UNIVERSITY OF ZÜRICH

Algebraic and geometric studies of combinatorial substructures and chromatic invariants

by

Raul Penaguiao

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Faculty of Science
Department of Mathematics

March 2020

“What is your favorite Hopf algebra?”

Stephanie van Willigenburg

UNIVERSITY OF ZÜRICH

Abstract

Faculty of Science
Department of Mathematics

Doctor of Philosophy

by Raul Penaguiao

In this thesis the reader can find results in algebraic combinatorics, along with its connections with convex geometry, extremal combinatorics and the study of substructures. We start by making the reader comfortable with the context of algebraic combinatorics and introducing the central object in this thesis: Hopf algebras.

This leads us to talk about the common ways in which these Hopf algebras are constructed from combinatorial objects and present an original one, the pattern algebras. We introduce the notion of combinatorial presheaf, by adapting the algebraic framework of species to the study of substructures in combinatorics. Afterwards, we consider functions that count the number of patterns of objects and endow the linear span of these functions with a product and a coproduct. In this way, any well behaved family of combinatorial objects that admits a notion of substructure generates a Hopf algebra, and this correspondence is functorial. For example, the Hopf algebra on permutations studied by Vargas in 2014 and the Hopf algebra on symmetric functions are particular cases of this construction.

From there, we consider three problems on Hopf algebras:

- Chromatic invariants and kernel problems. We study the chromatic symmetric function on graphs, and show that its kernel is spanned by the modular relations. We generalize this result to the chromatic quasi-symmetric function on hypergraphic polytopes, a family of generalized permutahedra. We use the description of the kernel of the chromatic symmetric function to find other graph invariants that may help us tackle the tree conjecture.

- Freeness and other structure results. We discuss the multiplicative structure of Hopf algebras, and the methods used to prove freeness. We show that all the pattern Hopf algebras corresponding to commutative presheaves are free. We also study a non-commutative presheaf on marked permutations, *i.e.* permutations with a marked element. These objects have an inherent product called inflation, which is an operation motivated by factorization theorems of permutations. In this thesis, we find new factorization theorems on marked permutations, and use them to show that this is another example of a pattern Hopf algebra that is free.
- Antipode formulas: Specifically, we consider cancellation-free and grouping-free antipode formulas. These formulas not only are economic and helpful to compute, but also reflect interesting combinatorial formulas like reciprocity results.

Finally, we address an unexpected application of this algebraic machinery in extremal combinatorics. The reader will find below a description of what is called a *feasible region* of classical and consecutive permutation patterns. This is a geometric body that encodes precisely to which extent the proportion of occurrences of a particular pattern affect the proportion of occurrences of other patterns. We see here that the description of the free generators of the permutation pattern Hopf algebra plays a role in the dimension of this geometric body. Furthermore, we show that in the context of consecutive occurrences, this feasible region is the cycle polytope of a particular graph, which allows us to compute its dimension, vertices and faces.

Contents

Symbols

Categories and functors

$\mathtt{Set}^{\hookrightarrow}$	Category of finite sets and injective maps	22
$\mathtt{Set}^{\times}$	Category of finite sets and bijective maps	16
$\mathtt{Set}$	Category of finite sets	16
$\mathtt{CPSh}$	Category of combinatorial presheaves	22
$\mathrm{Sp}_{\mathbb{K}}$	Category of vector species over $\mathbb{K}$	124
$\mathrm{Vec}_{\mathbb{K}}$	Category of vector spaces over $\mathbb{K}$	16
$\mathrm{Alg}_{\mathbb{K}}$	Category of algebras over $\mathbb{K}$	46
$\mathrm{BiAlg}_{\mathbb{K}}$	Category of bialgebras over $\mathbb{K}$	46
$\mathrm{Mon}(\mathcal{C})$	Category of monoids on $\mathcal{C}$	187
$\mathrm{gVec}_{\mathbb{K}}$	Category of graded vector spaces over $\mathbb{K}$	126
$\mathcal{A}$	The pattern algebra functor	24
$\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^{\vee}, \overline{\mathcal{K}}^{\vee}$	The Fock functors	17
$\Rightarrow$	A natural transformation	
$\Leftrightarrow$	A natural isomorphism	

Presheaves and species

$\mathtt{Gr}$	Associative presheaf on graphs	25
$\mathtt{Per}$	Associative presheaf on permutations	22
$\mathtt{MPer}$	Associative presheaf on marked permutations	39
$\mathtt{SPart}$	Associative presheaf on set partitions	56
$\mathtt{SComp}$	Associative presheaf on set compositions	57
$\mathbf{res}_J$ or $\|_J$	Restriction of objects	22
$\mathcal{G}$	Collection of coinvariants	23
$\mathbf{p}_a$	Pattern function	23
$\|\cdot\|$	The size or cardinality of an object	22
$\mathbb{X}(\cdot)$	The ground set of an object	37

$\odot$	The Cauchy product on species and presheaves	24
$\binom{a}{b,c}$	The number of quasi-shuffles of b, c that match a	41
$\mathcal{E}$	The unit species/presheaf	37
$\mathcal{E}(h)$	The fibers of an associative presheaf h	52
$*$	An associative operation on a presheaf	34
$\mathfrak{fac}(\cdot)$	Multiset of irreducible factors	51
$j(\cdot)$	Number of irreducible factors (with multiplicity)	51
$\mathcal{I}$	Family of irreducible objects	44
$\mathrm{Alg}(h)$	The algebra of an associative presheaf	48
$\cdot$	A product structure on $\mathcal{G}(h)$	42
$\oplus$	Direct sum of permutations	59
$\ominus$	Direct difference of permutations	39
$\star$	The inflation product on permutations and graphs	60, 55
$\mathrm{inc}_{J,I}$	The inclusion map, whenever $J \subseteq I$	22
$\mathrm{rel}_{A,B}$	The order preserving bijection $A \to B$	12
$\sqcup$	The union of disjoint sets, graphs, posets and walks	12
$\leq_p$	A partial order in $\mathcal{G}(h)$	53
$\leq_{fac}$	Total order on $\mathcal{G}(\mathtt{MPer})$	65
$\leq_{per}$	Total order on $\mathcal{G}(\mathtt{Per})$	65

Permutations

$\pi, \tau, \sigma, \gamma$	Permutations	22
$\pi^*, \tau^*, \sigma^*, \gamma^*$	Marked permutations	39
$\mathrm{beg}(\pi)$	The first pattern of a given size in π	164
$\mathrm{end}(\pi)$	The last pattern of a given size in π	164
$\mathcal{S}$	Set of finite permutations	29
$\mathcal{S}_{\leq k}$	Set of permutations of size at most k	144
$\mathcal{S}_k$	Set of permutations of size k	17
Π_π	Proportion of classical patterns in a permution	29
Γ_π	Proportion of consecutive patterns in a permution	29
$\widetilde{\mathrm{occ}}, \widetilde{\mathrm{occ}}_k$	Proportion of classical patterns	27, 145
$\widetilde{\text{c-occ}}, \widetilde{\text{c-occ}}_k$	Proportion of consecutive patterns in a permutation	28, 145
$\pi^* \backslash_{I^*}$	A permutation in I^c	71
$\mathfrak{rk}(\alpha^*)$	The rank of a marked permutation	65
clP_k	The feasible region on clasical patterns	27
P_k	The feasible region on consecutive patterns	28
$P_k^{\mathcal{C}}$	The restricted feasible region on consecutive patterns	175
$\mathrm{Av}(\pi)$	The set of permutations avoiding π	175

Partitions and other objects alike

Words

Graphs

$G/_F$	The flat quocient of a graph	4
$P(G)_H$	A face of the cycle polytope of G	157
$L(G)$	Incidence matrix of a graph G	147
$a(e)$	The arriving vertex of an edge on an oriented graph	146
$s(e)$	The starting vertex of an edge on an oriented graph	146
$\bullet$	The concatenation of walks, of (set) compositions	146, 8
$\mathrm{lb}(e)$	The label of an edge on a graph	146
$\deg^i_G(v)$	Number of incoming edges to a vertex v	146
$\deg^o_G(v)$	Number of outgoing edges from a vertex v	146
$N_G(v)$	The neighbouring edges to e	146
$B^G_\delta(v)$	The ball centered at a vertex v	176
G^c	The complement graph of G	98
$K_{\boldsymbol{\pi}}$	The complete graph on a set partition	98

Hopf algebras

$\mathbb{K}[G]$	Group Hopf algebra	5
$\mathbb{K}[\mathbf{x}]$	Polynomial ring on the variables $\mathbf{x} = (x_1, \dots)$	6
Sym	Hopf algebras of symmetric functions	8
$QSym$	Hopf algebras of quasi-symmetric functions	7
WSym	Hopf algebras of word symmetric functions	9
WQSym	Hopf algebras of word quasi-symmetric functions	9
G	Incidence Hopf algebra on graphs	12
Pos	Incidence Hopf algebra on posets	12
GP	Generalized permutahedra Hopf algebra	15
HGP	Hypergraphic polytopes Hopf algebra	15
SC	Singleton commuting Hopf algebra	108
x_f	Monomial of a coloring in commuting variables	84
$\mathbf{a}_f$	Monomial of a coloring in non-commuting variables	84
Ψ_H	Universal Hopf algebra morphism $H \to QSym$	19
Υ_h	Universal Hopf species morphism $\mathtt{h} \to \mathtt{WQSym}$	129
$\tilde{\Psi}$	Augmented chromatic symmetric function in **G**	100
comu	The projections **WSym** $\to Sym$, **WQSym** $\to QSym$	9
η, η_0	Characters in Hopf algebras	18
$P(C)$	The primitive space of a coalgebra C	183
$\star$	The convolution product on $\mathrm{Hom}(C, A)$	181
A^*	The dual vector space, the dual coalgebra	48
A°	The Sweedler dual of an algebra	48
twist	The natural map $A \otimes B \to B \otimes A$	6

id_A	The identity map on A	177
$\{m_\lambda\}_{\lambda\in\mathbf{C}_n}$	The monomial basis in Sym	8
$\{M_\alpha\}_{\alpha\in\mathcal{P}_n}$	The monomial basis in $QSym$	7
$\{\mathbf{m}_\pi\}_{\pi\in\mathbf{P}_n}$	The monomial basis in **WSym**	9
$\{\mathbf{M}_{\vec{\pi}}\}_{\vec{\pi}\in\mathbf{C}_n}$	The monomial basis in **WQSym**	9
$\{\mathbb{M}_{\vec{\pi}}\}_{\vec{\pi}\in\mathbf{C}_n}$	The monomial basis in `WQSym`	127
$\{\mathbf{N}_{[\vec{\pi}]}\}_{[\vec{\pi}]\in\mathbf{C}_n/\sim}$	The monomial basis in **SC**	108
$\Delta_{\vec{\pi}}$	The multi-coproduct corresponding to $\vec{\pi}$	130
$\bigoplus$	Direct sum of vector spaces	9

Polytopes

conv	The convex hull of a set	145
Aff	The affine span of a set	145
span	The linear span of a set	145
$A\vec{x}\geq\vec{b}$	Coordinatewise vector inequality	145
dim	Dimension of a vector space, affine space or a polytope	145
$\sum, +, -$	Minkowski sum of polytopes	13
$\mathfrak{s}_J$	The simplex generated by J	13
$\mathfrak{p}, \mathfrak{q}$	Polytopes	145
$\mathfrak{q}_f$	Face of $\mathfrak{q}$ minimizing the functional f	87
$\mathcal{N}_\mathfrak{q}(F)$	The normal fan of a face $F\subseteq\mathfrak{q}$	96

Generalized permutahedra

$\mathcal{L}(\mathfrak{q})$	Coefficients of a generalized permutahedron	95
$\mathfrak{q}\vert_A, \mathfrak{q}\backslash_A$	Comonoid structure on generalized permutahedra	98
$\mathcal{F}(\mathfrak{q})$	Summands of a generalized permutahedron	86
$\mathcal{F}^{-1}(A)$	The hypergraphic polytope generated by A	86
$\preceq$	Singleton commuting preorder in $\mathbf{C}_I$	107
sc_n	Singleton commuting numbers	115
$\mathrm{dist}(A,B)$	Distance between two convex sets	

Miscelaneous

e	Napier constant	115
γ	Unique positive solution of $e^{2\gamma}=1+(1+\gamma)e^\gamma$	115
$[n]$	The set $\{1,\dots,n\}$	8
$[m,n]$	The set $\{m,\dots,n\}$, whenever $m\leq n$	8

$\frac{\partial}{\partial x}P$ or P'	The differential operator on a formal power series P	103, 81
$\mathcal{F}(A,B), B^A$	The set of functions $A \to B$	23
I^*	If I is a set, $I \sqcup \{*\}$	39
$\mathrm{Mat}_{a\times b}(R)$	$a \times b$ matrices with coefficients on R	1
A^G	The invariant space of A over a group G	17
A_G	The coinvariant space of A over a group G	17
$\binom{I}{k}$	Subsets of I of size k	12
$\sim$	An equivalence relation	
$[a], [a]_\sim$	Equivalence class of a	
$\|\| \cdot \|\|$	Norm of a vector	
$\emptyset$	The empty set, other empty objects	
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}_{\geq 0}$	Usual sets of numbers	
$\max A, \min A$	Maximum and minimum of a set A	
$\mathrm{argmin}_{a\in A} f$	Element of A that minimizes f	
I^c	The complement set of I	
$A \setminus B$	The set difference	
$\lim$	Limit of a sequence	
$\mathbb{E}[\cdot]$	Expected value of a random variable	

Dedicated to Einstein

Chapter 1

Introduction

Dealing with combinatorial objects is of great importance to the working mathematician. This transpires both when the combinatorial objects are in plain sight, as the graph that is the underlying structure of a network, and when they can be introduced in our mathematical vocabulary to solve problems and point the way to correct solutions.

This thesis deals with the study of the interaction between combinatorial objects and algebraic operations. The methods underlined in this thesis are twofold: associating a suitable polytope to the combinatorial objects and studying the geometric properties of this polytope; or endowing algebraic structures on our combinatorial objects and studying the resulting algebraic structures. The hope is that either the features of the algebraic structures or the geometric properties allow us to more cleverly handle the original combinatorial structures, in order to extract information.

Take as an example the problem of enumerating matrices with integer entries and a fixed row sum, called *magic squares*. Specifically, we want to compute

$$H_n(r) = \left|\{A \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_{\geq 0}) \,|\, A \text{ has row sum } r\}\right| .$$

By considering a suitable embedding of matrices in a polynomial ring, namely $A \mapsto \prod_{i,j=1}^{n} x_{i,j}^{A_{i,j}}$, and a meticulous algebraic treatment, we can show that $H_n(r)$ is a polynomial in r for any integer n, meeting the expectation that algebraic structures can give us information about combinatorial objects. Details on this and many other examples are given in [Sta07].

The type of algebraic application that we develop in this thesis comes from recognizing *Hopf algebra* structures in combinatorics. The notion of Hopf algebra was introduced in the study of topological spaces, as an algebraic structure over the cohomology ring of certain topological spaces. It seems to be the case that the specific definition was never explicitly introduced by Hopf, but it came to be understood that this algebraic structure was the one that he was intending to use in his work in the 1950s. This has then grown to be a purely algebraic object, leaving the nuances of topological structures behind. We refer the reader to Appendix A, where the basics of Hopf algebras are given.

The introduction of Hopf algebras in combinatorics was brought forth by [JR79], where the relation between the study of "merging and splitting combinatorial objects" and bialgebras was pointed out. Specifically, consider a family of objects $\mathcal{O}$, where for three given objects $\bowtie, \diamond, \triangleright \in \mathcal{O}$ we can count how many ways we can split $\bowtie$ into the $\diamond, \triangleright$. For instance, how many ways can we split a collection of four white balls and two black balls into two collections of two white balls and a black ball. The resulting number is the so called *section coefficient*, denoted $(\bowtie|\diamond, \triangleright)$ in the literature. It is usually the case that these satisfy complicated algebraic relations. Depending on the specific equations, this often means that from the following definition

$$\Delta(\bowtie) = \sum_{\diamond, \triangleright} (\bowtie|\diamond, \triangleright) \diamond \otimes \triangleright \,, \tag{1.1}$$

a comultiplication Δ arises, *i.e.*, this is an algebraic operation that satisfies specific comultiplication properties presented in Definition A.1.2.

A product may be defined in a similar way. Thus, each family of combinatorial objects with a notion of splitting and merging yields a product or a coproduct. This is a different strategy from the usual one in enumerative combinatorics: we will not try to find relations and equations that these *splitting* coefficients satisfy, nor describe any generating function that relates to these coefficients. We will instead establish properties of these algebraic operations.

For instance, we are interested in the question of freeness of a Hopf algebra. A Hopf $\mathbb{K}$-algebra H is said to be free commutative (or simply free) over a ring R, if it is commutative and there is a set of generators $\mathcal{G} \subseteq H$ satisfying two properties: any element of the algebra H can be expressed as a polynomial over $\mathcal{G}$ with coefficients on the ring R; and there is no non-zero polynomial that annihilates over the set $\mathcal{G}$. Usually the ring is the field $\mathbb{K}$. Ditters' conjecture, proposed in the 1970s [Dit72], asks whether the Hopf algebra of quasi-symmetric functions is free over the integers. This conjecture is crucial for the classification theory of non-commutative formal groups, see [Dit72, Sch96]. Ditters' conjecture was proven in [Haz01].

The algebraic treatment of combinatorial objects also helps us to explain old invariants and find new ones. Take the case of graphs, posets and matroids, that admit the very well-known chromatic polynomial (see [Bir12]), the strict order polynomial (see

[Sta72]) and the Billera-Jia-Reiner polynomial (see [BJR09]), respectively. These were seen to be specializations of a particular Hopf algebra morphism to the Hopf algebra of quasi-symmetric functions in [ABS06]. The construction of such a Hopf algebra morphisms is mechanical and immediate in most combinatorial Hopf algebras, and results from a universal property in a suitable category. Consider the case of scheduling problems, a combinatorial abstraction of the usual NP-complete problem that in some ways generalize the notions of graph and matroids. In [BK16] a polynomial was associated to it to study these objects. Unsurprisingly, in the same paper the authors also give the corresponding quasi-symmetric version which comes from a suitable Hopf algebra structure.

