\sum_{y\in S}\sum_{i=0}^{T_1-1} f(y)\, 1\!\!1[X_i = y]\right]$

$$\overset{*}{=} \sum_{y\in S} f(y)\, \mathbb{E}_x\left[\sum_{i=0}^{T_1-1} 1\!\!1[X_i = y]\right] = \sum_{y\in S} f(y)\, \nu^{(x)}(y).$$

$$\int f\, d\mu = \sum_{y\in S} f(y)\cdot\mu(y) = \sum_{y\in S} f(y)\frac{\nu^{(x)}(y)}{\nu^{(x)}(S)} = \mu(x)\sum_{y\in S} f(y)\nu^{(x)}(y)$$

$$= \mu(x)\cdot \mathbb{E}_x[Z_0]$$

On $*$ we use the MCT, together with $f \geq 0$, whenever $|S| = +\infty$.

As a conclusion, we have from (1) that $(Z_i)_{i=0}^{k-1}$ are iid

$$\frac{1}{k}\sum_{i=0}^{T_k-1} f(X_i) = \frac{1}{k}\sum_{i=0}^{k-1} Z_i \underset{k\to+\infty}{=} \mathbb{E}_x[Z_i] \quad \text{a.s.}$$

(3) $\quad \frac{1}{N_n}\sum_{i=0}^{n} f(X_i) = \left[\frac{1}{N_n}\sum_{i=0}^{N_n-1} f(X_i)\right] + \frac{1}{N_n}\left(\sum_{i=N_n}^{n} f(X_i)\right)$

Claim 1 $\quad N_n \to +\infty$ a.s.

Proof: We have seen that $T_k \overset{iid}{\sim} T_1$, so fix $k \in \mathbb{Z}$:

$$\mathbb{P}(N_n \leq k \quad \forall n) = \mathbb{P}(T_1 = +\infty \text{ or } T_2 = +\infty \text{ or } \cdots \text{ or } T_k = +\infty)$$

$$= \prod_{i=1}^{k} \mathbb{P}(T_i = +\infty) = \mathbb{P}(T_1 = +\infty)^k = 0 \quad \text{because } x \text{ is recurrent.}$$

Claim 2: $\dfrac{N_n}{N_{n+1}} \to 1$ a.s.

Proof: $$\frac{N_n}{N_{n+1}} = \frac{T_1 + \cdots + T_k}{T_1 + \cdots + T_{k+1}} = \frac{k+1}{T_1 + \cdots + T_{k+1}} \times \frac{T_1 + \cdots + T_k}{k} \times \frac{k}{k+1}$$

where $k = N_n$.

Thus, for $n \to +\infty$, because the $T_i \overset{iid}{\sim} T_1$ we have that

$$\frac{N_n}{N_{n+1}} \to \frac{1}{\mathbb{E}[T_1]} \times \mathbb{E}[T_1] \times \underbrace{1}_{\text{from claim 1}} = 1$$

Because $\mathbb{E}[T_1] < +\infty$ by assumption

Let $\quad a_n := \dfrac{1}{N_n} \sum_{i=0}^{n} f(X_i) \quad$ and $\quad A_k = \dfrac{1}{k} \sum_{i=0}^{T_k - 1} f(X_i)$

Then observe that, because $f \geq 0$, $\quad a_n \geq A_k \quad$ and $\quad a_n \leq \dfrac{N_n}{N_{n+1}} A_{k+1}$

where $k = N_n$.

Thus, for $n \to +\infty$, we have that $\limsup_n a_n \leq 1 \cdot \lim A_{k+1} \overset{②}{=} \dfrac{\int f \, d\mu}{\mu(x)}$ a.s.

and $\liminf_n a_n \geq \lim A_k = \dfrac{\int f \, d\mu}{\mu(x)}$.

It follows that $\quad \lim_n a_n = \dfrac{\int f \, d\mu}{\mu(x)}$

④ Take $f \equiv 1$, then $\int f \, d\mu = 1 \cdot \mu(S) = 1$ so

$$\lim_n \frac{1}{N_n} \sum_{i=0}^{n} 1 = \lim_n \frac{n+1}{N_n} = \frac{1}{\mu(x)}$$

It follows that $\quad \lim_n \dfrac{1}{n} \sum_{i=0}^{n} f(X_i) = \lim_n \dfrac{N_n}{n} \dfrac{1}{N_n} \sum_{i=0}^{n} f(X_i)$

$$= \mu(X) \cdot \frac{\int f \, d\mu}{\mu(X)} = \int f \, d\mu .$$

(5) We showed that it holds for any $f \geq 0$. Let $f \in \mathcal{L}^1(S, \mu)$

Then $f = f^+ - f^-$ with $f^+, f^- \geq 0$. So

$$\frac{1}{n} \sum_{i=0}^{n} f(X_i) = \left(\frac{1}{n} \sum_{i=0}^{n} f^+(X_i) \right) - \left(\frac{1}{n} \sum_{i=0}^{n} f^-(X_i) \right)$$

$$\longrightarrow \int f^+ d\mu - \int f^- d\mu = \int f \, d\mu \qquad \text{a.s} \qquad \square$$