Probability 2 - Ex 0 Solution

Ex 1

(a) $0.4+0.2+0.1+a+b+c=1 \Rightarrow a+b+c=0.3$

$$\mathbb{E}[X] = 0.4+2\times 0.2+3\times 0.1+4a+5b+6c$$
$$= 0.4+0.4+0.3+4\underbrace{(a+b+c)}_{0.3}+b+2c$$
$$= 2.3+b+2c = 2.3+\overset{0.3}{(b+c)}+c \leq 2.3+0.3+0.3 = 2.9$$

False

(b) False

(c) No

(d) $\mathbb{E}[X] = 2.3+b+2c$

$a+b+c = b+2c = 0.3$

So $\mathbb{E}[X] = 2.6$.

Ex 2

(a) $\mathbb{P}(X=1) = \mathbb{P}\left(\bigcup_{y\in[0,1]} X=1 \ \&\ Y=y\right) =$

$$= \sum_{j=1}^{n} \mathbb{P}\left(X=1 \ \&\ Y\in\left[\tfrac{j-1}{n}, \tfrac{j}{n}\right)\right)$$

$$= \sum_{j=1}^{n} \mathbb{P}\left(Y\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right)\right)\cdot\mathbb{P}\left(X=1 \mid Y\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right)\right)$$

$$\leq \sum_{j=1}^{n} \frac{1}{n}\times\frac{j}{n} = \frac{1}{n^2}\sum_{j=1}^{n} j = \frac{n(n+1)}{2n^2} \xrightarrow[n\to+\infty]{} \frac{1}{2}$$

Similarly, $\mathbb{P}(X=1) = \sum_{j=1}^{n}\mathbb{P}\left(Y\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right)\right)\cdot\mathbb{P}\left(X=1 \mid Y\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right)\right)$

$$\geq \sum_{j=1}^{n} \frac{1}{n}\times\frac{j-1}{n} = \frac{1}{n^2}\sum_{j=1}^{n} j-1 = \frac{n(n-1)}{2n^2} \to \frac{1}{2}$$

So $\mathbb{P}(X=1) = 1/2$

$\boxed{\text{Other way of computing } \mathbb{P}(X=1)}$

$$\mathbb{P}(X=1) = \int \mathbb{P}(X=1 \mid Y=y)\, f_Y(y)\, dy$$

$$= \int_0^1 y\, dy = \frac{1}{2}$$

Note: these two techniques are essencially the same

(b) Because Y is a continuous r.v. we have $\mathbb{P}(Y=1)=0$.

So $\mathbb{P}(Y=1 \mid X=1) = \mathbb{P}(X=1 \mid Y=1) \dfrac{\mathbb{P}(Y=1)}{\mathbb{P}(X=1)} = 1 \times \dfrac{0}{1/2} = 0$

(c) $$\mathbb{P}(Y \geq 0.75 \mid X=1) = \frac{\mathbb{P}(Y \geq 0.75 \ \& \ X=1)}{\mathbb{P}(X=1)} =$$

$$= \int_{0.75}^{1} \mathbb{P}(X=1 \mid Y=y) f_Y(y)\, dy \frac{1}{\mathbb{P}(X=1)} = \int_{0.75}^{1} 1-y \, dy \times 2 =$$

$$= 2 \left[y - \tfrac{1}{2} y^2 \right]_{y=3/4}^{y=1} = 2\left(1 - \frac{1}{2} - \frac{3}{4} + \frac{9}{32}\right) = 2 - 1 - \frac{3}{2} + \frac{9}{16}$$

$$= -\frac{1}{2} + \frac{9}{16} = \frac{1}{16}$$

Ex3

(a) $Y \sim \mathrm{Unif}(0,1)$, $\quad f_X(x) = 2x \leftarrow$ density function for $x \in (0,1)$

For $t \in (0,1)$

$$\mathbb{P}(X^2 \leq t) \underset{t \geq 0}{=} \mathbb{P}(|X| \leq \sqrt{t}) \underset{X \geq 0 \text{ a.s}}{=} \mathbb{P}(X \leq \sqrt{t})$$

$$= \int_0^{\sqrt{t}} 2x\, dx \overset{0 \leq t \leq 1}{=} \left[x^2\right]_{x=0}^{x=\sqrt{t}} = \sqrt{t}^2 = t$$

$\mathbb{P}(Y \leq t) = t$ for $t \in (0,1)$.

So $\mathbb{P}(X^2 \leq t) = \mathbb{P}(Y \leq t)$ is true.

(b) $\mathbb{P}(X \leq t) = \int_0^t 2x\, dx = 2t^2$ for $t \in (0,1)$

$\mathbb{P}(Y^2 \leq t) = \int_0^{\sqrt{t}} 1\, dx = \sqrt{t}$ for $t \in (0,1)$

So $\mathbb{P}(X \le t) \ne \mathbb{P}(Y^2 \le t)$ in general.
is false

(c) We just have information on the law of X^2 and Y, not the full information, so we don't know if $\mathbb{P}(X^2 = Y) = 1$

(d) False, because $X = Y^2$ a.s. $\Rightarrow X \overset{(d)}{=} Y^2$, but $X \overset{(d)}{=} Y^2$ is false (see (b))

Ex 4

(a) False. Consider a r.v. $X \in (1, +\infty)$ with density $f_X(x) = \frac{1}{x^2}$.