In general, finding universal properties in combinatorially flavored categories gives us new invariants of combinatorial objects. In [Agu00] another result was presented, where the author observed that there is a universal object in the category of *infinitesimal Hopf algebras*, an algebra on non-commutative polynomials, and the map that arises encodes some information of the **cd**-index, an invariant of polytopes. These results are often called universality results, because they portray a universal property of an object in a suitable category.

The chromatic symmetric function is such an example. For a given graph G, this is a graph invariant $\Psi_{\mathbf{G}}(G)$ introduced in [Sta95] that generalizes the chromatic polynomial of a graph. It was observed that $\Psi_{\mathbf{G}}$ arises precisely as a morphism of combinatorial Hopf algebras from the incidence Hopf algebra on graphs $\mathbf{G}$, that we introduce in Section 1.1 below.

An important conjecture related to the chromatic symmetric function of graphs, $\Psi_{\mathbf{G}}$, is that this graph invariant distinguishes trees, a claim commonly referred to as the *tree conjecture*. When the chromatic symmetric function is seen as a Hopf algebra morphism, the tree conjecture is reduced to the following: if T_1, T_2 are two trees such that $T_1 - T_2 \in \ker \Psi_{\mathbf{G}}$, then the trees T_1, T_2 are isomorphic. This observation kick-starts the *kernel problem*, that is, to give generators for the kernel of Ψ. This problem makes sense for any combinatorial Hopf algebra, and we present a solution for the incidence Hopf algebra on graphs and for the Hopf algebra on **hypergraphic polytopes**, **HGP**, in Chapter 4, opening a possible avenue to tackle the tree conjecture.

Another interesting problem in combinatorial Hopf algebras is to obtain **cancellation-free** and **grouping-free** formulas for the antipode. This was done, for instance, in [HM12, ML14, MP15, Pat15, BS17, AA17]. These types of antipode formulas are specially interesting because they usually relate unexpected statistics to the Hopf algebra structure.

This can be applied to explain a result regarding the well-known chromatic polynomial of a graph. This is defined as

$$\chi_G(n) = |\{\text{proper vertex-colorings with } n \text{ colors}\}| ,$$

where a proper coloring of a graph is an assignment of a color to a vertex, in such a way that connected vertices have distinct colors. This is known to be a polynomial in n, so we can evaluate it at $n = -1$, obtaining the surprising identity (usually called a **reciprocity result**):

$$\chi_G(-1) = (-1)^{|G|} |\{\text{acyclic orientations of } G\}| . \tag{1.2}$$

Now we describe the antipode formula for graphs and compare with the above. Let $G = (V, E)$ be a graph. The rank of an edge set $E' \subseteq E$, written $\text{rank}(E')$, is the size of its maximal acyclic subset. A *flat* $F \subseteq E$ of the graph G is a set of edges such that $\text{rank}(F') > \text{rank}(F)$ for any $F' \supsetneq F$. If F is a flat of G, we define the graph $G/_F$ on the connected components of (V, F), connected by an edge if the components are connected by an edge in G. Then the following antipode formula was obtained in [HM12]:

$$S(G) = \sum_{F \subseteq E \text{ flat}} (-1)^{n - \text{rank}(F)} a(G/_F) G_{V,F} , \tag{1.3}$$

where $a(H)$ stands for the number of acyclic orientations of a graph H, and $G_{V,F}$ stands for the graph (V, F). There, we can see that the number of acyclic orientations and the concept of flats show up, despite having no immediate relation with the Hopf algebra structure. Based on the theory of Hopf algebras, we can in fact derive (1.2) from this antipode formula. Similarly, several **reciprocity results** are explained by what are called **cancellation-free and grouping-free formulas** for antipodes in combinatorial Hopf algebras, like the one in (1.3). These are formulas that are economic in that have as few terms as possible and no underlying cancellation of terms occurs.

As a concluding note, it is safe to say that Hopf algebras are widespread in mathematics. They made their debut in topology as describing the fundamental structure of the cohomology ring of path-connected Lie groups, and even nowadays have a great deal of influence in the study of algebraic topology. Hopf algebras have also made some unexpected appearances. They were discovered in quantum field theory as early as 1969 and were used to reinterpret some aspects of renormalization theory in the influential work of Conner and Kreimer in [CK99], see [Bro09] for a descriptive work. In knot theory, Hopf algebras are used to extract new knot invariants, for example the Jones polynomial, see a practical survey in [Saw96].

This introduction is structured as follows: in Section 1.1 we introduce most of the Hopf algebras that will play an important role in the thesis, along with some fundamental ones that are not necessary for the remaining of the thesis, but are nonetheless useful in algebraic combinatorics; in Section 1.2 we present the category theory point of view of combinatorial Hopf algebras. There, we see that the notion of **species** allows us to construct combinatorial Hopf algebras in a seamless way via the **Fock functors**; a method to obtain invariants in combinatorial structures from a Hopf algebra is presented in Section 1.3; in Section 1.4 we describe some of the reasons why simple antipode formulas for combinatorial Hopf algebras are an interesting topic of research in algebraic combinatorics; a general construction of pattern algebras from a **combinatorial presheaf** is described in Section 1.5; in Section 1.6 we present some freeness results on pattern algebras; in Section 1.7 we introduce the problem of describing feasible regions for patterns, and its connection to extremal combinatorics and to the algebra of permutation patterns. This leads us to the study of some graph polytopes.

1.1 Common Hopf algebras

Here we introduce some common Hopf algebras in combinatorics that play an important role in this thesis and in the landscape of algebraic combinatorics. We refer the reader to Appendix A for an introduction of Hopf algebras.

Note: for sake of clarity, we have been using boldface for non-commutative Hopf algebras in power series, their elements, and the associated combinatorial objects, like word symmetric functions and set compositions. We try and maintain that notational convention throughout the thesis.

The group Hopf algebra

We start with the group Hopf algebra. Despite not having a direct appearance in this thesis, it is perhaps the simplest generic construction of a Hopf algebra. It is a useful one: when suitably adapted, it can be used to introduce important Hopf algebras like quantum groups, see [Dri86]. These correspond to *deformations* of the group Hopf algebras. More specifically, they differ from the latter by a parameter. This was a surprising construction, as it has been known that compact Lie groups and Lie algebras do not admit deformations, that is, they are *rigid* objects. It turns out that finite groups are not like that, and admit deformations in the category of Hopf algebras.

Let $\mathbb{K}$ be a field, which we assume to have characteristic zero, and consider a finite group G. Let $\mathbb{K}[G]$ be a vector space, with a basis $\{e_g\}_{g\in G}$ indexed by G. The product is defined as

$$e_g e_h = e_{g \cdot h} \,,$$

and the coproduct is defined as

$$\Delta(e_g) = g \otimes g \,.$$

In this way, the linear map S defined on the basis as $S(e_g) = e_{g^{-1}}$ is the antipode map and endows $\mathbb{K}[G]$ with a Hopf algebra structure.

If 1 is the identity of G, then e_1 is the unit of $\mathbb{K}[G]$. The counit map ε is defined in the basis elements as $\varepsilon(e_g) = 1$. In Appendix A we saw that a group-like element x satisfies $\Delta(x) = x \otimes x$, and a primitive element x satisfies $\Delta(x) = 1 \otimes x + x \otimes 1$. In this way, the group Hopf algebra has no primitive elements. In addition, the set $\{e_g\}_{g \in G}$ is precisely the set of group-like elements. This Hopf algebra is cocommutative, *i.e.*, if we consider the natural map $\mathbf{twist} : \mathbb{K}[G] \otimes \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G]$, we have that $\mathbf{twist} \circ \Delta = \Delta$.

With this example we can explain the common adage "Hopf algebras are to bialgebras as groups are to monoids", as mentioned for instance in [VdL19]. Indeed, for a finite monoid G that is not closed under inversion, it can be shown that the corresponding bialgebra does not admit an antipode, and what we obtain is not a Hopf algebra.

We remark that the Hopf algebras presented in the remaining of this section have a fundamental difference comparing with the group Hopf algebra, as all of them are filtered, that is, the product and coproduct respect a "grading" of the vector space. See Definition A.2.3.

The polynomial Hopf algebra

The following Hopf algebra is another example of a simple construction. This Hopf algebra plays an important role in Theorem A.4.4, where a classification of cocommutative Hopf algebras is presented. Consider $\{x_i\}_{i \in I}$, a family of commuting variables indexed by a generic set. The polynomial Hopf algebra $\mathbb{K}[\mathbf{x}]$ is the usual algebra structure on the polynomial ring $\mathbb{K}[\mathbf{x}]$, together with the coalgebra map defined by $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$. In this way, because Δ is a multiplicative map, we have that

$$\Delta(x_i^n) = \Delta(x_i)^n = \sum_{k=0}^{n} \binom{n}{k} x_i^k \otimes x_i^{n-k} \,.$$

The counit ε is further defined with $\varepsilon(x_i) = 0$. The multiplicative map $S(x_i) = -x_i$ defines an antipode. This makes $\mathbb{K}[\mathbf{x}]$ a commutative and cocommutative Hopf algebra.

The space of primitive elements is $P(\mathbb{K}[\mathbf{x}]) = \text{span}\{x_i\}_{i\in I}$. There are no group-like elements other than the unit.

Symmetric functions and quasi-symmetric functions

The ring of symmetric functions has played an important role in combinatorics and algebra, see [Sta00] for some examples. In representation theory, the Frobenius map encodes characters of representations of the symmetric group S_n, as symmetric functions. Specifically, it maps irreducible representations to the celebrated Schur functions basis. With this encoding, we can relate the product in the ring to *induced representations*, a classical construction in representation theory, see for instance [Mac98]. Thus, the Hopf algebra structure arises naturally: the coproduct can be constructed via the restriction of representations.

The Hopf algebra on quasi-symmetric functions arises as a natural generalization of symmetric functions. It was introduced in [Ges84], to explain a combinatorial relation on P-partitions from Stanley, in [Sta72], by algebraic means. This Hopf algebra also arises in representation theory via the representations of the Hecke algebra, see [Hiv00].

To introduce these Hopf algebras, we first describe some relevant combinatorial objects. An *integer composition*, or simply a composition, of n is an ordered list $\alpha = (\alpha_1, \dots, \alpha_k)$ of positive integers whose sum is n. We write $\alpha \models n$. We denote the length of the list by $l(\alpha)$ and we denote the set of compositions of n by $\mathcal{C}_n$.

An *integer partition*, or simply a partition, of n is a non-increasing list of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ whose sum is n. We write $\lambda \vdash n$. We denote the length of the list by $l(\lambda)$ and we denote the set of partitions of n by $\mathcal{P}_n$. By disregarding the order of the parts on a composition α we obtain a partition $\lambda(\alpha)$.

The Hopf algebra of quasi-symmetric functions is a graded Hopf algebra $QSym = \bigoplus_{n\geq 0} QSym_n$. Each homogeneous component $QSym_n$ has a basis $\{M_\alpha\}_{\alpha\models n}$ indexed by compositions of n. If $\alpha = (\alpha_1, \dots, \alpha_k)$, we identify M_α with the formal power series in the family $\{x_n\}_{n\geq 1}$ of commuting variables

$$\sum_{i_1 < \dots < i_k} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k} .$$

The coproduct structure is defined as follows:

$$\Delta M_\gamma = \sum_{\gamma = \alpha \bullet \beta} M_\alpha \otimes M_\beta ,$$

where $\alpha \bullet \beta$ is the concatenation of compositions.

The Hopf algebra Sym is a Hopf subalgebra of $QSym$, spanned by the elements

$$m_\tau = \sum_{\lambda(\alpha)=\tau} M_\alpha \, .$$

It is also a graded Hopf algebra and its homogeneous components have a basis indexed by partitions $\{m_\tau\}_{\tau \vdash n}$.

Symmetric functions and quasi-symmetric functions on non-commuting variables

We now introduce the non-commutative versions of the Hopf algebras above. These are the Hopf algebra of word symmetric functions, also known as the Hopf algebra of symmetric functions in non-commutative variables, **WSym**, and the Hopf algebra of word quasi-symmetric functions, also known as the Hopf algebra of quasi-symmetric functions in non-commuting variables, or simply **WQSym**.

The Hopf algebra of word symmetric functions was introduced in [W$^+$36], see [RS06] for a description of its most important bases, along with several extensions of the theory of the ring of symmetric functions to **WSym**. This Hopf algebra is not to be confused with the Hopf algebra of non-commutative symmetric functions, the graded dual of $QSym$ introduced by Gelfand in [GKL$^+$94], nor with the Hopf algebra of partially commutative symmetric functions introduced by Lascoux and Schützenberger, see [LS81].

The Hopf algebra of word quasi-symmetric functions is an analogue of $QSym$ in non-commutative variables introduced in [NT06]. This Hopf algebra has a basis indexed by set compositions, which is in fact in bijection with the faces of the permutahedron, a well studied polytope in combinatorics. Unsurprisingly, this algebraic structure will reflect some geometric aspects, as it is seen in [Cha00].

Let us start by setting some notation. We write $[n]$ for the set $\{1, \dots, n\}$, and $[n, m] = \{n, \dots, m\}$ whenever $0 \leq n \leq m$ are integers. A *set partition* $\boldsymbol{\pi} = \{\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_k\}$ of a set I is a collection of non-empty disjoint subsets of I, called *blocks*, whose union is I, *i.e.*, that covers I. We write $\boldsymbol{\pi} \vdash I$. We denote the number of parts of the set partition by $l(\boldsymbol{\pi})$ and call it its length. We denote the family of set partitions of I by $\mathbf{P}_I$, or simply by $\mathbf{P}_n$ if $I = [n]$. By counting the elements on each block of $\boldsymbol{\pi}$, we obtain an integer partition denoted by $\lambda(\boldsymbol{\pi}) \vdash |I|$. We identify a set partition $\boldsymbol{\pi} \in \mathbf{P}_I$ with an equivalence relation $\sim_{\boldsymbol{\pi}}$ on I, where, for $x, y \in I$, we say that $x \sim_{\boldsymbol{\pi}} y$ if they are on the same block of $\boldsymbol{\pi}$.

A *set composition* $\vec{\boldsymbol{\pi}} = S_1|\dots|S_l$ of I is an ordered list of non-empty disjoint subsets of I, which we also call *blocks*, that cover I. We write $\vec{\boldsymbol{\pi}} \models I$. We denote the length of the set composition by $l(\vec{\boldsymbol{\pi}})$ and call it its length. We write $\mathbf{C}_I$ for the family of set compositions of I, or simply $\mathbf{C}_n$ if $I = [n]$. By disregarding the order of a set composition $\vec{\boldsymbol{\pi}}$, we obtain a set partition $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}) \vdash I$. By counting the elements on each block of $\vec{\boldsymbol{\pi}}$, we obtain a composition denoted by $\alpha(\vec{\boldsymbol{\pi}}) \models |I|$. A set composition is naturally identified with a total preorder $R_{\vec{\boldsymbol{\pi}}}$ on I, where $xR_{\vec{\boldsymbol{\pi}}}y$ if $x \in S_i, y \in S_j$ for $i \leq j$.

A *coloring* of the set I is a function $f : I \to \mathbb{N}$. The set composition type $\vec{\boldsymbol{\pi}}(f)$ of a coloring $f : I \to \mathbb{N}$ is the set composition obtained after deleting the empty sets in the list $f^{-1}(1)|f^{-1}(2)|\dots$. This notation is extended to functions $f : I \to \mathbb{R}$. We denote by $\mathfrak{C}_I$ the set of colorings of the set I.

The Hopf algebra of word quasi-symmetric functions is a graded Hopf algebra, and we write its homogeneous components as $\mathbf{WQSym} = \oplus_{n\geq 0}\mathbf{WQSym}_n$. Each homogeneous component has a basis $\{\mathbf{M}_\alpha\}_{\vec{\boldsymbol{\pi}}\models[n]}$, indexed by set compositions of $[n]$, of power series in the family of **non-commuting** variables $\{\mathbf{a}_n\}_{n\geq 1}$, as

$$\mathbf{M}_{\vec{\boldsymbol{\tau}}} = \sum_{\substack{f \text{ colorings s.t.} \\ \vec{\boldsymbol{\pi}}(f)=\vec{\boldsymbol{\tau}}}} \mathbf{a}_{f(1)} \dots \mathbf{a}_{f(n)} \, .$$

This defines a product structure on $\mathbf{WQSym}$. For instance, we have that

$$\mathbf{M}_{1|2}\mathbf{M}_{12} = \mathbf{M}_{1|2|34} + \mathbf{M}_{1|234} + \mathbf{M}_{1|34|2} + \mathbf{M}_{134|2} + \mathbf{M}_{34|1|2} \, .$$

The coproduct structure is defined as follows:

$$\Delta(\mathbf{M}_{\vec{\boldsymbol{\pi}}}) = \sum_{\vec{\boldsymbol{\pi}}=\vec{\boldsymbol{\tau}}\bullet\vec{\boldsymbol{\delta}}} \mathbf{M}_{\vec{\boldsymbol{\tau}}} \otimes \mathbf{M}_{\vec{\boldsymbol{\delta}}} \, ,$$

where $\bullet$ stands for the concatenation of set compositions.

The Hopf algebra $\mathbf{WSym}$ is a Hopf subalgebra of $\mathbf{WQSym}$, spanned by the elements

$$\mathbf{m}_{\boldsymbol{\tau}} = \sum_{\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}})=\boldsymbol{\tau}} \mathbf{M}_{\vec{\boldsymbol{\pi}}} \, .$$

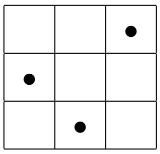
It is also a graded Hopf algebra and its homogeneous components have a basis indexed by set partitions $\{\mathbf{m}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\vdash[n]}$.

We write comu to denote the canonical maps $\mathbf{WQSym} \to QSym$ and $\mathbf{WSym} \to Sym$.

Permutation pattern Hopf algebra

Permutation patterns is an area of combinatorics that has been heavily studied. In [Wil02], the number of *occurrences of a permutation* τ in π is studied, and the concept of *pattern avoiding* permutation is described in [Knu68]. We introduce these concepts below. A survey about pattern counting and the combinatorics of patterns in permutations can be found in [Bón16]. In this way, we wish to endow a suitable vector space with an algebraic structure that resembles pattern counting. This structure, the permutation pattern Hopf algebra $\mathcal{A}(\texttt{Per})$, was introduced by Vargas in [Var14].

