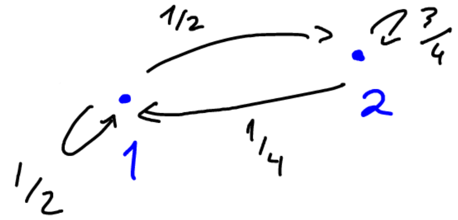
Exercise 1



1.1 $\quad \mathbb{P}(T_1 = k \mid X_0 = 1) = \mathbb{P}(X_0 = X_k = 1,\ X_1 = \dots = X_{k-1} = 2 \mid X_0 = 1)$

$(k=1) \quad = \mathbb{P}(X_1 = 1 \mid X_0 = 1) = 1/2$

$(k>1) \quad = \mathbb{P}(X_1 = 2 \mid X_0 = 1) \times \left[\prod_{j=2}^{k-1} \mathbb{P}(X_j = 2 \mid X_{j-1} = 2)\right] \times \mathbb{P}(X_k = 1 \mid X_{k-1} = 2)$

$= \frac{1}{2} \times \left(\frac{3}{4}\right)^{k-2} \cdot \frac{1}{4}$

Note that $\mathbb{P}(T_1 < +\infty \mid X_0 = 1) = \sum_{k \geq 1} \mathbb{P}(T_1 = k \mid X_0 = 1)$

$= \frac{1}{2} + \frac{1}{8} \cdot \sum_{k \geq 2} \left(\frac{3}{4}\right)^{k-2} = \frac{1}{2} + \frac{1}{8}\left(\frac{1}{1 - 3/4}\right)$

$= \frac{1}{2} + \frac{1}{8} \cdot 4 = 1.$ (This is because 1 is a rec. state)

This describes the distribution of T_1.

1.2 Find $\mu = (\mu_1, \mu_2)$ s.t. $\mu Q = \mu$ and $\mu \vec{1}^T = 1$

That is an eighenvector of Q^T for the eighenvalue 1.

$$\ker(Q^T - I) = \ker \begin{pmatrix} -1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$$

Thus $\mu = \left(\frac{1}{3}, \frac{2}{3}\right)$.

$$\mathbb{E}[T_1] = \frac{1}{2} \times 1 + \sum_{k \geq 2} k \cdot \frac{1}{8}\left(\frac{3}{4}\right)^{k-2} = \frac{1}{2} + \frac{1}{8} \cdot \left(\frac{4}{3}\right) \cdot \sum_{k \geq 2} k \cdot (3/4)^{k-1}$$

$= \frac{1}{2} + \frac{1}{6} f(3/4)$ where $f(x) := \sum_{k \geq 2} k \cdot x^{k-1} = g'(x)$

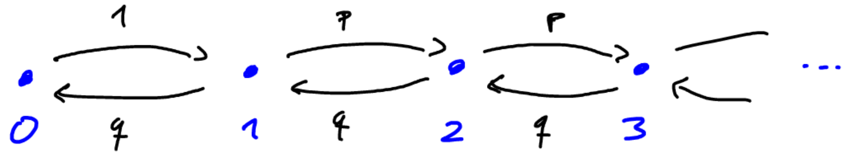
$g(x) := \sum_{k \geq 2} x^k = \frac{x^2}{1-x}$

Therefore, $f(x) = \frac{d}{dx} g(x) = \frac{2x(1-x) + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$

And $\mathbb{E}[T_1] = \frac{1}{2} + \frac{1}{6} \frac{2 \cdot \frac{3}{4} - \left(\frac{3}{4}\right)^2}{(1 - 3/4)^2} = \frac{1}{2} + \frac{1}{6} \frac{(24-9)/\cancel{16}}{1/\cancel{16}} = \frac{1}{2} + 4 - \frac{3}{2} = 3$

Note that $\mu(1)^{-1} = 3 = \mathbb{E}[T_1]$ $\square$

Exercise 2



$q = 1-p \in (0,1)$.

2.1 $\mu_i = \begin{cases} (p/q)^i & , \text{if } i \geq 1 \\ p & , \text{if } i = 0 \end{cases}$

Then
$$(\mu \cdot Q)_0 = \mu_1 Q_{1,0} = \frac{p}{q} \cdot q = p = \mu_0$$
$$(\mu \cdot Q)_1 = \mu_0 Q_{0,1} + \mu_2 Q_{2,1} = p \cdot 1 + \left(\frac{p}{q}\right)^2 \cdot q$$
$$= \frac{pq + p^2}{q} = \frac{p}{q} \underbrace{(p+q)}_{=1} = \mu_1$$

$j \geq 2$; $$(\mu \cdot Q)_j = \mu_{j-1} Q_{j-1,j} + \mu_{j+1} Q_{j+1,j}$$
$$= \left(\frac{p}{q}\right)^{j-1} \cdot p + \left(\frac{p}{q}\right)^{j+1} q = \left(\frac{p}{q}\right)^j \Big[\underbrace{\frac{q}{p} \cdot p + \frac{p}{q} \cdot q}_{=1} \Big]$$
$$= \mu_j$$

Thus $\mu = \mu Q$ is a stationary measure.

2.2 $$\mu(\mathbb{Z}_{\geq 0}) = p + \sum_{i \geq 1} (p/q)^i = p + \frac{p/q}{1 - p/q} < +\infty$$

$p/q < 1$ for $p < 0.5$

Therefore, by Theorem 16.3, because this is an irreducible MC, and has a finite measure, all states are recurrent.