Note that $\int_1^{+\infty} f_X(x)\,dx = \left[-x^{-1}\right]_1^{+\infty} = 1$ but
$\mathbb{E}[X] = \int_1^{+\infty} f_X(x) \cdot x\,dx = \left[\ln x\right]_1^{+\infty} = +\infty$.

Let $Y \perp\!\!\!\perp X$ be s.t. $\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = 1/2$
Then $\mathbb{E}[X \cdot Y]$ is not well defined.

The expected value of a r.v. X is well defined in $\overline{\mathbb{R}}$ when
- $X \ge 0$ or $X \le 0$ or
- $\mathbb{E}[|X|] < +\infty$ (that is, $X \in \mathcal{L}^1$).

(b) True. This is a consequence of Hölder's inequality

(c) True. This is a consequence of (b)

(d) Because $X \le 1$ a.s., $\|X\|_{\infty} = 1$ so $X \in \mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^\infty$

(e) Directly computing

$\|X\|_\infty = \infty \qquad \|X\|_2 = \infty \qquad \|X\|_1 = 2$

So $X \in \mathcal{L}_1$ but $X \notin \mathcal{L}^2, \mathcal{L}^\infty$

(f) $\mathbb{E}[|X|] = \sum_{j=0}^{+\infty} e^{-1}\frac{1}{j!}\cdot j = e^{-1}\sum_{j=1}^{+\infty}\frac{1}{(j-1)!} = e^{-1}\cdot e = 1$

$$\mathbb{E}[X^2] = \mathbb{E}[X(X-1) + X] =$$
$$= e^{-1}\left(\sum_{j=0}^{+\infty} \frac{1}{j!} j(j-1) + \frac{1}{j!} j\right)$$
$$= e^{-1}\left(\sum_{j=2}^{+\infty}\frac{1}{(j-2)!} + \sum_{j=1}^{+\infty}\frac{1}{(j-1)!}\right) = 2$$

$\|X\|_\infty =$ essencial supremum of $X = +\infty$

So $X \in \mathcal{L}^1, \mathcal{L}^2$ but $X \notin \mathcal{L}^\infty$.

(g) $\mathbb{E}[|X|] = \sum_{k=1}^{+\infty} \frac{6}{\pi^2}\frac{1}{k} = +\infty$ So $X \notin \mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^\infty$

Ex 5

(a) $X_n \xrightarrow{\mathcal{L}^p} Y$ if $\|X_n - Y\|_p \to 0$.

$$\|X_n - Y\|_p = \|X_n\|_p = \mathbb{E}[|X_n|^p]^{\frac{1}{p}} = \mathbb{E}[X_n]^{\frac{1}{p}} = \frac{1}{n^{1/p}} \to 0$$

So $X_n \xrightarrow{\mathcal{L}^p} Y$ for any $p \geq 1$.

(b) We simply observe that $Y_n \sim \mathrm{Ber}(1/n)$. That is

$\mathbb{P}(Y_n = 1) = \mathbb{P}(Y \leq 1/n) = 1/n$

$\mathbb{P}(Y_n = 0) = \mathbb{P}(Y > 1/n) = 1 - 1/n$

From (a) and the diagram (Figure 1), we have that

$$X_n \xrightarrow{L^p} Y \Rightarrow X_n \xrightarrow{(d)} Y$$

Because $X_n \overset{(d)}{=} Y_n$, $Y_n \xrightarrow{(d)} Y$.

(c) From (a) and the diagram (Figure 1), $X_n \xrightarrow{P} Y$

$Y_n \xrightarrow{P} Y$ if $\mathbb{P}(|Y_n - Y| < \varepsilon) \xrightarrow[n]{} 1$ for any $\omega > 0$.

$$\mathbb{P}(|Y_n - Y| < \varepsilon) = \mathbb{P}(|Y_n| < \varepsilon) = \mathbb{P}(|Y_n| = 0) = 1 - \frac{1}{n} \to 1$$

(d) First, note that $|Y_n| \leq 1$ a.s, so Y_n is a dominated sequence of r.v.

Also, for any $\omega \in \Omega$, $Y_n = 0$ whenever $n \geq \frac{1}{U(\omega)}$, by definition of $Y_n = \mathbb{1}[U \leq 1/n]$.

Since $\mathbb{P}(U = 0) = 0$, $\mathbb{P}(\lim Y_n = Y) = 1$.

So by the dominated convergence theorem we have that $Y_n \xrightarrow{L^1} Y$.

(e) We use again the DCT. $|Z_n| \leq 1$ a.s and $Z_n = \frac{1}{n} Z \xrightarrow[n]{} 0$, so $Z_n \xrightarrow{L^1} 0$ and $\mathbb{E}[Z_n] \to \mathbb{E}[0] = 0$.

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