We represent a permutation σ of size n in its one-line notation, that is, as $\sigma(1)\dots\sigma(n)$, or as an $n \times n$ diagram, where the i-th column is marked in the box of height $\sigma(i)$. So, for instance, we have the following representations of the same permutation of size three:

213 =  . (1.4)

Definition 1.1.1 (Patterns in permutations)**.** If $w = w_1 \dots w_k$ is a word of distinct positive integers, we denote by $\operatorname{std}(w)$ the order preserving relabeling of these integers, so that we obtain a standard one-line notation of a permutation. If π, τ are two permutations of sizes k and n, respectively, we say that $i_1 < i_2 < \cdots < i_k$ is an occurrence of π in τ if $\pi = \operatorname{std}(\tau(i_1)\dots\tau(i_k))$. In this way, if $\tau = 1423$ and $\pi = 21$, then $\{2, 3\}$ and $\{2, 4\}$ are occurrences of π in τ. We say that a permutation π is a *pattern* of another permutation τ if there is an occurrence of π in τ.

We can now introduce the pattern Hopf algebra on permutations, $\mathcal{A}(\texttt{Per})$, brought forth in [Var14]. Consider $\mathcal{S}$ the set of all permutations, and for each permutation π let $\mathbf{p}_\pi : \mathcal{S} \to \mathbb{Q}$ be defined as

$$\mathbf{p}_\pi(\sigma) = |\{\text{occurrences of } \pi \text{ in } \sigma\}| \,.$$

We also define the *cover number* $\binom{\tau}{\pi_1, \pi_2}$ of the permutations π_1, π_2 and τ, as the number of pairs of occurrences in τ, one of π_1, another of π_2, whose union is precisely $[|\sigma|]$.

Specifically, if $\pi_1 = 132$, $\pi_2 = 23$ and $\tau = 1324$, then

$$\binom{1324}{132, 23} = |\{(\mathbf{132}4, 13\mathit{24}), (\mathbf{132}4, 1\mathit{3}2\mathit{4}), (\mathbf{132}4, \mathit{1}32\mathit{4})\}| = 3 \,.$$

Consider, as another example, $\sigma = 24135$. The occurrences of $\pi_1 = \mathbf{123}$ in σ are **24**1**35**, **2**41**35**, 24**135**. On the other hand, there is a unique occurrence of $\pi_2 = \mathit{132}$ in σ, namely *24*1*3*5.

Thus,

$$\binom{24135}{\mathbf{132},\, \mathit{123}} = |\{(24\mathbf{135}, \mathit{24}1\mathit{3}5)\}| = 1\,.$$

In a similar way,

$$\binom{4726135}{\mathbf{2413},\, \mathit{35142}} = |\{(\mathbf{47}26\mathbf{1}3\mathbf{5}, \mathit{4726}1\mathit{3}5), (47\mathbf{26135}, \mathit{4726}1\mathit{3}5)\}| = 2\,.$$

These functions span a vector subspace $\mathrm{span}\{\mathbf{p}_\pi \,|\pi \in \mathcal{S}\} \subseteq \mathcal{F}(\mathcal{S}, \mathbb{Q})$ of the space of functions from $\mathcal{S}$ to $\mathbb{Q}$. In Theorem 2.2.3, we see that this vector space is closed with respect to the pointwise multiplication. More precisely, we have for any permutations π_1, π_2, τ and σ that

$$\mathbf{p}_{\pi_1}(\tau)\, \mathbf{p}_{\pi_2}(\tau) = \sum_\sigma \binom{\sigma}{\pi_1, \pi_2} \mathbf{p}_\sigma(\tau)\,,$$

where the sum runs over all permutations σ.

This is a commutative algebra where the unit is indexed by the unique permutation of size zero, $\mathbf{p}_\emptyset \equiv 1$, *i.e.*, the constant function that is equal to one. It can be further endowed with a coproduct structure by means of the *direct sum of permutations*, introduced below in Example 3.1.1, and setting

$$\Delta(\mathbf{p}_\pi) = \sum_{\pi = \tau_1 \oplus \tau_2} \mathbf{p}_{\tau_1} \otimes \mathbf{p}_{\tau_2}\,,$$

where the sum runs over all the *direct sum* factorizations of π.

Observe, in particular, that if $|\tau| > |\pi_1| + |\pi_2|$, then $\binom{\tau}{\pi_1, \pi_2} = 0$. We can describe combinatorially the "top component": if $|\tau| = |\pi_1| + |\pi_2|$, the covers counted by the cover coefficient are the *bishuffles* of π_1 and π_2 that match τ. That is, the number of ways to partition σ into two occurrences $i_1 < \cdots < i_{|\pi_1|}$ and $j_1 < \cdots < j_{|\pi_2|}$ of π_1 and π_2, respectively. In this way, comparing the product structures between $\mathcal{A}(\texttt{Per})$ and the shuffle algebra (see [DK92] for more on this algebra), the product structure presented has more terms, in general. So, for instance, we have

$$\mathbf{p}_1\, \mathbf{p}_{21} = \mathbf{p}_{132} + \mathbf{p}_{312} + \mathbf{p}_{321} + \mathbf{p}_{213} + 2\,\mathbf{p}_{231} + \mathbf{p}_{312} + 2\,\mathbf{p}_{321} + 2\,\mathbf{p}_{21}\ .$$

In [Var14], it is shown that this is a free algebra and a free generator set is $\{\mathbf{p}_\pi \,|\pi \in \mathcal{L}\}$, where $\mathcal{L}$ is the set of the so-called *Lyndon permutations*. The combinatorics of Lyndon words is developed in [CFL58] and it has been ever since serviceable in establishing the freeness of algebras.

Incidence Hopf algebras on graphs and posets

The incidence Hopf algebras on graphs and posets have been studied ever since the introduction of Hopf algebras in combinatorics, with [JR79]. A great description of the Hopf algebra on graphs, usually called the *incidence Hopf algebra on graphs*, was presented in [HM12], where a very simple antipode formula was shown. This antipode formula is expressed in terms of acyclic orientations of a graph, explaining some *reciprocity formulas*. On posets, the general results presented by [Sch93] provide a determinantal formula for its antipode.

The Hopf algebras on graphs $\mathbf{G}$ and on posets $\mathbf{Pos}$ are graded and connected Hopf algebras that portray simple construction of Hopf algebras through combinatorial operations.

The homogeneous components of these Hopf algebras $\mathbf{G}_n$, respectively $\mathbf{Pos}_n$, are the linear span of the graphs with vertex set $[n]$, respectively partial orders in the set $[n]$.

In these graded vector spaces, one defines the products and coproducts in the basis elements. For that, when A, B are sets of integers with the same cardinality, we let $\mathrm{rel}_{A,B}$ be the canonical relabeling of combinatorial objects on A to combinatorial objects on B that preserves the order of the labels.

We introduce now the underlying combinatorial constructions in order to present a product and a coproduct on graphs and posets. The disjoint union of graphs $(V_1, E_1), (V_2, E_2)$, where $V_1 \cap V_2 = \emptyset$, is $(V_1 \sqcup V_2, E_1 \sqcup E_2)$. Given a graph $G = (V(G), E(G))$ and a set $I \subseteq V(G)$, the restriction of a graph $G|_I$ is $(I, E(G) \cap \binom{I}{2})$. Given two graphs G_1, G_2 with vertices labeled in $[n], [m]$ respectively, the product is the relabeled disjoint union

$$G_1 \cdot G_2 = G_1 \sqcup \mathrm{rel}_{[m],[n+1,n+m]}(G_2)\,.$$

For the coproduct, let G be a graph labeled in $[n]$, then

$$\Delta(G) = \sum_{[n]=I \sqcup J} \mathrm{rel}_{I,[|I|]}(G|_I) \otimes \mathrm{rel}_{J,[|J|]}(G|_J)\,.$$

To define a Hopf algebra on posets, consider two posets $P_1 = (S_1, R_{P_1}), P_2 = (S_2, R_{P_2})$, where R represents the set of pairs (x, y) such that $x \leq y$ in the respective poset. The disjoint union of posets is written $P_1 \sqcup P_2$ and defined as $(S_1 \sqcup S_2, R_{P_1} \sqcup R_{P_2})$, and the restriction of a poset $P = (S, R_P)$ is written $P|_I$ and defined as $(I, R_P \cap (I \times I))$. A subset S of I is said to be an ideal of P if, whenever $x \leq y$ and $x \in S$, then $y \in S$. Define the product between partial orders P, Q in the sets $[n], [m]$, respectively, as

$$P \cdot Q = P \sqcup \mathrm{rel}_{[m],[n+1,n+m]}(Q)\,,$$

and the coproduct for a partial order P in $[n]$ as

$$\Delta(P) = \sum_{S \text{ ideal of } P} \mathrm{rel}_{S,[|S|]}(P|_S) \otimes \mathrm{rel}_{S^c,[n-|S|]}(P|_{S^c})\,.$$

These operations define a Hopf algebra structure in **G** and **Pos**, as described in [GR14].

Hopf algebra of generalized permutahedra

Our last specific example of a Hopf algebra introduces the strong geometric flavor that this thesis has been alluding to. The family of generalized permutahedra is a family of polytopes that has been introduced recently, in [PRW08]. There, Postnikov presented combinatorial formulas for their volume and showed that some classical families of polytopes are also generalized permutahedra. Most notably, the graphical zonotope (see [Gru16]) and the matroid polytope (see [Grö04]) associated to each graph, respectively matroid, are generalized permutahedra. In [AA17], a Hopf algebra structure on generalized permutahedra was introduced and a cancellation-free formula was presented. Remarkably, this Hopf algebra structure is compatible with the graphical zonotopes and the matroid polytopes, therein resulting two Hopf algebra morphisms. Other combinatorial objects present the same feature (see [AA17] for a small collection of such objects). This naturally implies that we obtain cancellation-free formulas on these Hopf algebras as well.

We start with some notational remarks: For a set composition $\vec{\pi} = S_1|\dots|S_k$ on $[n]$, recall that $R_{\vec{\pi}}$ is a partial order on $[n]$. For a non-empty set $J \subseteq [n]$, define the set $J_{\vec{\pi}} = \{\text{minima of J in } R_{\vec{\pi}}\} = J \cap S_{i(J)}$, where $i(J)$ is the smallest index i with $J \cap S_i \neq \emptyset$. In convex combinatorics, we identify a coloring $f : [n] \to \mathbb{R}$ with the linear function $f : \mathbb{R}^{[n]} \to \mathbb{R}$.

$$x \mapsto \sum_{i=1}^{n} f(i) x_i\,.$$

We denote the convex hull of a set A, the set of all convex combinations of points of A (see Section 6.1.7) by conv A. In the space $\mathbb{R}^{[n]}$, we define the simplices $\mathfrak{s}_J = \mathrm{conv}\{e_v |\, v \in J\}$ for each $J \subseteq [n]$.

In convex geometry, we also define the Minkowski sum, difference and scalar multiplication. Given two convex sets $A, B \subseteq \mathbb{R}^I$, the Minkowski sum is

$$A + B = \{a + b \,|\, a \in A, b \in B\}\,,$$

and we define $A - B$ to be the convex set C such that $B + C = A$. It is known that if such set exists, it is unique. Finally, if $\lambda \in \mathbb{R}_{\geq 0}$, then we define λA as $\{\lambda a | a \in A\}$.

A *generalized permutahedron* is a Minkowski sum and difference of the form

$$\mathfrak{q} = \left(\sum_{\substack{J \neq \emptyset \\ a_J > 0}} a_J \mathfrak{s}_J \right) - \left(\sum_{\substack{J \neq \emptyset \\ a_J < 0}} |a_J| \mathfrak{s}_J \right)\,,$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$ that can be either positive, negative or zero.

A *hypergraphic polytope* is a generalized permutahedron of the form

$$\mathfrak{q} = \sum_{J \neq \emptyset} a_J \mathfrak{s}_J\,,$$

for non-negative reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$.

For a polytope $\mathfrak{q}$ and a real coloring f on $[n]$, we denote by $\mathfrak{q}_f$ the subset of $\mathfrak{q}$ on which f is minimized, that is,

$$\mathfrak{q}_f := \arg\min_{x \in \mathfrak{q}} \sum_{i \in I} f(i) x_i\,.$$

A face of $\mathfrak{q}$ is a subset of $\mathfrak{q}$ of the form $\mathfrak{q}_f$ for some f. A real coloring is said to be $\mathfrak{q}$-generic if the corresponding face is a point.

Example 1.1.2. Consider the hypergraphic polytope $\mathfrak{q} = \mathfrak{s}_{\{1,2,3\}} + \mathfrak{s}_{\{1,2\}}$ in $\mathbb{R}^3$. If we take the coloring of $\{1, 2, 3\}$ given by $f(1) = f(2) = 1$ and $f(3) = 3$, then $\mathfrak{q}_f = 2\mathfrak{s}_{\{1,2\}}$. If we consider the coloring $g(1) = g(3) = 2$ and $g(2) = 1$, then $\mathfrak{q}_g = 2\mathfrak{s}_{\{2\}}$ is a point, so g is $\mathfrak{q}$-generic.

For a generalized permutahedron $\mathfrak{q}$ in $\mathbb{R}^I$ and $A \subseteq I$, let $B = I \setminus A$ pick $f \in \mathbb{R}^I$ given by $f(x) = 1$ if $x \in A$, and $f(x) = 0$ otherwise. It was shown in [AA17] that the corresponding face decomposes in the following way:

$$\mathfrak{q}_f =: \mathfrak{q}|_A \times \mathfrak{q}\backslash_A\,,$$

where $\mathfrak{q}|_A$ is a generalized permutahedron in $\mathbb{R}^A$ and $\mathfrak{q}\backslash_A$ is a generalized permutahedron on $\mathbb{R}^B$. Note that $B = A^c$, so the dependence of $\mathfrak{q}|_A$ and $\mathfrak{q}\backslash_A$ on B is implicit. In fact, in the proof of Proposition 4.2.10, we obtain explicit expressions for $\mathfrak{q}|_A$ and $\mathfrak{q}\backslash_A$.

Finally, recall that we use $\mathrm{rel}_{I,J}$ to denote the relabeling of combinatorial objects via the order preserving map between I, J. In this way, if $\mathfrak{q} \subseteq \mathbb{R}^I$, we have that $\mathrm{rel}_{I,J}(\mathfrak{q}) \subseteq \mathbb{R}^J$.

We have now all the material to endow the space of generalized permutahedra with a Hopf algebra structure as done in [AA17]: let $\mathbf{GP} = \oplus_{n\geq 0}\mathbf{GP}_n$, where $\mathbf{GP}_n$ is the free linear space on generalized permutahedra in $\mathbb{R}^{[n]}$. Notice that each $\mathbf{GP}_n$ is infinite dimensional, for $n \geq 1$.

The **GP** linear space has the following product, when $\mathfrak{q}_1, \mathfrak{q}_2$ are generalized permutahedra in $\mathbb{R}^n, \mathbb{R}^m$ respectively:

$$\mathfrak{q}_1 \cdot \mathfrak{q}_2 = \mathfrak{q}_1 \times \mathrm{rel}_{[m],[n+1,n+m]}(\mathfrak{q}_2)\,,$$

where we note that $\mathrm{rel}_{[m],[n+1,n+m]}(\mathfrak{q}_2)$ is a polytope in $\mathbb{R}^{[n+1,n+m]}$.

The **GP** linear space has the following coproduct, when $\mathfrak{q}$ is a generalized permutahedron in $\mathbb{R}^n$:

$$\Delta\mathfrak{q} = \sum_{A\subseteq[n]} \mathrm{rel}_{A,[|A|]}(\mathfrak{q}|_A) \otimes \mathrm{rel}_{A^c,[n-|A|]}(\mathfrak{q}\backslash_A)\,.$$

Remark 1.1.3. The span of the hypergraphic polytopes also forms a Hopf algebra, that we denote by **HGP**. This is a Hopf subalgebra of **GP**.

Cohomology ring of a Lie group

The remaining of the thesis does not depend on this section. It only illustrates a rich source of Hopf algebras in topology, yet another motivation to study Hopf algebras. This was in fact the first motivation to study Hopf algebras, in the 1950s, see [Jam99, Chapter 26].

A Lie group is a group X that also has a smooth manifold structure, and the group structure maps are smooth. This means that not only we can do group theory, but also analysis and topology. In particular, we can talk about the cohomology ring $H^*(\mathbb{K}, X)$ of a Lie group X, that is induced from its topology, see [Hat05, Chapter 3.1], where the product corresponds to the *cup product*. In [Hat05, Chapter 3.C], the cohomology ring of a Lie group is further endowed with a Hopf algebra structure, via the dual of the group product. The inverse map gives us an antipode, making the cohomology ring $H^*(\mathbb{K}, X)$ a Hopf algebra. This entails a topological group invariant that we can associate to topological spaces.

Central problems in Hopf algebras

On describing the structure of the operations of a Hopf algebra, in algebraic combinatorics, there are three objectives that we will focus on this thesis: to study the unique Hopf algebra morphism to $QSym$ (see [ABS06, BJR09, Gru16] for instance), to find a simple formula for the antipode map (see [BS17, HM12, AA17] for instance), and to establish a structure theorem on the product or coproduct maps, usually establishing that it is free or cofree (see [Haz01, BZ09, Var14] for instance). We will deal with each of these questions in the following sections, and point to the relevant results in this thesis.

1.2 Species and Hopf monoids

Species were introduced in [Joy81] as a tool to transform generating function equalities into bijective proofs in an automatic way. A bijective proof is deemed a more desirable type of proof in enumerative combinatorics, because it usually provides finer equalities, by keeping track of suitable statistics.

The benefits of species, however, go further than obtaining bijective proofs: this treatment allows us to construct Hopf algebras easily, via the *Fock functors*. In this manner we introduce *combinatorial species*.

Combinatorial species

Let $\mathtt{Set}$ be the category of finite sets and maps between finite sets. Define as well $\mathtt{Set}^{\times}$ to be the category of finite sets and bijections between finite sets, and let $\mathtt{Vec}_{\mathbb{K}}$ be the category of vector spaces over $\mathbb{K}$ and linear maps.

