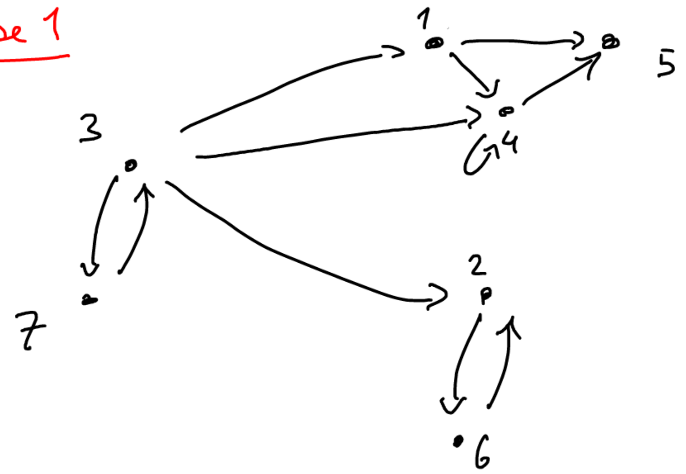
Exercise 1



① Observe that $7 \leftrightarrow 3$, $2 \leftrightarrow 6$ and $1 \leftrightarrow 4 \leftrightarrow 5$.

Further $\{2,6\}$ and $\{1,4,5\}$ are closed. Being both closed, irreducible and finite, we get that all these states are recurrent.

Also, because $3 \rightarrow 2 \not\rightarrow 3$, we have that $3,7$ are transient.

$S_1 = \{2,6\}$ $\quad T = \{3,7\}$ is the desired decomposition

$S_2 = \{1,4,5\}$

② Recall that the period is a property of the indecomposible classes.

Observe that $A_3 = \{2,4,6,\dots\}$ since the only paths that return to 3 go through 7. Thus $p(3) = 2 = p(7)$

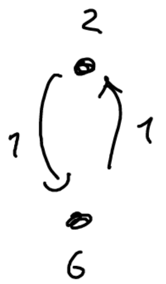
Similarly, $A_2 = A_3$ and $p(2) = p(6) = 2$

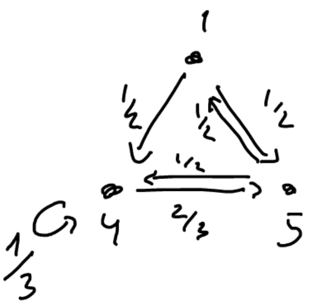
Since $Q^7_{4,4} > 0$, $p(4) = 1 = p(1) = p(5)$.

③ We find one for each closed irreducible component.

For $\{2,6\}$ we can take $\mu_2 = \mu_6 = \frac{1}{2}$, and $\mu_i = 0$ otherwise

For $\{1,4,5\}$ we have the equations





$$\mu_1 + \mu_4 + \mu_5 = 1 \qquad (a)$$

$$\frac{1}{2}\mu_5 = \mu_1 \qquad (b)$$

$$\frac{1}{2}\mu_1 + \frac{2}{3}\mu_4 = \mu_5 \qquad (c)$$

$$\frac{1}{2}\mu_1 + \frac{1}{2}\mu_5 + \frac{1}{3}\mu_4 = \mu_4 \qquad (d)$$

$$\frac{1}{2}(b) + (c) \Rightarrow \frac{2}{3}\mu_4 = \frac{3}{4}\mu_5 \Rightarrow \mu_4 = \frac{9}{8}\mu_5 \qquad (e)$$

Using (b) & (e) in (a) gives
$$\mu_5\left(\frac{1}{2} + \frac{9}{8} + 1\right) = 1$$
$$\Rightarrow \mu_5 = \frac{8}{21}$$
$$\mu_4 = \frac{9}{21} \qquad \mu_1 = \frac{4}{21}$$

with $\mu_i = 0$ otherwise.

Exercise 2

(1) For $v_i \neq v_0$, we have

$$\mathbb{E}_{v_j}\left[T_{v_0} \mid X_1 = v_i\right] = \mathbb{E}_{v_i}\left[T_{v_0} + 1\right] = m_i + 1$$

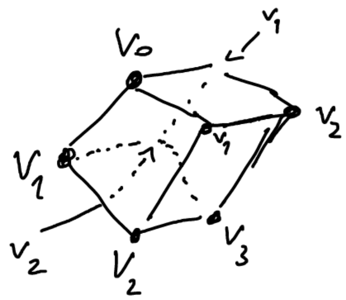
Strong Markov property, $T=1$

For $v_i = v_0$, it is immediate that $T_{v_0} = 1$, that is

$$\mathbb{E}_{v_i}\left[T_{v_0} \mid X_1 = v_0\right] = 1$$

(2)
$$m_1 = \mathbb{E}_{v_1}\left[T_{v_0}\right] = \mathbb{E}_{v_1}\left[\sum_v T_{v_0} \mathbb{1}_{X_1 = v}\right]$$

$$= \sum_v \mathbb{E}_{v_1}\left[T_{v_0} \mid X_1 = v\right] \cdot \mathbb{P}_{v_1}\left[X_1 = v\right]$$



$$\underset{MP}{=} \sum_v \mathbb{E}_{v_1}\left[T_{v_0} \mid X_1 = v\right] \cdot Q_{v_1, v}$$

$$= \mathbb{E}_{v_1}\left[T_{v_0} \mid X_1 = v_0\right]\frac{1}{3} + 2\,\mathbb{E}_{v_1}\left[T_{v_0} \mid X_1 = v_2\right]\frac{1}{3}$$