Exercise 3 $H_0 = \inf_n \{X_n = 0\}$, $H_1 = \inf_n \{X_n = 1\}$

$$\phi(s) := \mathbb{E}[s^{H_0} \mid X_0 = 1]$$

3.1 The strong Markov property says that $(\{X_{n+H_1}\}_{n \geq 0}, X_0 = 2)$ has the same distribution as $(\{X_n\}_{n \geq 0}, X_0 = 1)$.

Further, $(\{X_{n+H_1}\}_{n \geq 0}, X_0 = 2) \perp\!\!\!\perp \{X_0, \dots, X_{H_1}\}$.

It follows that $\mathbb{P}_1(H_0 = k) = \mathbb{P}_2(H_1 + \tilde{H}_0 = H_1 + k) \quad \forall k$
and that $\tilde{H}_0 \perp\!\!\!\perp H_1$.

Also $\left(\{X_n - 1\}_{n=0}^{H_1}, X_0 = 2\right) \sim \left(\{X_n\}_{n=0}^{H_0}, X_0 = 1\right)$ because the Markov chains have the same transition matrix and inicial dist. under the identification $k \mapsto k-1$.

Thus $\mathbb{P}_2(H_1 = k) = \mathbb{P}_1(H_0 = k)$.

It follows that $\mathbb{E}_2[s^{H_0}] = \mathbb{E}_2[s^{\tilde{H}_0 + H_1}] = \mathbb{E}_2[s^{\tilde{H}_0}]\,\mathbb{E}_2[s^{H_1}] = \phi(s)^2$

$\mathbb{E}_1[s^{H_0} \mathbb{1}[X_1 = 2]] = \mathbb{P}_1(X_1 = 2) \cdot \mathbb{E}_1[s^{H_0} \mid X_1 = 2] = p \cdot \mathbb{E}_2[s^{H_0 + 1}] = p \cdot s \cdot \phi(s)^2$

3.2

$$\mathbb{E}[s^{H_0} \mid X_0 = 1] = p\,\mathbb{E}[s^{H_0} \mid X_1 = 2] + q\,\mathbb{E}[s^{H_0} \mid X_1 = 0]$$

$$= p \cdot s\phi(s)^2 + q \cdot s \qquad \text{as desired}$$

$$\phi^2 ps - \phi + qs = 0 \implies \phi(s) = \frac{1 \pm \sqrt{1 - 4pqs^2}}{2ps} \qquad (**)$$

From $\sqrt{1-x} = 1 - \frac{x}{2} + O(x^2)$, for $s \to 0$ we get

$$\phi(s) = \frac{1 \pm (1 - 2pqs^2 + O(s^4))}{2ps}$$

Now $\phi(0) = \mathbb{P}(H_0 = 0) \leq 1$ so for s close to zero we have

$$\phi(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}$$

Let us now study when do the positive and negative roots of $**$ coincide. This happens when $1 - 4pqs^2 = 0$, or for $s^2 = \frac{1}{4pq}$

Now $pq \leq \left(\frac{p+q}{2}\right)^2 = 1/4$ so $\frac{1}{4pq} \geq 1$. $(*_1)$

The positive and the negative root of $(**)$ are distinct for $1-4pqs^2 \neq 0$, so for $s \in [0,1)$ they are always distinct according to $(*_1)$. They also vary continuously wrt s, so

$$\phi(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}$$

3.3 From $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} + O(x^3)$ we have

$$\phi(s) = \frac{2pqs^2 + 2p^2q^2s^4 + O(s^6)}{2ps} = qs + pqs^3 + O(s^5)$$

It follows that $\mathbb{P}(H_0 = 3 \mid X_0 = 1) = p \cdot q$

3.4 $\phi(s) = \dfrac{1-\sqrt{1-4pqs^2}}{2ps}$, so:

$$\lim_{s \to 1^-} \phi(s) = \frac{1-\sqrt{1-4pq}}{2p} = \frac{1-\sqrt{1-4p(1-p)}}{2p} = \frac{1-\sqrt{(1-2p)^2}}{2p} = \begin{cases} 1, & \text{if } p \leq 1/2 \\ \frac{-2p+2}{2p}, & \text{if } p \geq 1/2 \end{cases}$$

$$= \begin{cases} 1, & \text{if } p \leq 1/2 \\ q/p, & \text{if } p \geq 1/2 \end{cases} = \mathbb{P}(H_0 < +\infty \mid X_0 = 1)$$

3.5 Because $\frac{d}{ds}\left(1-\sqrt{1-4pqs^2}\right) = \frac{8pqs}{2\sqrt{1-4pqs^2}} = \frac{4pqs}{\sqrt{1-4pqs^2}}$, we have

$$\phi'(s) = \frac{2ps \cdot \frac{4pqs}{\sqrt{1-4pqs^2}} - 2p\left(1-\sqrt{1-4pqs^2}\right)}{4p^2s^2} = \frac{2q}{\sqrt{1-4pqs^2}} - \frac{1-\sqrt{1-4pqs^2}}{2ps^2}$$

Thus, $\lim_{s \to 1^-} \phi'(s) = \dfrac{2q}{|1-2p|} - \dfrac{1-|1-2p|}{2p} \underset{p<0.5}{=} \dfrac{2q}{1-2p} - 1$

$$= \frac{2q-1+2p}{1-2p} = \frac{1}{1-2p} \quad \square$$