A **set species** is a functor from $\mathtt{Set}^{\times}$ to $\mathtt{Set}$. Hence, a species h is described by an assignment of each finite set I to a finite set $h[I]$, together with some relabeling map for each bijection $f : I \to J$, which we usually refer to $\mathrm{rel}_{I,J}$, disregarding the dependence on f. A *vector species* is a functor $h : \mathtt{Set}^{\times} \to \mathtt{Vec}_{\mathbb{K}}$. We will refer to vector species as simply species. The categories of species and set species are what is called a *monoidal category*.

In category theory, the abstract notion of an algebra arises with the study of monoidal categories. A monoidal category is a pair $(\mathcal{C}, \bullet)$ where $\mathcal{C}$ is a category, and $\bullet$ is a bifunctor $\bullet : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that satisfies some specific associativity properties. In Appendix B, the reader can find basic definitions in monoidal categories.

Monoidal categories were defined by Mac Lane in [Mac63]. Mac Lane shows that specifying a *monoidal category* structure is enough to be able to define *monoids*, *comonoids* and *Hopf monoids*, that are generalizations of the usual concepts of algebras, coalgebras and Hopf algebras.

The Fock functors

In this section we will explore constructions on combinatorial species, called the **Fock functors**, $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}^\vee$, $\overline{\mathcal{K}}^\vee$. These functors build a clear bridge between Hopf monoids in the monoidal category of vector species and graded Hopf algebras. In particular, this provides a fairly simple way to build Hopf algebras from the aforementioned "merge and split" operations that are common in combinatorics. Therefore, it is not surprising that most of the Hopf algebras in combinatorics result from applying one of the Fock functors to a suitable species.

We denote the set of permutations of size n by $\mathcal{S}_n$. The Fock functors $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}^\vee$, and $\overline{\mathcal{K}}^\vee$ were introduced in [Sto93]. These are functors from the category of species to the category of graded vector spaces. Fix a species $\mathbf{q}$ and define

$$\mathcal{K}^\vee(\mathbf{q}) = \mathcal{K}(\mathbf{q}) = \bigoplus_{n\geq 0} \mathbf{q}[n]\,,$$

$$\overline{\mathcal{K}}^\vee(\mathbf{q}) = \bigoplus_{n\geq 0} \mathbf{q}[n]^{\mathcal{S}_n}\,,$$

$$\overline{\mathcal{K}}(\mathbf{q}) = \bigoplus_{n\geq 0} \mathbf{q}[n]_{\mathcal{S}_n}\,,$$

where $\mathbf{q}[n]^{\mathcal{S}_n}$ is the subspace of $\mathbf{q}[n]$ that is invariant under the action of $\mathcal{S}_n$, and $\mathbf{q}[n]_{\mathcal{S}_n}$ is the quotient of $\mathbf{q}[n]$ through the action of $\mathcal{S}_n$.

If $\mathbf{q}$ is a monoid or a comonoid, then we can endow the vector spaces $\mathcal{K}^\vee(\mathbf{q}), \mathcal{K}(\mathbf{q}), \overline{\mathcal{K}}^\vee(\mathbf{q})$ and $\overline{\mathcal{K}}(\mathbf{q})$ with an algebra, respectively coalgebra structure.

To illustrate this concept, we detail it for the functor $\mathcal{K}$. Assume that the species $\mathbf{q}$ is equipped with a structure of Hopf monoid: $(\mathbf{q}, \mu, \iota, \Delta, \varepsilon, S)$. Then we can define a product of two elements $a \in \mathbf{q}([n]), b \in \mathbf{q}([m])$ by using the structure maps as follows:

$$a \cdot b = \mu_{[n],[n+1,n+m]}(a, \mathrm{rel}_{[m],[n+1,n+m]}(b))\,,$$

and a coproduct

$$\Delta(a) = \sum_{A\sqcup B=[n]} \mathrm{rel}_{A,[|A|]} \otimes \mathrm{rel}_{B,[|B|]} \circ \Delta_{A,B}(a)\,.$$

The fact that these operations, together with the correct unit and counit, form a Hopf algebra is encoded in the next theorem:

Theorem 1.2.1 ([AM10, Theorem 15.12.])**.** If $\mathbf{q}$ is a Hopf species, then each of the constructions above gives a graded Hopf algebra.

Example 1.2.2. We consider the species of graphs $\mathtt{Gr}$. This associates to a set I, the vector space $\mathtt{Gr}[I]$ spanned by graphs with vertex set I. Furthermore, for each bijection $f : I \to J$, we also consider the natural relabeling maps $\mathrm{rel}_{I,J} : \mathtt{Gr}[I] \to \mathtt{Gr}[J]$ that map a graph in the vertex set I to the corresponding graph in the vertex set J via the relabeling f.

This species can be endowed with a Hopf monoid structure in a very similar way as done in Section 1.1. In this way, the Hopf algebra $\mathcal{K}(\mathtt{Gr})$ is precisely the Hopf algebra on graphs $\mathbf{G}$ introduced above.

Other Hopf algebras arise as an application of Fock functors. The Hopf algebras on posets and generalized permutahedra are prime examples, as are the Hopf algebras in symmetric functions, in quasi-symmetric functions, in word symmetric functions, and in word quasi-symmetric functions. Details on how to construct such Hopf algebras via the Fock functors can be found in [AM10].

1.3 Chromatic invariants as characters of Hopf algebras

In this section we describe a method to obtain invariants in combinatorial structures from Hopf algebras. This method was introduced in [ABS06] and can be used to obtain, for instance, the chromatic symmetric function of graphs, introduced by Stanley in [Sta95]. We call these invariants *chromatic invariants*. Particular specializations of these chromatic invariants have been studied in several places of combinatorics, including the chromatic polynomial of graphs, the Billera–Jia–Reiner polynomial in matroids and others.

To present the construction, we first clarify the algebraic structure on our combinatorial objects. A *combinatorial Hopf algebra* is a pair (H, η), where $H = \bigoplus_{n\geq 0} H_n$ is a graded connected Hopf algebra and $\eta : H \to \mathbb{K}$ is a character, *i.e.*, a multiplicative linear functional. A morphism of combinatorial Hopf algebras $\phi : (H, \eta) \to (H', \eta')$ is a graded Hopf algebra morphism that satisfies $\eta = \eta' \circ \phi$. In [ABS06], it was shown that the *terminal object* in the category of combinatorial Hopf algebras is $QSym$, the graded Hopf algebra of quasi-symmetric functions endowed with the character $\eta_0 : QSym \to \mathbb{K}$ defined as $\eta_0(M_\alpha) = 1$ if α is a composition with at most one part and $\eta_0(M_\alpha) = 0$ otherwise. Concretely, this means the following:

Theorem 1.3.1 (Terminal object in combinatorial Hopf algebras - [ABS06, Theorem 4.1.]). If (H, η) is a combinatorial Hopf algebra, then there is a unique combinatorial Hopf algebra morphism $\Psi_H : (H, \eta) \to (QSym, \eta_0)$.

Specifically, if $H = \bigoplus_{n \geq 0} H_n$ is the grading on the Hopf algebra H, the map Ψ_H can be constructed at each H_n as follows: For each $\alpha \models n$, define η_α by the following composition of functions:

$$H \xrightarrow{\Delta^{k-1}} H^{\otimes k} \to H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_k} \xrightarrow{\eta^{\otimes k}} \mathbb{K} .$$

In this way, for $x \in H_n$, we define $\Psi_H(x) = \sum_{\alpha \models n} \eta_\alpha(x) M_\alpha$.

For several combinatorial Hopf algebras, this recovers a studied invariant. For example, consider the following character on $\mathbf{G}$, the graph Hopf algebra:

$$\eta(G) = \mathbb{1}[G \text{ has no edges}] .$$

Then, $\eta_\alpha(G)$ counts how many set compositions $\vec{\pi}$ of $V(G)$ such that $\alpha(\vec{\pi}) = \alpha$ and that each block of $\vec{\pi}$ has no edges in G. In this way, we obtain precisely the classic chromatic symmetric function:

$$\Psi_{\mathbf{G}}(G) = \sum_{\substack{\vec{\pi} \models V(G) \\ \vec{\pi} \text{ stable partition}}} M_{\alpha(\vec{\pi})} = \sum_{\substack{f \text{ stable coloring} \\ \text{of } G}} x_{f(1)} \cdots x_{f(n)} ,$$

where a stable coloring of a graph $f : V(G) \to \mathbb{Z}_{\geq 0}$ is a coloring such that $f(v) \neq f(w)$ whenever v, w are connected by an edge in G.

A classic problem pertaining the chromatic symmetric function $\Psi_{\mathbf{G}}$ is the *tree conjecture*. It claims that this graph invariant distinguishes trees. A complete invariant for general graphs is an invariant F such that any two non-isomorphic graphs G_1, G_2 satisfy $F(G_1) \neq F(G_2)$. The fact that the chromatic symmetric function is not a complete invariant can be readily observed: in [Sta95], Stanley identified two non-isomorphic graphs with the same chromatic symmetric function. Some efforts have been done in establishing that the tree conjecture is true. For instance, in [APZ14], it is shown that all proper caterpillars have distinct chromatic symmetric functions. This is done by suitably manipulating the combinatorial interpretation of the coefficients of the chromatic symmetric function into the power-sum basis of Sym.

The map $\Psi_{\mathbf{G}}$ satisfies linear relations, called *modular relations on graphs*, established independently in [GP13] and [OS14]. In Theorem 4.1.1, we show that these are essentially the only linear relations that these maps satisfy. This is done by describing generators for the kernel of $\Psi_{\mathbf{G}}$. A similar problem has been already debated in the context of posets, specifically for the graded Hopf algebra **Pos** and its corresponding chromatic function $\Psi_{\mathbf{Pos}}$, in [Fér15]. This is called the *kernel problem*.

An application for this problem is to find new invariants that are as strong as the chromatic symmetric function. For instance, in Section 4.3.1 we describe a new graph invariant $\tilde{\Psi}$ that seems stronger than the chromatic symmetric function. That is, for two graphs G_1, G_2 such that $\tilde{\Psi}(G_1) = \tilde{\Psi}(G_2)$, we can easily see that we also have $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2)$. However, by showing that $\tilde{\Psi}$ also satisfies the modular relations, we establish that it is as strong as the chromatic symmetric function. In particular, the tree conjecture in $\tilde{\Psi}$ is equivalent to the tree conjecture in $\Psi_{\mathbf{G}}$. This has synergy with the classic strategies to tackle the tree conjecture, where we use other graph invariants to establish the tree conjecture for particular classes of trees, instead of using directly $\Psi_{\mathbf{G}}$, see [APZ14] for instance.

The role played by generalized permutahedra in this is ubiquitous. We have observed that many Hopf algebras H in combinatorics can be embedded in **GP** via an injective Hopf algebra morphism $Z : H \to \mathbf{GP}$. Because of the uniqueness in Theorem 1.3.1, we can in fact observe that $\Psi_{\mathbf{GP}} \circ Z = \Psi_H$. Thus, from the kernel of $\Psi_{\mathbf{GP}}$ we can extract satisfactory intuition about the kernel of Ψ_H. This motivates us to try to find generators of the vector space $\ker \Psi_{\mathbf{GP}}$. A partial result is presented in Theorem 4.1.4. There, we address the kernel problem on the Hopf algebra **HGP**, a Hopf subalgebra of **GP** that embeds a significant amount of relevant Hopf algebras, for instance the Hopf algebra on graphs introduced above or the Hopf algebra on matroids introduced in [GR14].

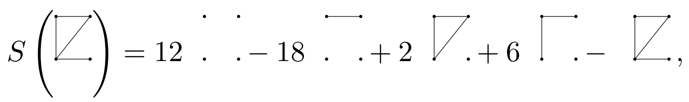
In [GS01], a generalization of $\Psi_{\mathbf{G}}$ was presented in the Hopf algebra of symmetric functions in non-commutative variables. There it is shown that this chromatic invariant satisfies a *deletion-contraction property*, like the chromatic polynomial χ_G does, but unlike $\Psi_{\mathbf{G}}$. This allows some progress in relevant problems on chromatic symmetric functions, for instance in the (3+1)-free conjecture; see [SS93, Sta95] for more details on that conjecture. This construction is in fact a general one: the chromatic invariants can be generalized to an invariant in non-commutative variables. Motivated by this, in Chapter 5, we describe the notion of combinatorial Hopf species, which is the parallel notion of combinatorial Hopf algebra in the context of species. We present the Hopf species of word quasi-symmetric functions, denoted by `WQSym`. Finally, in Theorem 5.4.1, we show the following:

Theorem 1.3.2 (Terminal object in combinatorial Hopf monoids)**.** Given a combinatorial Hopf monoid $\mathtt{h}$ with a character $\eta : \mathtt{h} \Rightarrow \mathtt{Exp}$, there is a unique combinatorial Hopf monoid morphism $\boldsymbol{\Upsilon}_{\mathtt{h}} : \mathtt{h} \Rightarrow \mathtt{WQSym}$.

1.4 Antipode formulas: reciprocity results, power series and the character group

There is a rather general antipode formula given in Theorem A.2.2, that dates back to [Tak71]. However, this formula is neither **cancellation-free** nor **grouping-free**. In algebraic combinatorics, interest has been shown to obtain such antipode formulas, see [HM12, ML14, MP15, Pat15, BS17, AA17].

In (1.5), we give some examples of antipodes on particular objects in the incidence Hopf algebra on graphs and in the permutation pattern Hopf algebra.



$$S(\mathbf{p}_{132}) = 3\,\mathbf{p}_{321} + \mathbf{p}_{312} + \mathbf{p}_{213} + 2\,\mathbf{p}_{231} + 2\,\mathbf{p}_{21}\ . \tag{1.5}$$

Describing the antipode of a Hopf algebra may have applications outside algebraic combinatorics. For instance, in [AA17], the relation between the antipode of the Faá di Bruno Hopf algebra and an inversion formula for the composition of generating functions is described.

In combinatorics, it has been observed that some formulas come in pairs, usually under an umbrella term "reciprocity results" that dates back to [Sta75]. For instance, two distinct ways of describing the chromatic polynomial $p_G(x)$ are through stable colorings of the given graph G and through its acyclic orientations. These descriptions of the chromatic polynomial $p_G(x)$ are related by the transformation $x \mapsto -x$. The antipode plays a crucial role in explaining this relation: in the case of the graph Hopf algebra, a cancellation-free formula for the antipode is described precisely by counting acyclic orientations. In general, having simple formulas for the antipode of a Hopf algebra presents itself as a direct method of obtaining such reciprocity results.

A notable recent result in cancelation-free and grouping-free antipode formulas is set in [AA17], where an antipode formula is described for the Hopf algebra of generalized permutahedra, **GP**. This helps explaining old antipode formulas and finding new ones in several combinatorial Hopf algebras, because many combinatorial Hopf algebras can be embedded in **GP**. For instance, the graphical zonotope gives precisely such an embedding from the incidence Hopf algebra on graphs to **GP**, and the same can be done for matroids, set partitions, paths, and many others.

In this thesis we discuss open problems on cancellation-free and grouping-free formulas for the antipode of the pattern Hopf algebras, in Section 7.3. Specifically, we debate a preliminary result of the author, where an antipode formula of the pattern Hopf algebra in permutations is found, and possible avenues for future projects starting from there.

1.5 Pattern algebras

The permutation pattern Hopf algebra is the starting point of Chapter 2, where we find a general construction of pattern algebras. We understand that the ingredients that make $\mathcal{A}(\texttt{Per})$ a Hopf algebra are present not only in permutations, but in many combinatorial objects such as matroids, graphs, marked permutations and set partitions. All these examples fit into the framework of *combinatorial presheaves*, introduced first in [Sch93] as *species with restrictions*. These are in essence species with a notion of patterns, and this is how we start to construct a Hopf algebra on combinatorial objects that allow for the notion of pattern:

Definition 1.5.1 (Combinatorial presheaves)**.** Let $\texttt{Set}_{\hookrightarrow}$ be the category of finite sets and injective maps between finite sets. Recall that $\texttt{Set}$ is the usual category on finite sets. A *combinatorial presheaf*, or a *presheaf* for short (also called *species of finite sets with restrictions*), is a contravariant functor from $\texttt{Set}_{\hookrightarrow}$ to $\texttt{Set}$. A morphism of combinatorial presheaves is a natural transformation of functors.

In this way, we have the category $\texttt{CPSh}$ of combinatorial presheaves. For a presheaf h and a set I, we refer to the elements of $h[I]$ as the h-objects on I, or simply objects on I, when h is clear from the context. Two objects $a \in h[I], b \in h[J]$ are said to be *isomorphic* if there is a bijection $f : J \to I$ such that $h[f](a) = b$. In this case, we write $a \sim b$.

Given a combinatorial presheaf h and two finite sets I, J such that $J \subseteq I$, the inclusion map $\text{inc}_{J,I} : J \to I$ play an important role in describing the presheaf h. Whenever the combinatorial presheaf is clear, we write $h[\text{inc}_{J,I}] = \mathbf{res}_{J,I}$, or simply $\mathbf{res}_J$.

Example 1.5.2 (The presheaf on permutations)**.** To fit the framework of presheaves and define a combinatorial presheaf on permutations $\texttt{Per}$, we use a rather unusual definition of permutations introduced in [ABF18]. There, a permutation on a set I is seen as a pair of total orders in I. Two permutations π_1, π_2 in the sets I_1, I_2 respectively are said to be *isomorphic* if there is a bijection $f : I_1 \to I_2$ that maps both orders of π_1 to the respective orders of π_2.

This notion of permutation relates to the usual definition as bijection in the following way: if we are given a permutation $\pi = (\leq_P, \leq_V)$ on the set I, we order the elements of $I = \{a_1 \leq_P \cdots \leq_P a_k\} = \{b_1 \leq_V \cdots \leq_V b_k\}$ therein defining a bijection $\tilde{\pi} : I \to I$ via $a_i \mapsto b_i$. Conversely, for any bijection $\tilde{\pi}$ on I, there are several pairs of orders $(\leq_P, \leq_V)$ that correspond to the bijection $\tilde{\pi}$, all of which are isomorphic. We consider the restriction map $\texttt{Per}[\mathrm{inc}_{J,I}] = \mathbf{res}_{J,I}$ from permutations on I to permutations on J by simply restricting both total orders. This maps a permutation $\tau \in \texttt{Per}[I]$ to the pattern $\pi \in \texttt{Per}[J]$ corresponding to the occurrence J (see Definition 1.1.1). This results in a presheaf structure Per. We write $|\sigma| = n$ whenever $\sim \in \texttt{Per}[I]$ such that $|I| = n$.