$$\underset{(1)}{=} \frac{1}{3} + \frac{2}{3}(m_2 + 1) = 1 + \frac{2}{3}m_2$$

$$m_0 = \sum_v \mathbb{E}_{v_0}[T_{v_0} \mid X_1 = v] \cdot Q_{v_0,v}$$

$$= 3 \cdot \mathbb{E}_{v_0}[T_{v_0} \mid X_1 = v_1] \cdot \frac{1}{3} \underset{1}{=} m_1 + 1$$

$$m_3 = \sum_v \mathbb{E}_{v_3}[T_{v_0} \mid X_1 = v] \cdot Q_{v_3,v}$$

$$= 3 \cdot \mathbb{E}_{v_3}[T_{v_0} \mid X_1 = v_2] \cdot \frac{1}{3} \underset{1}{=} m_2 + 1$$

$$m_2 = \sum_v \mathbb{E}_{v_2}[T_{v_0} \mid X_1 = v] \cdot Q_{v_2,v}$$

$$= \mathbb{E}_{v_2}[T_{v_0} \mid X_1 = v_3] \cdot \frac{1}{3} + 2\mathbb{E}_{v_2}[T_{v_0} \mid X_1 = v_1] \cdot \frac{1}{3}$$

$$\underset{1}{=} (m_3 + 1) \cdot \frac{1}{3} + 2\,\frac{m_1 + 1}{3} = \frac{1}{3} m_3 + \frac{2}{3} m_1 + 1$$

That is
$$\begin{cases} m_0 = m_1 + 1 & (a) \\ m_1 = 1 + \frac{2}{3} m_2 \\ m_2 = \frac{1}{3} m_3 + \frac{2}{3} m_1 + 1 \\ m_3 = m_2 + 1 & (b) \end{cases} \Rightarrow \begin{cases} m_1 = 1 + \frac{2}{3} m_2 \\ m_2 = \frac{1}{3}(m_2 + 1) + \frac{2}{3}(1 + \frac{2}{3} m_2) + 1 \end{cases}$$

$$\Rightarrow \begin{cases} m_1 = 1 + \frac{2}{3} m_2 \\ m_2\left(1 - \frac{1}{3} - \frac{4}{9}\right) = \frac{1}{3} + \frac{2}{3} + 1 \end{cases} \Rightarrow \begin{cases} m_1 = 1 + \frac{2}{3} \cdot 9 = 7 \\ m_2 = \frac{9}{\cancel{2}} \times \cancel{2} \end{cases}$$

Thus, $m_3 \underset{b}{=} 10$ and $m_1 \underset{a}{=} 8$

(4) Because the MC is uniform, $\mu(i) = \mu(v_0) = m_0^{-1} = \frac{1}{8}$ is a candidate for a stationary probability measure.

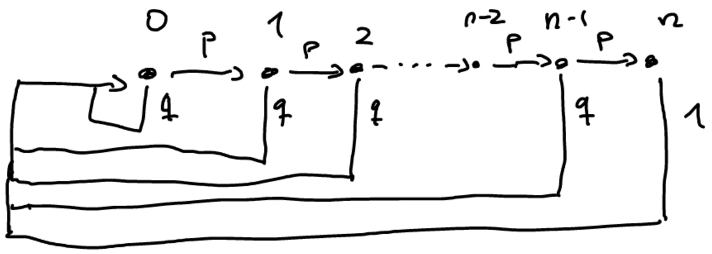
Indeed, this is a reversible measure, as if v, w are two neighbouring vertices,

$$\mu(v)\, Q_{v,w} = \frac{1}{8} \cdot \frac{1}{3} = 1/24$$
$$\mu(w)\, Q_{w,v} = \frac{1}{8} \cdot \frac{1}{3} = 1/24$$

so the prob. measure is stationary.

$\boxtimes$

Exercise 3



(1) $0 \to 1 \to \cdots \to n \to 0$

So this is irreducible MC.

Since it is finite, it is recurrent.

Since $Q^1_{0,0} > 0$, $p(0) = 1$ so $p(i) = 1$ $\forall i \in S$. Thus, this is an aperiodic MC.

(2) Observe that because this is a recurrent finite MC, it has a stationary prob. measure $\mu = (\mu_0, \dots, \mu_n)$.

Since $\mu = \mu Q$, this satisfies for $i > 0$

$$(\mu Q)_i = \mu_{i-1} Q_{i-1,i} = \mu_i$$

That is $\quad \mu_i = p\,\mu_{i-1} \quad (*)$

Claim: $\quad \mu_i = p^i \mu_0 \quad \forall\, i \in S$

Proof: For $i = 0$ this is trivial. We act by induction and use $(*)$ to get

$$\mu_{i+1} = p \cdot \mu_i \overset{\text{induction hypothesis}}{=} p \cdot p^i \mu_0 = p^{i+1} \mu_0 \quad \text{as desired} \quad \square$$

Because μ is a probability measure, $\sum_{i=0}^n \mu_i = 1$ so

$$\sum_{i=0}^n \mu_0 \cdot p^i = 1 \implies \mu_0 = \frac{1}{\sum_{i=0}^n p^i} = \frac{1-p}{1-p^{n+1}}$$

From the claim it follows that $\quad \mu_i = \dfrac{p^i - p^{i+1}}{1 - p^{n+1}}$ for each $i \in S$.

Thus,

$$\mathbb{E}_n[T_n] = \mu_n^{-1} = \frac{1-p^{n+1}}{p^n - p^{n+1}} = p^{-n} + p^{-n+1} + \cdots + p^{-1} + 1$$

$\square$

Example: If $p = 1/2$, $\mathbb{E}_n[T_n] = 2^{n+1} - 1$