To simplify notation we refer to the Per-objects in $[n]$ simply by $\texttt{Per}[n]$. We will use the same convention for all the other combinatorial presheaves and species in this thesis.

As usual we represent permutations in I as square diagrams. In this case we label the entries of the diagram by elements of I. This is done in the following way: we place the elements of I in an $|I| \times |I|$ grid so that the elements are placed horizontally according to the $\leq_P$ order, and vertically according to the $\leq_V$ order. For instance, the permutation $\pi = \{1 <_P 2 <_P 3, 2 <_V 1 <_V 3\}$ in $\{1, 2, 3\}$ can be represented as

$$\begin{array}{|c|c|c|} \hline & & 3 \\ \hline 1 & & \\ \hline & 2 & \\ \hline \end{array} \; . \tag{1.6}$$

In this way, there are $(n!)^2$ many elements in $\texttt{Per}[n]$. Up to isomorphism, we can represent a permutation as a diagram with one dot in each column and row.

Given a presheaf h, we let $\mathcal{G}(h) = \sqcup_{n \geq 0} h[n]/_{\sim}$, where $h[n]/_{\sim}$ is the collection of h-objects on $[n]$ up to isomorphism. This is called the *collection of coinvariants.*

Definition 1.5.3 (Patterns in presheaves)**.** Let h be a presheaf, and consider two objects $a \in h[I], b \in h[J]$. We define the *pattern function*

$$\mathbf{p}_a(b) := \left| \{J' \subseteq J \text{ s.t. } \mathbf{res}_{J',J}(b) \sim a\} \right| .$$

A crucial observation is that this notion recovers the usual concept of permutation pattern counting already present in the literature, described in [Wil02].

In Proposition 2.2.1, we observe that this definition only depends on the isomorphism classes of a and b. Hence, we can consider $\{\mathbf{p}_a\}_{a \in \mathcal{G}(h)}$ as a family of functions from $\mathcal{G}(h)$ to $\mathbb{Q}$, indexed by $\mathcal{G}(h)$. We remark that the choice to work with functions over $\mathbb{Q}$ is not important, as we can also choose to work with functions over $\mathbb{Z}$, for instance.

Denote the family of functions $A \to B$ by $\mathcal{F}(A, B)$. If h is a combinatorial presheaf, then the linear span of the pattern functions is a linear subspace $\mathcal{A}(h) \subseteq \mathcal{F}(\mathcal{G}(h), \mathbb{Q})$ of the space of functions on $\mathcal{G}(h)$ taking rational values.

Theorem 1.5.4. The vector space $\mathcal{A}(h)$ is closed under pointwise multiplication and has a unit. It forms an algebra, called the *pattern algebra*. More precisely, we have the product rule

$$\mathbf{p}_a \, \mathbf{p}_b = \sum_c \binom{c}{a, b} \mathbf{p}_c \,, \tag{1.7}$$

where the sum runs over elements of $\mathcal{G}(h)$, and we define the coefficients $\binom{c}{a,b}$ below in (2.4) as the number of "quasi-shuffles" of a, b that result in c.

Quasi-shuffles of objects have been studied in several contexts as a notion of merging objects together. For details on quasi-shuffles of combinatorial objects the interested reader can see [Hof00, AM10, FPT16].

We now address the coproduct of the pattern algebra. For that, we will exploit the duality in the world of functions, and instead define a **product** on our combinatorial objects. This allows us to define a coproduct by essentially requiring it to satisfy the following equation, see (2.7):

$$\mathbf{p}_a(b \cdot c) = \Delta \, \mathbf{p}_a(b \otimes c) \,.$$

For two presheaves f, g, define the presheaf $f \odot g$ by $f \odot g[I] = \bigcup_{A \sqcup B = I} f[A] \times g[B]$. This is called the **Cauchy product** on combinatorial presheaves, and makes $\texttt{CPSh}$ a *monoidal category*. In particular, we can talk about monoids in this category, or simply an *associative presheaf*. An *associative presheaf* is a presheaf h together with a map $* : h \odot h \to h$ and a unit. More details on the axioms and properties of $*$ and 1 are given in Observation 2.1.5.

Examples of associative operations on combinatorial presheaves are the disjoint union of graphs and the direct sum of permutations. Another less standard example which we study in Chapter 3, is the inflation of marked permutations.

If $(h, *, 1)$ is an associative presheaf, the associative product $*$ in our combinatorial objects is a natural transformation. Thus this induces a product on $\mathcal{G}(h)$, which we denote by $\cdot$ (see Definition 2.2.6 for the details of this construction). More importantly, it allows us to introduce a coproduct in the pattern algebra $\mathcal{A}(h)$:

$$\Delta \, \mathbf{p}_a = \sum_{\substack{b, c \in \mathcal{G}(h) \\ a = b \cdot c}} \mathbf{p}_b \otimes \mathbf{p}_c \,. \tag{1.8}$$

Theorem 1.5.5. If $(h, *, 1)$ is an associative presheaf such that $|h[\emptyset]| = 1$, then the pattern algebra of h together with this coproduct, and a natural choice of counit, forms a Hopf algebra.

Some known Hopf algebras can be constructed as the pattern algebra of a combinatorial presheaf. An example is *Sym*, the Hopf algebra of *symmetric functions*. This Hopf algebra has a basis indexed by partitions, and corresponds to the pattern Hopf algebra of the presheaf on set partitions (see details in Section 2.3.4). The pattern Hopf algebra corresponding to the presheaf on permutations described above is precisely the permutation pattern Hopf algebra already introduced.

Some other Hopf algebras constructed here, like the ones on graphs and on marked permutations below, are new to the knowledge of the author. However, we conjecture that the Hopf algebra of quasi-symmetric functions arises as a pattern algebra, see Conjecture 2.3.15.

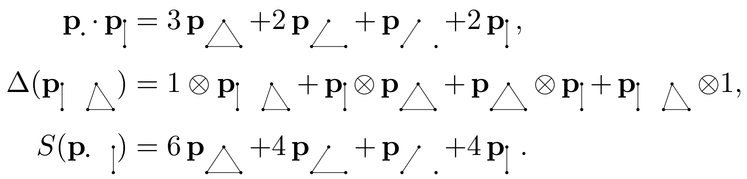
In Chapter 2, we also describe the primitive space of any pattern Hopf algebra, and prove that some properties that hold in $\mathcal{A}(\texttt{Per})$, established in [Var14], are also true in general.

1.6 The freeness of pattern algebras

Having observed such a general construction of Hopf algebras via patterns, in this section we delve into the algebraic properties of some examples of pattern algebras. We start with an example of an associative structure on graphs.

Example 1.6.1 (Associative presheaf on graphs)**.** For each set I, let $\texttt{Gr}[I]$ be the family of graphs with vertex set I. Recall that given a graph $G = (I, E(G))$ and a subset $J \subseteq I$ of vertices, we can define the restriction $\mathbf{res}_{J,I}(G) = G|_J$. This defines a presheaf structure on graphs.

Furthermore, we can endow $\texttt{Gr}$ with a structure of associative presheaf via the disjoint union of graphs $\sqcup : \texttt{Gr} \odot \texttt{Gr} \to \texttt{Gr}$. In this way, we have a pattern Hopf algebra $\mathcal{A}(\texttt{Gr})$. Some simple calculations are given in (1.9).



(1.9)

This example has a quite general property: the underlying product (disjoint union) on the presheaf is commutative. An associative presheaf $(h, \cdot, 1)$ is said to be *commutative* if the product $\cdot$ is commutative. In particular, observe that the graph presheaf presented above is commutative. In Chapter 2, we establish the main result for pattern Hopf algebras of commutative presheaves:

Theorem 1.6.2. Let $(h, *, 1)$ be a commutative presheaf. Then, the pattern Hopf algebra $\mathcal{A}(h)$ is free. Furthermore, the free generators are precisely $\{\mathbf{p}_a\}_{a \in \mathcal{I}(h)}$, where $\mathcal{I}(h)$ is the set of irreducible elements of $\mathcal{G}(h)$ under the product $\cdot$ defined in Definition 2.2.6.

With this, the graph pattern Hopf algebra $\mathcal{A}(\texttt{Gr})$ is free, and the pattern functions of connected graphs are the free generators, that is:

$$\mathcal{A}(\texttt{Gr}) = \mathbb{Q}[\mathbf{p}_G \,|\, G \text{ connected graph}] \,.$$

This was already proved in [Whi32, Theorem 3], though in a largely different language.

This theorem in particular totally describes the *coradical filtration* of a Hopf algebra, via Corollary 2.3.5. The coradical filtration is, roughly speaking, the “largest filtration that suits the coalgebra structure”, see Definition A.4.2 for a concrete definition.

Remarkably, the algebraic structure of a pattern Hopf algebra does not depend on the associative structure that it is endowed with - this only affects the coproduct. Thus, any presheaf that admits a commutative product has a free pattern algebra.

In this way, we obtain some other examples of free Hopf algebras, for instance the pattern Hopf algebra of marked graphs under the inflation product, defined in Definition 2.3.10.

On the presheaf of permutations, it is a result from [Var14] that the algebra $\mathcal{A}(\texttt{Per})$ is free. Observe that this is an example of an associative presheaf that is not commutative. In Chapter 3, we extend the methods used on that paper and show that $\mathcal{A}(\texttt{MPer})$ is also free, and similarly construct a set of generators.

Theorem 1.6.3. The pattern Hopf algebra on marked permutations $\mathcal{A}(\texttt{MPer})$ is free.

We conjecture that any pattern Hopf algebra arising from an associative presheaf is free.

Conjecture 1.6.4 (Freeness conjecture)**.** Let $(h, *, 1)$ be an associative presheaf. Then the pattern algebra $\mathcal{A}(h)$ is free.

1.7 The feasible region of patterns and the dimension conjecture

In this section we introduce the concept of feasible region. This is a geometrical object that explores the interplay between the proportion of occurrences of different patterns. This has been studied in the community of extremal combinatorics, by generalizing questions like "what is the maximum number of triangles in a graph with a limited number of edges?". In [BCL+08, Lov12, HKM+13, Bor19] this has been studied by introducing notions of a *limiting object* in graphs and permutations, called *graphons* and *permutons*. This transforms a combinatorial problem into an analytic one. More detail is given in Section 1.7.1.

We denote by $\mathcal{S}_n$ the set of all permutations of size n, and by $\mathcal{S} = \biguplus_{n\geq 0}\mathcal{S}_n$ the set of all permutations of finite size. For two permutations σ and π, we the proportion of occurrences of π as a pattern of σ as $\widetilde{\text{occ}}(\pi,\sigma) = \frac{\mathbf{p}_\pi(\sigma)}{\binom{|\sigma|}{|\pi|}}$. We will be considering the following family of feasible regions for each fixed $k \geq 0$:

$$\begin{aligned} clP_k = \{\vec{v} \in [0,1]^{\mathcal{S}_k} | \exists \{\sigma^m\}_{m\geq 1} \subseteq \mathcal{S} \text{ s.t. } \lim_{n\to\infty} |\sigma^n| = +\infty \\ , \lim_{n\to\infty} \widetilde{\text{occ}}(\pi,\sigma^n) = \vec{v}_\pi \text{ for } \pi \in \mathcal{S}_k\}\,. \end{aligned} \tag{1.10}$$

This set clP_k was studied in [GHK+17]. There, it was shown that it contains an open ball B with dimension $|I_k|$, where I_k is the set of $\oplus$-indecomposable permutations of size at most k. This work opened the problem of finding the maximal dimension of an open ball contained in clP_k, and placed a lower bound on it.

We now introduce another point of view in this problem, by recalling the pattern Hopf algebra in permutations discussed above. Indeed, the construction of free generators in the pattern algebra on permutations $\mathcal{A}(\texttt{Per})$, studied in [Var14], allows us to derive an upper bound for this maximal dimension. Let $\mathcal{L}_k$ be the set of *Lyndon permutations* of size at most k. Then, for any permutation π that is not a Lyndon permutation, $\mathbf{p}_\pi(\sigma)$ can be expressed as a polynomial on the functions $\{\mathbf{p}_\tau(\sigma) | \tau \in \mathcal{L}_k\}$ that does not depend on σ. It follows that clP_k sits inside an algebraic variety of dimension $|\mathcal{L}_k|$, given by these polynomial equations. We expect that this bound is sharp since this is the case for small values of k.

Conjecture 1.7.1. The feasible region clP_k is full-dimensional inside a manifold of dimension $|\mathcal{L}_k|$.

In this thesis we study the corresponding set for the *consecutive occurrences* of a pattern, defined as:

$$\text{c-occ}(\pi,\sigma) := \left|\left\{ I \subseteq [n] \middle| \, I \text{ is an interval}, \, \sigma|_I = \pi \right\}\right|.$$

Moreover, we denote by $\widetilde{\text{c-occ}}(\pi,\sigma) = \frac{\text{c-occ}(\pi,\sigma)}{|\sigma|}$ the proportion of consecutive occurrences of a pattern π in σ. In this way, the *feasible region for consecutive patterns* is:

$$\begin{aligned} P_k = \{\vec{v} \in [0,1]^{\mathcal{S}_k} | \exists \{\sigma^m\}_{m\geq 1} \subseteq \mathcal{S} \text{ s.t.} \\ \lim_{n\to\infty} |\sigma^n| = +\infty, \text{ and } \lim_{n\to\infty} \widetilde{\text{c-occ}}(\pi,\sigma^n) = \vec{v}_\pi \text{ for } \pi \in \mathcal{S}_k\}\,. \end{aligned} \tag{1.11}$$

We prove that this region is a polytope, *i.e.*, it is described by a finite family of linear inequalities. Its vertices, in fact, are related to the cycles of a specific graph called the *overlap graph*. Motivated by this, in Chapter 6 we define a graph invariant called the *cycle polytope*, and solve certain geometric problems for general graphs and their respective cycle polytope. With these results, for instance, we obtain that the dimension of the feasible region P_k is $k! - (k-1)!$, and also a structure theorem about the faces of the feasible region.

In the remaining of this section, we describe the main results related to the feasible region in Section 1.7.1, and introduce the cycle polytope construction and the relevant geometric results in Section 1.7.2.

1.7.1 Limiting objects in permutations

A notion of limiting object in combinatorics arises when a suitable notion of convergence is introduced. On permutations, two main notions of convergence have been defined: a global notion of convergence (called permuton convergence) and a local notion of convergence (called Benjamini–Schramm convergence).

The first notion of limit for permutations has been introduced in [HKM+13], where the **permuton** was introduced. A permuton is a probability measure on the unit square with uniform marginals, and represents the scaling limit of a permutation seen as a permutation matrix, as the size grows to infinity. The study of permuton limits is an active and exciting research field in combinatorics, see for instance [BBF+17, BBF+19, BBF+18, BBFS19, KKRW15, RVV16, Rom06, Sta09]. On the other hand, the notion of Benjamini–Schramm limit for permutations is more recent, having been introduced in [Bor19]. Informally, in order to investigate Benjamini–Schramm limits, we study the permutation in a neighborhood around a randomly marked point. Limiting objects for this framework are called *infinite rooted permutations* and are in bijection with total orders on the set of integer numbers. Benjamini–Schramm limits have also been studied in some other works, see for instance [Bev19, BBFS19, BS19]. We present the definition of a random *shift-invariant* permutation in Definition 6.3.3.

The following theorems provide relevant combinatorial characterizations of the two aforementioned notions of convergence. We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}$ the set of all permutations, and by $\widetilde{\text{occ}}(\pi,\sigma)$ (respectively $\widetilde{\text{c-occ}}(\pi,\sigma)$) the proportion of classical occurrences (respectively consecutive occurrences) of a permutation π in σ (see Section 6.1.7 for notation and basic definitions).

Theorem 1.7.2 ([HKM+13])**.** For any $n \in \mathbb{N}$, let $\sigma^n \in \mathcal{S}$ and assume that $|\sigma^n| \to \infty$. The sequence $(\sigma^n)_{n\in\mathbb{N}}$ converges to some limiting permuton P if and only if there exists a vector $(\Lambda_\pi(P))_{\pi\in\mathcal{S}}$ of non-negative real numbers (that depends on P) such that, for all $\pi \in \mathcal{S}$,

$$\widetilde{\text{occ}}(\pi,\sigma^n) \to \Lambda_\pi(P).$$

Theorem 1.7.3 ([Bor19])**.** For any $n \in \mathbb{N}$, let $\sigma^n \in \mathcal{S}$ and assume that $|\sigma^n| \to \infty$. The sequence $(\sigma^n)_{n\in\mathbb{N}}$ converges in the Benjamini–Schramm topology to some random infinite rooted permutation σ^∞ if and only if there exists a vector $(\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}}$ of non-negative real numbers (that depends on σ^∞) such that, for all $\pi \in \mathcal{S}$,

$$\widetilde{\text{c-occ}}(\pi,\sigma^n) \to \Gamma_\pi(\sigma^\infty).$$

With these limiting objects, we can describe the feasible region on classical patterns as follows:

$$\begin{aligned} clP_k &:= \left\{\vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists(\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{occ}}(\pi,\sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\right\} \\ &= \left\{(\Lambda_\pi(P))_{\pi\in\mathcal{S}_k} \middle| P \text{ is a permuton}\right\}, \end{aligned}$$

and the feasible region on consecutive patterns as:

$$\begin{aligned} P_k &:= \left\{\vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists(\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi,\sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\right\} \\ &= \left\{(\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}_k} \middle| \sigma^\infty \text{ is a random infinite rooted } \textit{shift-invariant} \text{ permutation}\right\}. \end{aligned}$$

Using this description of the feasible region in classical patterns, the dimension problem was addressed in [GHK+17], where explicit elements of the feasible region were constructed by giving suitable permutons, and a lower bound for its dimension was given.

1.7.2 The cycle polytope

In the literature we can find some examples of polytopes that are associated to graphs. This helps us to construct intricate polytopes whose geometric properties can be related to the properties of the original graphs (see Theorem 6.2.13). The *flow polytope*, introduced by [BV08], is such a polytope. Originally associated to a root system of type A_n, the flow polytope can be described from a labeled undirected graph on the vertex set $[n] := \{1, \dots, n\}$: thus, if we are given a graph $G = ([n], E)$ and a flow vector $\vec{a} \in \mathbb{R}^n$, its corresponding flow polytope is

$$\mathcal{F}_G(\vec{a}) := \left\{ \vec{x} \in \mathbb{R}^E \middle| \sum_{\{j<i\} \in E} \vec{x}_{\{j<i\}} - \sum_{\{i<j\} \in E} \vec{x}_{\{i<j\}} = \vec{a}_i,\, i \in [n] \right\}.$$

Classical examples of polytopes that are flow polytopes are the *Stanley–Pitman polytope*, also called the *parking functions polytope*, and the *Chan–Robbins–Yuen polytope*, a polytope on the space of doubly stochastic square matrices. In [BV08], a formula was obtained for the volume and the number of integer points in its interior. In praticular, they recovered a formula of the volume of the Chan–Robbins–Yuen polytope, due to Zeilberger, in his very short paper [Zei99].

Polytopes related to cycles have also been around in the literature. The *cycle polytope* was introduced in [BP19] and in this thesis, see Chapter 6, which we present now:

Definition 1.7.4. Let $G = (V, E)$ be a directed multigraph. For each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^E$ so that

$$(\vec{e}_{\mathcal{C}})_e := \frac{|\{\text{occurrences of } e \text{ in } \mathcal{C}\}|}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E.$$

The *cycle polytope* of G is the polytope $P(G) := \text{conv}\{\vec{e}_{\mathcal{C}} | \mathcal{C} \text{ is a simple cycle of } G\}$. This is a polytope in the affine space $\mathbb{R}^{E(G)}$.

Polytopes introduced in [BO00] are similar to the cycle polytopes (here referred to U-cycle polytopes). In fact, in Balas & Oosten [BO00] and Balas & Stephan [BS09] the dimension of the U-cycle polytopes of the *complete graph* (that is, the complete directed graph without loops) is computed, and its facets described.

Our main result is Theorem 6.2.13, and relates the structure of the cycle polytope with the corresponding graph. Before introducing this result, we define a *full* subgraph H of a directed graph G. A subgraph H of G is said to be full if any edge of H is contained in some cycle of H.

Theorem 1.7.5. The face poset of $P(G)$ is isomorphic to the poset of non-empty full subgraphs of G according to the following identification:

$$H \mapsto P(G)_H := \{\vec{x} \in P(G) | x_e = 0 \text{ for } e \notin E(H)\} .$$

Also, if we identify $P(H)$ with its image under the canonical inclusion $\mathbb{R}^{E(H)} \hookrightarrow \mathbb{R}^{E(G)}$, we have that $P(H) = P(G)_H$.

Furthermore, $\dim(P(G)_H) = |E(H)| - |V(H)| + |\{\text{connected components of } H\}| - 1$.

1.8 Outline of thesis

In what follows of the thesis, we will present works based on [Pen20, Pen18, BP19]. However, Section 4.5 is an original work and exposed for the first time in this thesis.

In Chapter 2, we introduce the construction of pattern Hopf algebras from combinatorial presheaves, presenting the pattern Hopf algebra functor $\mathcal{A}$. We also describe the primitive space of any pattern Hopf algebra, and prove that some properties of $\mathcal{A}(\texttt{Per})$ are also true in general pattern Hopf algebras. In Chapter 3, we study a specific pattern Hopf algebra on marked permutations. We show that this pattern algebra is free, and construct explicitly the free generators. In Chapter 4 we find a description of the kernel of chromatic symmetric functions, and of the chromatic quasi-symmetric function on hypergraphic polytopes. We also present new graph invariant $\tilde{\Psi}$. In Chapter 5 we present the word quasi-symmetric function species $\texttt{WQSym}$, and a construction of a universal Hopf monoid morphism from $\texttt{WQSym}$. In Section 4.5, we describe the image of $\Upsilon_{\mathbf{GP}}$. In Chapter 6, we show that the feasible region for consecutive occurrences is a polytope that can be described as the cycle polytope of a specific graph. Next we present Chapter 7, a section dedicated to further work and open questions left in this thesis. Finally, in Appendix A we discuss some basic facts about Hopf algebras and about monoidal categories.

Chapter 2

Pattern Hopf algebras

This chapter is based on the article [Pen20], which is submitted for publication. The work in [Pen20] is split into this chapter and Chapter 3.

2.1 Introduction

The notion of substructure is important in mathematics, and particularly in combinatorics. In graph theory, minors and induced subgraphs are the main examples of studied substructures. Substructures of other objects also got some attention: set partitions, trees, paths and, to a bigger extent, permutations, where the study of patterns leads to the concept of permutation class.

A priori unrelated, it has been shown that Hopf algebras are a natural tool in algebraic combinatorics to study discrete objects, like graphs, set compositions and permutations. For instance, the celebrated Hopf algebra on permutations named after Malvenuto and Reutenauer sheds some light on the structure of shuffles in permutations. Other examples of Hopf algebras in combinatorics that are relevant to this work are the Hopf algebra on symmetric functions (described for instance in [Sta00]), and the permutation pattern Hopf algebra introduced by Vargas in [Var14].

With that in mind, we build upon the notion of species, as presented in [AM10] by Aguiar and Mahajan, in order to connect these two areas of algebraic combinatorics. We propose to enrich a species with restriction maps to obtain what we call a *combinatorial presheaf*. With this combinatorial data, we show a functorial construction of a pattern algebra $\mathcal{A}(h)$ from any given combinatorial presheaf h. By further considering an associative product in our objects, we can endow $\mathcal{A}(h)$ with a coproduct that makes it a bialgebra, and under specific circumstances a Hopf algebra. Main examples of combinatorial presheaves are words, graphs and permutations. Examples of associative products on combinatorial objects are the disjoint union on graphs or the direct sum on permutations.

The algebras obtained from a combinatorial presheaf are always commutative. In analyzing Hopf algebras, it is of particular interest to show that such algebras are free commutative (henceforth, we simply say *free*), and to construct free generators of the algebra structure. The fact that a Hopf algebra is a free algebra has several applications. For instance, in [Foi12], it was shown that any graded free and cofree Hopf algebra is self dual. Moreover, the self dual Hopf algebras are characterized by studying their primitive elements. The freeness of a Hopf algebra also allows us to gain some insight on the character group of the Hopf algebra, see for instance [Sup19]. It can also be used under duality maps to establish cofreeness of Hopf monoids, as described in the methods of Möbius inversion in [San19]. If the Hopf algebra $H = \bigcup_{n\geq 0} H^{(n)}$ is a filtered Hopf algebra with $P(H)$ primitive space, computing the dimension of $H^{(n)} \cap P(H)$ is a classical problem in Hopf algebras and may have applications, for instance in proving that a give Hopf algebra is not a Hopf subalgebra of another one.

In this chapter, we show that any commutative combinatorial presheaf gives rise to a pattern algebra that is free commutative. We also study a non-commutative combinatorial presheaf on marked permutations, where we establish the freeness, construct the free elements with help of Lyndon words, and enumerate the primitive elements of the pattern Hopf algebra on marked permutations. In the remaining part of this section, we present these results with more detail, and describe the methods for proving them.

2.1.1 Pattern Hopf algebras from monoids in presheaves

Let $\mathtt{Set}_{\hookrightarrow}$ be the category whose objects are finite sets and morphisms are injective maps between finite sets. Let also $\mathtt{Set}_{\times}$ be the category whose objects are finite sets and morphisms are bijective maps between finite sets. Write $\mathtt{Set}$ for the usual category on finite sets. A **set species** is a functor from $\mathtt{Set}_{\times}$ to $\mathtt{Set}$, whereas a **combinatorial presheaf** is a contravariant functor from $\mathtt{Set}_{\hookrightarrow}$ to $\mathtt{Set}$.

In this form, a presheaf is simply a species enriched with restriction maps $\mathbf{res}_J : h[I] \to h[J]$ for each inclusion $J \subseteq I$ in a way that is functorial, that is if $J_1 \subseteq J_2$, then $\mathbf{res}_{J_2} \circ \mathbf{res}_{J_1} = \mathbf{res}_{J_1}$. The notion and the name of presheaves has been around in category theory and geometry for some time, where it generally refers to contravariant functors from the category of open sets of a topology with inclusions as morphisms. Main examples of combinatorial presheaves are graphs (see Example 1.6.1), set partitions (see

Definition 2.3.12), and permutations (see Example 1.5.2). In general, any combinatorial object that admits a notion of restriction admits a presheaf structure.

In presheaves, two objects $a \in h[I], b \in h[J]$ are said to be *isomorphic objects*, or $a \sim b$, if there is a bijection $f : I \to J$ such that $h[f](b) = a$. The equivalence classes are also called *coinvariants*. Let $h[n]$ denote the objects of type h on the set $[n] = \{1, \dots, n\}$. The collection of coinvariants of a presheaf h is denoted by $\mathcal{G}(h) = \bigcup_{n \geq 0} h[n]/_{\sim}$. Recall from Definition 1.5.3 the definition of the pattern functions:

Definition 2.1.1 (Patterns in presheaves)**.** Let h be a presheaf, and consider two objects $a \in h[I], b \in h[J]$. We say that $J' \subseteq J$ is a *pattern* of a in b if $b|_{J'} \sim a$. We define the *pattern function*

$$\mathbf{p}_a(b) := \left| \{ J' \subseteq J \text{ s.t. } \mathbf{res}_{J'}(b) \sim a \} \right| .$$

In Proposition 2.2.1, we observe that this definition only depends on the isomorphism classes of a and b. Hence, we can consider $\{\mathbf{p}_a\}_{a \in \mathcal{G}(h)}$ as a family of functions from $\mathcal{G}(h)$ to $\mathbb{Q}$, indexed by $\mathcal{G}(h)$.

Recall that we denote the family of functions $A \to B$ by $\mathcal{F}(A, B)$. If h is a combinatorial presheaf, then we saw in Theorem 1.5.4 that the linear span of the pattern functions is a linear subspace $\mathcal{A}(h) \subseteq \mathcal{F}(\mathcal{G}(h), \mathbb{Q})$ of the space of functions in $\mathcal{G}(h)$ taking rational values. We will prove this result in this section.

The Cauchy product gives rise to the notion of a monoid structure in the category of presheaves, or simply an *associative presheaf*. This is a triple $(h, *, 1)$, where h is a combinatorial presheaf, $*$ is a natural transformation $h \odot h \Rightarrow h$, and $1 \in h[\emptyset]$ a unit that satisfy classical axioms of associativity and unit. We detail further in Section 2.1.4 below. Examples of associative operations on combinatorial presheaves are the disjoint union of graphs and the direct sum of permutations. Another less standard example, which we study in Chapter 3, is the inflation of marked permutations, defined in Definition 3.1.3.

Observe that the associative product $*$ in our combinatorial objects is a natural transformation. This means that the product is stable with respect to relabelings and restrictions, so we can also define the corresponding product on $\mathcal{G}(h)$, which we denote by $\cdot$ for the sake of distinction (see Definition 2.2.6 for details). With this, we introduce the following coproduct in the pattern algebra $\mathcal{A}(h)$:

$$\Delta \, \mathbf{p}_a = \sum_{\substack{b,c \in \mathcal{G}(h) \\ a = b \cdot c}} \mathbf{p}_b \otimes \mathbf{p}_c \, . \tag{2.1}$$

As we have claimed in Theorem 1.5.5, and prove below, whenever $(h, *, 1)$ is an associative presheaf such that $|h[\emptyset]| = 1$, then $\mathcal{A}(h)$ forms a Hopf algebra. The reason we need the presheaf to be connected is so we can find an antipote through the so called *Takeuchi formula*, introduced in [Tak71].

Some known Hopf algebras can be constructed as the pattern algebra of a combinatorial presheaf, like the Hopf algebra of *symmetric functions*, which arises as the pattern algebra of set partitions (see details in Section 2.3.4). The pattern Hopf algebra corresponding to the presheaf on permutations described above was introduced by Vargas in [Var14]. Some other Hopf algebras constructed here, like the ones on graphs and on marked permutations below, are new.

In this chapter, we also establish some general properties of the pattern Hopf algebras, like describing its primitive elements, finding the inverse of the so called *pattern action* in $\mathcal{A}(h)$, and relating $\mathcal{A}(h)$ with the Sweedler dual of an algebra generated by $*$.

Remark 2.1.2. It is common to use category theory tools to construct Hopf algebras in a mechanical way, as it gives us more algebraic tools to understand combinatorial objects. This is the case with the Fock functors (see [AM10, Chapter 15]), where from a Hopf monoid in species we construct four distinct Hopf algebras.

It is then meaningful to compare the construction of a pattern algebra with the Fock functors. In fact, a cocommutative comonoid in set species is precisely a presheaf. This was already observed in [AM10, Section 8.7.8]. Furthermore, an associative presheaf is a cocomutative bimonoid in set species, and this is established in [AM10, Proposition 8.29]. The coalgebra structure of the pattern Hopf algebras that we construct here is a subcoalgebra of the dual algebra of the so called *bosonic Fock functor* of these comonoids in linearized set species. However, the algebra structure is in general different.

Specifically, on the combinatorial presheaf on graphs introduced below, the corresponding coalgebra structure is the dual of the well known incidence Hopf algebra introduced in [Sch94].

2.1.2 Commutative presheaves

In this chapter, we focus on the problem of proving the freeness of some pattern algebras. The first case that we want to explore is the one of commutative presheaves. An associative presheaf $(h, *, 1)$ is called *commutative* if $*$ is commutative, that is for any $a \in h[I], b \in h[J]$ we have that $a * b = b * a$, see Definition 2.1.4.

As it turns out, having a commutative monoid structure is enough to guarantee the freeness of the pattern Hopf algebra. This is the main result of this chapter, already mentioned in Theorem 1.6.2.

The proof of this result is presented in Section 2.3. The main ingredient for this result is Corollary 2.3.4, a surprising structure result on associative presheaves. If one wishes to describe the associative structure of $\mathcal{G}(h)$ under the product $\cdot$, it can be done as it is for groups: by prescribing a set of generators, whose role is played by the *irreducible coinvariants*, and a collection of relations that these satisfy. Corollary 2.3.4 says that on the case of associative presheaves, the collection of relations is very restricted, allowing only for relations that use the same factors but with different orders. This in particular also describes the *coradical filtration* of a Hopf algebra, for instance, via Corollary 2.3.5 (see Definition A.4.2 for definition of coradical filtration).

We remark that the algebraic structure of the pattern Hopf algebra $\mathcal{A}(h)$ is defined independently from its associative product. It follows that the pattern Hopf algebra of a combinatorial presheaf is free whenever we *can* endow h with a commutative associative product, regardless of whether that is the associative presheaf at hand. In other words, given $(h, *_1, 1)$ and $(h, *_2, 1)$ connected associative presheaf structures on the same presheaf h, such that $*_2$ is commutative, then $\mathcal{A}(h) = \mathcal{A}(h, *_1, 1)$ is a free Hopf algebra. An example of this is presented on the presheaf of marked graphs, in Section 2.3.3 below. Notwithstanding, all freeness proofs on pattern Hopf algebras uses both the pattern structure and the associative structure. In fact, the role played by the associative structure is streamlined and inflexible, as described in Section 2.1.3, whereas the role of the pattern structure is somewhat harder to deal with.

This was already proved in [Whi32, Theorem 3], where if a function satisfies (1.7), the connected graphs (there called non-separable) are enough to determine its values, that the remaining values are obtained via polynomial expressions, and that no other polynomial expressions hold for such a generic function.

2.1.3 Strategy for establishing the freeness of a pattern algebra

We now discuss the general strategy that we employ when establishing the freeness of a pattern Hopf algebra. In particular, we clarify what is the relation between unique factorization theorems and freeness of the algebra of interest. Let $\mathcal{S} \subseteq \mathcal{G}(h)$ be a collection of objects in a presheaf h. Then the set $\{\mathbf{p}_s \,|s \in \mathcal{S}\}$ is a set of free generators of $\mathcal{A}(h)$ if the set

$$\left\{ \prod_{s \in S} \mathbf{p}_s \,\middle|\, S \text{ multiset of elements of } \mathcal{S} \right\},$$

is a basis of $\mathcal{A}(h)$. This is usually established by connecting this set with the set $\{p_a | a \in \mathcal{G}(h)\}$, which is known to be a basis by Remark 2.2.2. This connection is done with the following ingredients:

- An order $\preceq$ in $\mathcal{G}(h)$.
- A bijection f between $\{\prod_{s\in S}\mathbf{p}_s\,|S$ multiset of elements of $\mathcal{S}\}$ and $\mathcal{G}(h)$, which is usually phrased in terms of a *unique factorization theorem*. See for instance Theorem 3.3.9.
- The property that, for any S multiset of $\mathcal{S}$,

$$\prod_{s\in S}\mathbf{p}_s = \sum_{t\preceq f(S)} c_{t,S}\,\mathbf{p}_t\,, \tag{2.2}$$

  with non-negative coefficients $c_{t,s}$ such that $c_{f(S),S}\neq 0$.

These are enough to establish the desired freeness. In the commutative presheaf case, the unique factorization theorem is the one that we naively would expect, see Theorem 2.3.3. For the case of presheaf on permutations and the presheaf on marked permutations, the construction of the set $\mathcal{S}$ is more technical. In general, we conjecture that any associative presheaf is free, see Conjecture 1.6.4.

2.1.4 Notation and preliminaries

2.1.4.1 Species and monoidal functors

If h is a combinatorial presheaf, and if $a\in h[I]$, we define the *size* of a as $|a|:=|I|$ and the *indexing set* of a as $\mathbb{X}(a)=I$.

Example 2.1.3. The unit for the Cauchy product is the unique presheaf that satisfies $\mathcal{E}[A]=\emptyset$ for $A\neq\emptyset$, and $\mathcal{E}[\emptyset]=\{\diamond\}$. The presheaf of graphs

$$\texttt{Gr}[I]=\{\text{graphs with vertex set } I\}\,,$$

results from the usual species structure by adding the natural graph restrictions.

In this way, $\mathbb{X}(\diamond)=\emptyset$, and if G is a graph in the vertex set V, we have that $\mathbb{X}(G)=V$ and $|G|=|V|$.

Recall that we are given a bifunctor $\odot$ that endows the category of combinatorial presheaves with a monoidal category structure, as introduced in [AM10].

Definition 2.1.4 (Associative presheaf)**.** An *associative presheaf* is a monoid in $\texttt{CPSh}$, that is, is a combinatorial presheaf h together with natural transformations $\eta : h\odot h\Rightarrow h$ and $\iota:\mathcal{E}\Rightarrow h$ that satisfy associativity and unit conditions. We use, for $a\in h[I]$ and $b\in h[J]$, the notation $\eta_{I,J}(a,b)=a*b$. We also denote the unit by $1:=\iota[\emptyset](\diamond)\in h[\emptyset]$.

This is said to be *commutative* if, for any $a \in h[I], b \in h[J]$ with $I \cap J = \emptyset$, we have $a * b = b * a$.

Thus, a product on a presheaf way simply describes how to *merge* objects of a certain type h that are based in disjoint sets.

Observation 2.1.5 (Naturality axioms in associative presheaves)**.** The naturality of η, the associativity and unit conditions correspond to, respectively,

- For all I, J disjoint sets, all $a \in h[I], b \in h[J]$ and all $A \subseteq I, B \subseteq J$, we have $(a * b)|_{A \sqcup B} = a|_A * b|_B$.
- For all I, J, K disjoint sets and all $a \in h[I], b \in h[J], c \in h[K]$, we have $(a * b) * c = a * (b * c)$.
- For any set I, and $a \in h[I]$, we have $a * 1 = 1 * a = a$.

Remark 2.1.6 (Monoidal product and quasi-shuffle)**.** In an associative presheaf $(h, *, 1)$, let $a \in h[I], b \in h[J]$ with I, J disjoint sets. Then it may not be the case that $a|_\emptyset = 1$, that $(a * b)|_I = a$ or that $(a * b)|_J = b$.

This is the case, however, when h is a connected presheaf. It follows that in an associative connected presheaf h, we have that $\binom{a*b}{a,b} \geq 1$.

Definition 2.1.7. Let $(h, *_h, 1_h)$ and $(j, *_j, 1_j)$ be associative presheaves, and let f be a *presheaf morphism* between $(h, *_h, 1_h)$ and $(j, *_j, 1_j)$. This is an *associative presheaf morphism* if it preserves the unit and the associative product of the associative presheaves.

That is, $f : h \Rightarrow j$ is an associative presheaf morphism if it is a presheaf morphism that satisfies $f(1_h) = 1_j$ and $f(b' *_h c') = f(b') *_j f(c')$ for any $b' \in h[I], c' \in h[J]$.

2.1.4.2 Preliminaries on permutations and marked permutations

We can write a permutation in its two-line notation, as $\substack{a_1,\dots,a_k \\ b_1,\dots,b_k}$ where $a_1 \leq_V a_2 \leq_V \dots$ and $b_1 \leq_P b_2 \leq_P \cdots \leq_P b_k$. If we identify $b_1, \dots, b_k$ with $1, \dots, k$, respectively, we can disregard the bottom line. This also disregards the indexing set I, and in fact any two isomorphic permutations have the same representation with the one line notation.

To the unique permutation in the empty set we call the *trivial permutation* and denote it $\emptyset$.

Definition 2.1.8 (The $\ominus$ operation)**.** Given two permutations, π, σ, we have already introduced the product $\pi \oplus \sigma$. We now define the permutation $\pi \ominus \tau \in \mathtt{Per}[I \sqcup J]$ as the pair of total orders $(\leq_P^{\ominus}, \leq_V^{\ominus})$ extending the respective ones from π, τ to $I \sqcup J$ by forcing that $i \leq_P^{\ominus} j$ and $i \geq_V^{\ominus} j$ for any $i \in I, j \in J$. Correspondingly, the diagram of $\pi \ominus \tau$ results from the ones from π, τ as

$$\pi \ominus \tau = \begin{array}{|c|c|} \hline \pi & \\ \hline & \tau \\ \hline \end{array} \, .$$

It is a routine observation to check that both $\oplus, \ominus$ are associative products on $\mathtt{Per}$, and that $\emptyset$ is the unit of both operations, by simply checking that all properties in Observation 2.1.5 are fulfilled.

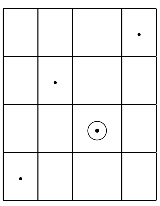
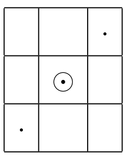
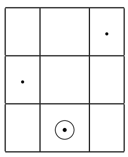
Definition 2.1.9 (Marked permutations)**.** Given a set I, we use the shorthand notation $I^* = I \sqcup \{*\}$. A marked permutation π^* on a set I is a pair $(\leq_P, \leq_V)$ of total orders in I^*, we write $\mathbb{X}(\pi^*) = I$. If $f : J \to I$ is an injective map, this can be extended canonically to an injective map $f^* : J^* \to I^*$. Thus, the preimage of each order $\leq_P, \leq_V$ under f^* is well defined and is also a total order in J^*. This defines the marked permutation $\mathtt{MPer}[f](\pi^*)$.

Note that a relabeling of the permutation π^* in I^* is a relabeling of the corresponding marked permutation in I if the relabeling preserves the marked elements.

We can also write marked permutations in a one line notation, where we add a marker over the position of $*$. The resulting notation only disregards the indexing set I, and so any two isomorphic marked permutations have the same one line notation. Note that for each permutation of size n it corresponds n different non-isomorphic marked permutations of size $n-1$, one for each possible marked position.

Example 2.1.10. If we consider $(1 <_P 2 <_P * <_P 4,\ 1 <_V * <_V 4 <_V 2)$, a marked permutation on $\{1, 2, 4\}$, its representation with the one line notation is $14\bar{2}3$.

The marked permutation $\tau^* = (73 <_P x <_P * <_P 47,\ 73 <_V * <_V x <_V 47)$ is based on the set $I = \{x, 47, 73\}$ and has a one line representation $13\bar{2}4$. Consider now $\pi^* = (1 <_P * <_P 2, 1 <_V * <_V 2)$ and $\sigma^* = (1 <_P * <_P 2, * <_V 1 <_V 2)$ marked permutations in $\{1, 2\}$. So the marked permutations τ^*, π^*, σ^* correspond to the one line notations below

$\tau^* = 13\bar{2}4 =$  , $\pi^* = 1\bar{2}3 =$  , $\sigma^* = 2\bar{1}3 =$  . (2.3)

Then, we have that π^*, σ^* are patterns of τ^*, because $J = \{73, 47\}$ is an occurrence of π^* in τ^*, and $J' = \{x, 47\}$ is an occurrence of σ^* in τ^*.

2.2 Substructure algebras

In this section, we present the properties of the framework of combinatorial presheaves and associative presheaves, introduced above, and describe the construction of the pattern Hopf algebra mentioned in Theorem 1.5.5. We establish that for a combinatorial presheaf h, $\mathcal{A}(h)$ is an algebra (see Theorem 2.2.3). Moreover, if h is connected and endowed with an associative structure, then $\mathcal{A}(h)$ is a Hopf algebra (see Theorem 2.2.8). We also describe the space of primitive elements in $\mathcal{A}(h)$, clarify that $\mathcal{A}$ is in fact functorial (see Theorem 2.2.11), and find some identities and properties of the pattern functions in Propositions 2.2.14 and 2.2.15.

2.2.1 Coinvariants and pattern algebras

Given a combinatorial presheaf h, let $[n] = \{1, \dots, n\}$ for n non-negative integer. Then, recall that we define the *coinvariants* of h as the family

$$\mathcal{G}(h) = \biguplus_{n \geq 0} h[n]_{\sim} \, .$$

For an object $a \in \biguplus_J h[J]$, we define the *pattern function* $\mathbf{p}_a : \biguplus_J h[J] \to \mathbb{Q}$ as follows: if $b \in h[I]$, then

$$\mathbf{p}_a(b) := \left| \{ J' \subseteq I \text{ s.t. } b|_{J'} \sim a \} \right| \, .$$

The following proposition allows us to consider a, b to be coinvariants, showing that this pattern function is still well defined for $a \in \mathcal{G}(h)$ as function in $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})$.

Proposition 2.2.1. This definition only depends on the isomorphism type of a, b. That is, if $a_1 \in h[I_1], a_2 \in h[I_2], b_1 \in h[J_1], b_2 \in h[J_2]$ are so that $a_1 \sim a_2, b_1 \sim b_2$ then $\mathbf{p}_{a_1}(b_1) = \mathbf{p}_{a_2}(b_2)$.

Proof. Because $a_1 \sim a_2$, we have that

$$\{ J' \subseteq J_1 \text{ s.t. } b_1|_{J'} \sim a_1 \} = \{ J' \subseteq J_1 \text{ s.t. } b_1|_{J'} \sim a_2 \} \, ,$$

so $\mathbf{p}_{a_1}(b_1) = \mathbf{p}_{a_2}(b_1)$. It remains to prove that $\mathbf{p}_{a_2}(b_1) = \mathbf{p}_{a_2}(b_2)$.

Because $b_1 \sim b_2$, there exists some bijective map $f : J_1 \to J_2$ such that $h[f](b_2) = b_1$. This bijection lifts to a bijection between $\{\tilde{J} \subseteq J_2\}$ and $\{J' \subseteq J_1\}$, via $\tilde{J} \mapsto f^{-1}(\tilde{J})$.

We claim that this in fact restricts to a bijection between $\{\tilde{J} \subseteq J_2 \text{ s.t. } b_2|_{\tilde{J}} \sim a_2\}$ and $\{J' \subseteq J_1 \text{ s.t. } b_1|_{J'} \sim a_2\}$. Indeed, let $\tilde{J} \subseteq J_1$, let $\mathrm{inc}_{J_2,\tilde{J}} : \tilde{J} \to J_2$, $\mathrm{inc}_{J_1,f^{-1}(\tilde{J})}$ denote the inclusion maps of $\tilde{J}$ in J_2 and $f^{-1}(\tilde{J})$ in J_1, respectively. Then

$$b_1|_{f^{-1}(\tilde{J})} = h[f^{-1} \circ \mathrm{inc}_{J_2,\tilde{J}}](b_2) = h[\mathrm{inc}_{J_1,f^{-1}(\tilde{J})} \circ f^{-1}](b_2) = h[f^{-1}](b_2|_{\tilde{J}}) \,,$$

so $b_2|_{\tilde{J}} \sim b_1|_{f^{-1}(\tilde{J})}$. This proves that $\mathbf{p}_{a_2}(b_1) = \mathbf{p}_{a_2}(b_2)$. □

Recall that we write

$$\mathcal{A}(h) := \mathrm{span}\{\mathbf{p}_a \,|\, a \in \mathcal{G}(h)\} \subseteq \mathcal{F}(\mathcal{G}(h), \mathbb{Q}) \,,$$

for the linear space spanned by all pattern functions.

Remark 2.2.2. If two coinvariants a, b are such that $|a| \geq |b|$ and $a \neq b$, then $\mathbf{p}_a(b) = 0$. We also have $\mathbf{p}_b(b) = 1$. Hence, the set $\{\mathbf{p}_a \,| a \in \mathcal{G}(h)\}$ is a basis of $\mathcal{A}(h)$.

For a, b objects and $c \in h[C]$, we defined the *quasi-shuffle* number as follows:

$$\binom{c}{a, b} = |\{(I, J) \, s.t. \; I \cup J = C \,, \; c|_I \sim a, \, c|_J \sim b\}| \,. \tag{2.4}$$

And a *quasi-shuffle* as a pair that contributes to the coefficient above. This is invariant under the equivalence classes of $\sim$, as it can be show in a similar way to Proposition 2.2.1.

In the following proposition we observe that $\mathcal{A}(h)$ is a subalgebra of $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})$ with the pointwise multiplication structure.

Theorem 2.2.3. Let h be a presheaf. Then the pattern functions satisfy the following identity:

$$\mathbf{p}_a \, \mathbf{p}_b = \sum_{c \in \mathcal{G}(h)} \binom{c}{a, b} \mathbf{p}_c \,. \tag{2.5}$$

In particular, the pattern functions of h span a subalgebra of the function algebra $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})$, where the unit is $\sum_{c \in h[\emptyset]} \mathbf{p}_c$. We say that $\mathcal{A}(h)$ is the *pattern algebra* of h.

Proof. Fix $x \in h[I]$, and note that $\mathbf{p}_a(x)\,\mathbf{p}_b(x)$ counts the following

$$\begin{aligned}\mathbf{p}_a(x)\,\mathbf{p}_b(x) &= |\{A \subseteq I \text{ s.t. } x|_A \sim a\} \times \{B \subseteq I \text{ s.t. } x|_B \sim b\}| \\ &= |\{(A,B) \text{ s.t. } A, B \subseteq I,\, x|_A \sim a,\, x|_B \sim b\}| \\ &= \sum_{C \subseteq I} |\{(A,B) \text{ s.t. } A \cup B = C,\, x|_A \sim a,\, x|_B \sim b,\}| \qquad (2.6) \\ &= \sum_{C \subseteq I} \binom{x|_C}{a, b} = \sum_{c \in \mathcal{G}(h)} \binom{c}{a, b} \mathbf{p}_c(x)\,.\end{aligned}$$

Hence, the space $\mathcal{A}(h)$ is closed for the product of functions. Further, it is easy to observe that $\sum_{a \in h[\emptyset]} \mathbf{p}_a$ is the constant function equal to one, so this is a unit and $\mathcal{A}(h)$ is an algebra, concluding the proof. □

Corollary 2.2.4 (Products and quasi-shuffles)**.** Let $c \in h[I]$ and $b_i \in h[J_i]$ for $i = 1, \ldots, k$, and let

$$\binom{a}{b_1, \ldots, b_k} := \left| \left\{ (J_1, \ldots, J_k) \text{ s.t. } \bigcup_{i=1}^{k} J_i = I,\ a|_{J_i} \sim b_i\ \forall i = 1, \ldots, k \right\} \right| .$$

Then we have that

$$\prod_{i=1}^{k} \mathbf{p}_{b_i} = \sum_c \binom{c}{b_1, \ldots, b_k} \mathbf{p}_c\ .$$

Definition 2.2.5 (Shuffles and quasi-shuffles)**.** If an object a is such that $\binom{a}{b_1,\ldots,b_k} > 0$ we say that a is a quasi-shuffle of $b_1, \ldots, b_k$. In addition, if $|a| = \sum_{i=1}^{k} |b_i|$, we say that a is a *shuffle* of $b_1, \ldots, b_k$.

2.2.2 Coproducts on pattern algebras

In this section we consider an associative presheaf $(h, *, 1)$. Concretely, our combinatorial presheaf h is endowed with an associative product $*$ and a unit $1 \in h[\emptyset]$.

Definition 2.2.6 (Product structure in $\mathcal{G}(h)$)**.** If $(h, *, 1)$ is an associative presheaf, then $\mathcal{G}(h)$ inherits an associative product. If a is an object, we denote its equivalence class under $\sim$ by $\bar{a}$ in this remark. The associative product in $\mathcal{G}(h)$ is defined as follows:

Let $a \in h[n_1], b \in h[n_2]$ and denote $[n_1 + 1, n_1 + n_2] = \{n_1 + 1, \ldots, n_1 + n_2\}$. Consider st the order preserving map $st : [n_1 + 1, n_1 + n_2] \to [n_2]$, and let $b' = h[st](b)$. Then we define the product in $\mathcal{G}(h)$ as $\bar{a} \cdot \bar{b} := \overline{a * b'} \in h[n_1 + n_2]_{\sim}$.

It is a direct computation to see that $\overline{a * b}$ does not depend on the representative chosen for $\overline{a}$ and $\overline{b}$. Thus we have a well defined operation in $\mathcal{G}(h)$.

Definition 2.2.7. Let $a \in \mathcal{G}(h)$. Then, define

$$\Delta\, \mathbf{p}_a := \sum_{\substack{b,c \in \mathcal{G}(h) \\ b \cdot c = a}} \mathbf{p}_b \otimes \mathbf{p}_c \,.$$

Note that the right hand side is a finite sum, because $b \cdot c = a$ implies that $|b|, |c| \leq |a|$, so this is well defined. Define further the map $\varepsilon : \mathcal{A}(h) \to \mathbb{Q}$ that sends $\mathbf{p}_a$ to $\mathbb{1}[a = 1]$, where 1 stands for the unit in the associative presheaf h.

We recall and prove Theorem 1.5.5 here.

Theorem 2.2.8 (The pattern Hopf algebra)**.** Let $(h, *, 1)$ be an associative presheaf. Then, the maps Δ and ε give $\mathcal{A}(h)$ a structure of a coalgebra. Furthermore, together with pointwise multiplication of functions, this defines a bialgebra structure in $\mathcal{A}(h)$.

Further, $\mathcal{A}(h)$ is a Hopf algebra whenever h is a connected presheaf.

Remark that we refer to the pattern Hopf algebra of $(h, *, 1)$ simply by $\mathcal{A}(h)$ instead of $\mathcal{A}(h, *, 1)$, for simplicity of notation, whenever emphasis on the role of $*$ is not needed.

Proof of Theorem 1.5.5. First, we note that Δ is trivially coassociative, from the associativity axioms of $*$ described in Observation 2.1.5. That ε is a counit follows from the unit axioms on $(h, *, 1)$.

We first claim that, for $a, x, y \in \mathcal{G}(h)$,

$$\Delta\, \mathbf{p}_a(x, y) = \mathbf{p}_a(x \cdot y)\,, \tag{2.7}$$

using the natural inclusion $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})^{\otimes 2} \subseteq \mathcal{F}(\mathcal{G}(h)^2, \mathbb{Q})$.

Indeed, take representatives $x \in h[n_1]$ and $y \in h[n_2]$ with no loss of generality, and write $B = [n_1]$, $C = \{n_1 + 1, \dots, n_1 + n_2\}$. Let st be the order preserving map between C and $[n_2]$. Then

$$\begin{aligned}
\mathbf{p}_a(x * y) &= |\{J \subseteq B \sqcup C \text{ s.t. } (x * y)|_J \sim a\}| \\
&= |\{J \subseteq B \sqcup C \text{ s.t. } x|_{J \cap B} * y|_{J \cap C} \sim a\}| \\
&= \sum_{\substack{b,c \in \mathcal{G}(h) \\ a \sim b \cdot c}} |\{J \subseteq B \sqcup C \text{ s.t. } x|_{J \cap B} \sim b,\ y|_{J \cap C} \sim c\}| \\
&= \sum_{\substack{b,c \in \mathcal{G}(h) \\ a \sim b \cdot c}} |\{J \subseteq B \text{ s.t. } x|_J \sim b\} \times \{J \subseteq C \text{ s.t. } y|_J \sim c\}| \\
&= \sum_{\substack{b,c \in \mathcal{G}(h) \\ a \sim b \cdot c}} \mathbf{p}_b(x)\, \mathbf{p}_c(y) = \Delta\, \mathbf{p}_a(x, y)\,.
\end{aligned} \tag{2.8}$$

Since both functions take the same values on $\mathcal{G}(h)^2$, we conclude that (2.7) holds.

The following are the bialgebra axioms that we wish to establish:

$$\Delta(\mathbf{p}_a\,\mathbf{p}_b) = \Delta(\mathbf{p}_a)\Delta(\mathbf{p}_b)\,,$$
$$\Delta\left(\sum_{a\in h[\emptyset]}\mathbf{p}_a\right) = \left(\sum_{a\in h[\emptyset]}\mathbf{p}_a\right)\otimes\left(\sum_{a\in h[\emptyset]}\mathbf{p}_a\right)\,,$$
$$\varepsilon(\mathbf{p}_a\,\mathbf{p}_b) = \varepsilon(\mathbf{p}_a)\varepsilon(\mathbf{p}_b)\,,$$
$$\varepsilon\left(\sum_{a\in h[\emptyset]}\mathbf{p}_a\right) = 1\,.$$

The last three equations are direct computations. For the first equation, we use (2.7) as follows: take $a, b, x, y \in \mathcal{G}(h)$, then

$$\Delta(\mathbf{p}_a\,\mathbf{p}_b)(x,y) = (\mathbf{p}_a\,\mathbf{p}_b)(x\cdot y) = \mathbf{p}_a(x\cdot y)\,\mathbf{p}_b(x\cdot y) = (\Delta\,\mathbf{p}_a\,\Delta\,\mathbf{p}_b)(x,y)\,.$$

This concludes the first part of the proof.

Now suppose that h is connected, so that the zero degree component $\mathcal{A}(h)_0$ is one dimensional. From [Tak71], because $\mathcal{A}(h)$ is commutative, it suffices to establish that the group-like elements of $\mathcal{A}(h)$ are invertible. Now it is a direct observation that any group-like element is in $\mathcal{A}(h)_0$, which is a one dimensional algebra, so all non-zero elements are invertible. This concludes that $\mathcal{A}(h)$ is a Hopf algebra. □

2.2.3 The space of primitive element

In this section we give a description of the primitive elements of any pattern Hopf algebra. Denote by $P(H) \coloneqq \{a \in H \,|\, \Delta a = a\otimes 1 + 1\otimes a\}$ the space of *primitive elements*, where 1 stands for the unit of the Hopf algebra.

Definition 2.2.9 (Irreducible objects)**.** Let $(h, *, 1)$ be a connected associative presheaf. An object $t \in h[I]$ with $t \neq 1$ is called *irreducible* if any two objects $a \in h[A], b \in h[B]$ such that $a * b = t$ and $A \sqcup B = I$ have either $a = 1$ or $b = 1$.

The notion of irreducibility lifts to $\mathcal{G}(h)$. That is, a coinvariant $t \in \mathcal{G}(h)$ with $t \neq 1$ is said to be *irreducible* if any $a, b \in \mathcal{G}(h)$ such that $a \cdot b = t$ have either $a = 1$ or $b = 1$. We have that $a \in h[I]$ is irreducible if and only if the corresponding equivalence class $\bar{a} \in \mathcal{G}(h)$ is irreducible. The family of irreducible equivalent classes in $\mathcal{G}(h)$ is denoted by $\mathcal{I}(h) \subseteq \mathcal{G}(h)$.

Proposition 2.2.10 (Primitive space of pattern Hopf algebras)**.** Let $(h, *, 1_h)$ be a connected associative presheaf, and $\mathcal{I}(h)$ the set of irreducible elements in $\mathcal{G}(h)$.

Then

$$P(\mathcal{A}(h)) = \text{span}\{\mathbf{p}_a \,|a \in \mathcal{I}(h)\}\,.$$

Proof. If f is irreducible, it is straightforward to observe that $\mathbf{p}_f$ is primitive. On the other hand, let $a = \sum_{f \in \mathcal{G}(h)} c_f \, \mathbf{p}_f$ be a generic primitive element from $\mathcal{A}(h)$. Then, the equation $\Delta a = a \otimes \mathbf{p}_{1_h} + \mathbf{p}_{1_h} \otimes a$ becomes

$$\sum_{g_1, g_2 \in \mathcal{G}(h)} c_{g_1 \cdot g_2} \, \mathbf{p}_{g_1} \otimes \mathbf{p}_{g_2} = \sum_f c_f (\mathbf{p}_f \otimes \mathbf{p}_{1_h} + \mathbf{p}_{1_h} \otimes \mathbf{p}_f)\,.$$

From Remark 2.2.2, $\{\mathbf{p}_{g_1} \otimes \mathbf{p}_{g_2}\}_{g_1, g_2 \in \mathcal{G}(h)}$ is a basis of $\mathcal{A}(h)^{\otimes 2}$, so we have that for any $g_1 \neq 1_h, g_2 \neq 1_h$,

$$c_{g_1 \cdot g_2} = 0\,.$$

Thus we conclude that a is a linear combination of the set $\left\{\mathbf{p}_f \,|f \in \mathcal{I}(h)\right\}$, as desired. □

2.2.4 The pattern algebra functor

In this section, we see that the mapping $\mathcal{A}$ is in fact functorial, bringing the parallel between the pattern Hopf algebras and the Fock functors even closer.

Theorem 2.2.11 (Pattern algebra maps)**.** If $f : h \Rightarrow j$ is a morphism of combinatorial presheaves, then the following formula

$$\mathcal{A}[f](\mathbf{p}_a) := \sum_{f(b)=a} \mathbf{p}_b \in \mathcal{A}(h)\,, \tag{2.9}$$

defines an algebra map $\mathcal{A}[f] : \mathcal{A}(j) \to \mathcal{A}(h)$.

Further, if f is a morphism of associative presheaves, then $\mathcal{A}[f]$ is a bialgebra morphism. Consequently, if h, j are connected, this is a Hopf algebra morphism.

Remark that the sum in (2.9) is finite, so this functor is well defined.

Proof. The map $\mathcal{A}[f]$ is linear and sends the unit $\sum_{a \in j[\emptyset]} \mathbf{p}_a$ of $\mathcal{A}(j)$ to

$$\sum_{a \in j[\emptyset]} \sum_{\substack{b \in h[\emptyset] \\ f(b)=a}} \mathbf{p}_b = \sum_{b \in j[\emptyset]} \mathbf{p}_b\,.$$

Hence, to establish that $\mathcal{A}[f]$ is an algebra morphism, it suffices to show that it preserves the product on the basis, *i.e.*, that $\mathcal{A}[f](\mathbf{p}_a\,\mathbf{p}_b) = \mathcal{A}[f](\mathbf{p}_a)\mathcal{A}[f](\mathbf{p}_b)$.

It is easy to see that this holds if we have that

$$\binom{f(c')}{a,b} = \sum_{\substack{a',b' \in \mathcal{G}(h)\\ f(a')=a, f(b')=b}} \binom{c'}{a',b'},$$

for any $a, b \in \mathcal{G}(j), c' \in \mathcal{G}(h)$.

Indeed, if $a \in j[A], b \in j[B]$ and $c' \in h[C]$, then for any set I, by naturality of f, we have $f(c')|_I = f(c'|_I)$. Then

$$\begin{aligned}\binom{f(c')}{a,b} &= \left|\left\{(I,J) \text{ s.t. } f(c')|_I \sim a,\ f(c')|_J \sim b,\ I \cup J = C\right\}\right| \\ &= \sum_{\substack{a',b' \in \mathcal{G}(h)\\ f(a')=a,\ f(b')=b}} \left|\left\{(I,J) \text{ s.t. } c'|_I \sim a',\ c'|_J \sim b',\ I \cup J = C\right\}\right| \\ &= \sum_{\substack{a',b' \in \mathcal{G}(h)\\ f(a')=a,\ f(b')=b}} \binom{c'}{a',b'}.\end{aligned}$$

Now suppose further that f is an associative presheaf morphism between $(h, *_h, 1_h)$ and $(j, *_j, 1_j)$. That $\mathcal{A}[f]$ preserves counit follows from $f(1_h) = 1_j$. That $\mathcal{A}[f]$ preserves Δ follows because both $\mathcal{A}[f]^{\otimes 2}(\Delta\,\mathbf{p}_a)$ and $\Delta(\mathcal{A}[f]\,\mathbf{p}_a)$ equal

$$\sum_{\substack{b',c' \in \mathcal{G}(h)\\ f(b' \cdot_h c')=a}} \mathbf{p}_{b'} \otimes \mathbf{p}_{c'},$$

since $f(b' *_h c') = f(b') *_j f(c')$, concluding the proof. □

Write $\mathrm{Alg}_{\mathbb{Q}}$, $\mathrm{BiAlg}_{\mathbb{Q}}$ for the categories of algebras and bialgebras over $\mathbb{Q}$. Recall that, given a monoidal category $\mathcal{C}$, we denote by $\mathrm{Mon}(\mathcal{C})$ its subcategory of monoids objects, see Appendix B.1.

Definition 2.2.12 (The pattern algebra functor)**.** Because of Theorem 2.2.11, we can define the *pattern algebra* contravariant functor $\mathcal{A} : \mathtt{CPSh} \to \mathrm{Alg}_{\mathbb{Q}}$.

This functor when restricted to the subcategory of associative presheaves is also a functor to bialgebras as $\mathcal{A} : \mathrm{Mon}(\mathtt{CPSh}) \to \mathrm{BiAlg}_{\mathbb{Q}}$.

Example 2.2.13 (Graph patterns in permutations)**.** Consider again the associative presheaves $(\mathtt{Gr}, \oplus, \emptyset)$ on graphs and $(\mathtt{Per}, \oplus, \emptyset)$ on permutations. The inversion graph of a permutation is a presheaf morphism $\mathtt{Inv} : \mathtt{Per} \Rightarrow \mathtt{Gr}$ defined as follows: given a permutation $\pi = (\leq_P, \leq_V)$ on the set I, we take the graph $\texttt{Inv}(\pi)$ with vertex set I and an edge between $i, j \in I$, $i \neq j$ if

$$i \leq_P j \Leftrightarrow j \leq_V i \,.$$

It is a direct observation that this map is indeed an associative presheaf morphism. As a consequence, we have a Hopf algebra morphism $\mathcal{A}(\texttt{Inv}) : \mathcal{A}(\texttt{Gr}) \to \mathcal{A}(\texttt{Per})$.

2.2.5 Magnus relations and representability

The goal of the following two sections is to show that arithmetic properties of the permutation pattern algebra that are established in [Var14] are actually general properties of pattern algebras.

Proposition 2.2.14 (Magnus inversions)**.** Let h be a combinatorial presheaf, and consider the maps $M, N : \operatorname{span} \mathcal{G}(h) \to \operatorname{span} \mathcal{G}(h)$ given in the basis elements $a \in \mathcal{G}(h)$ by

$$M : a \mapsto \sum_{b \in \mathcal{G}(h)} \mathbf{p}_b(a) b \,,$$

$$N : a \mapsto \sum_{b \in \mathcal{G}(h)} (-1)^{|a|+|b|} \, \mathbf{p}_b(a) b \,,$$

and extended linearly to $\operatorname{span} \mathcal{G}(h)$. Then the maps M, N are inverses of each other.

This result was already known in the context of words in [Hof00] and [AM10] and in the context of permutations in [Var14].

Proof. We start by proving a relation on pattern functions. Let $a \in h[I], c \in \mathcal{G}(h)$. Then we claim that

$$\sum_{b \in \mathcal{G}(h)} (-1)^{|b|} \, \mathbf{p}_b(a) \, \mathbf{p}_c(b) = (-1)^{|a|} \mathbb{1}[a \in c] \,.$$

Indeed, for each pair of sets (J, B) such that $J \subseteq B \subseteq I$ and $a|_J \in c$ it corresponds an object $b = a|_B$, and the patterns B of b in a, and J of c in b. Observe that this is a bijective correspondence, so we have

$$\sum_{b \in \mathcal{G}(h)} (-1)^{|b|} \, \mathbf{p}_b(a) \, \mathbf{p}_c(b) = \sum_{\substack{J \subseteq I \\ a|_J \in c}} \sum_{J \subseteq B \subseteq I} (-1)^{|B|} = (-1)^{|I|} \mathbb{1}[a|_I \in c] = (-1)^{|a|} \mathbb{1}[a \in c] \,.$$

It follows that

$$\begin{aligned} N(M(a)) =& N\left(\sum_{b\in\mathcal{G}(h)} \mathbf{p}_b(a) b\right) = \sum_{b\in\mathcal{G}(h)} \mathbf{p}_b(a) \sum_{c\in\mathcal{G}(h)} (-1)^{|b|+|c|}\, \mathbf{p}_c(b) \\ =& \sum_{c\in\mathcal{G}(h)} c(-1)^{|c|} \sum_{b\in\mathcal{G}(h)} (-1)^{|b|}\, \mathbf{p}_b(a)\, \mathbf{p}_c(b) \\ =& \sum_{c\in\mathcal{G}(h)} c(-1)^{|c|}(-1)^{|a|} \mathbb{1}[a=c] = a\,, \end{aligned}$$

and that

$$\begin{aligned} M(N(a)) =& M\left(\sum_{b\in\mathcal{G}(h)} (-1)^{|a|+|b|}\, \mathbf{p}_b(a) b\right) = \sum_{b\in\mathcal{G}(h)} \mathbf{p}_b(a) \sum_{c\in\mathcal{G}(h)} (-1)^{|a|+|b|}\, \mathbf{p}_c(b) \\ =& \sum_{c\in\mathcal{G}(h)} c(-1)^{|a|} \sum_{b\in\mathcal{G}(h)} (-1)^{|b|}\, \mathbf{p}_b(a)\, \mathbf{p}_c(b) \\ =& \sum_{c\in\mathcal{G}(h)} c(-1)^{|a|}(-1)^{|a|} \mathbb{1}[a=c] = a\,, \end{aligned}$$

as desired. □

2.2.6 The Sweedler dual and pattern algebras

Let A be an algebra over a field $\mathbb{K}$. We denote by A^* the vector space $\mathcal{F}(A,\mathbb{K})$. Then the Sweedler dual A° is defined in [Swe69] as

$$A^\circ := \{g \in A^* \,|\, \ker g \text{ contains a cofinite ideal}\}\,,$$

where a *cofinite ideal* J of A is an ideal such that A/J is a finite dimensional vector space over $\mathbb{K}$. There, it is established that A° is a coalgebra, where the coproduct map is the transpose of the product map.

Consider the following right action of A on A^*: for $f \in A^*$ and $a, b \in A$,

$$(f \cdot b)(a) := f(ab)\,. \tag{2.10}$$

A description of A° is given in [Swe69, Proposition 6.0.3], as all *representable* elements $f \in A^*$, that is all f such that the vector space $\{f \cdot b\}_{b\in A}$ is finite dimensional.

Let $(h, *, 1)$ now be an associative presheaf, and consider the algebra generated by $\cdot$ in $\mathcal{G}(h)$:

$$\mathrm{Alg}(h) := \mathrm{span}(\mathcal{G}(h), \cdot)\,.$$

Then $\mathcal{A}(h) \subseteq \mathrm{Alg}(h)^* = \mathcal{F}(\mathcal{G}(h), \mathbb{Q})$. In fact, the following proposition guarantees that we have $\mathcal{A}(h) \subseteq \mathrm{Alg}(h)^\circ$. Remark that the coproduct in $\mathcal{A}(h)$ is precisely the transpose of the multiplication in $\mathrm{span}(\mathcal{G}(h), \cdot)$, therefore $\mathcal{A}(h) \subseteq \mathrm{Alg}(h)^\circ$ is an inclusion of coalgebras.

Proposition 2.2.15 (Pattern algebra and the sweedler dual)**.** Let h be an associative presheaf. Then, its pattern algebra satisfies $\mathcal{A}(h) \subseteq \mathrm{Alg}(h)^\circ$.

This result generalizes the one from [Var14], where it is shown that any pattern function on permutations is representable.

Proof. We claim that each pattern function is representable, which concludes the proof according to [Swe69, Proposition 6.0.3]. In fact, for $a, b, c \in \mathcal{G}(h)$:

$$\begin{aligned} \mathbf{p}_a \cdot b\,(c) = \mathbf{p}_a(b \cdot c) &= \Delta\, \mathbf{p}_a(b, c) \\ &= \sum_{\substack{a_1, a_2 \in \mathcal{G}(h) \\ a = a_1 \cdot a_2}} \mathbf{p}_{a_1}(b)\, \mathbf{p}_{a_2}(c) \end{aligned}$$

That is, $\mathbf{p}_a \cdot b = \sum_{a = a_1 \cdot a_2} \mathbf{p}_{a_1}(b)\, \mathbf{p}_{a_2}$. It follows that

$$\mathrm{span}\{\mathbf{p}_a \cdot b\}_{b \in \mathcal{G}(h)} \subseteq \mathrm{span}\{\mathbf{p}_{a_2}\}_{\substack{a_2 \in \mathcal{G}(h) \\ |a_2| \leq |a|}},$$

which is finite dimensional. □

2.3 Freeness of commutative presheaves

We start this section with a discussion on factorization theorems on combinatorial presheaves. We will observe that the factorizations of objects in connected associative presheaves into irreducibles is unique up to some possible commutativity. this is a general fact on associative presheaves, and is a central point in establishing freeness of any pattern Hopf algebra so far in the literature.

We also dedicate some attention to commutative presheaves. An almost immediate consequence of the general fact discussed above is that the pattern algebra of a commutative presheaf is free.

We also explore specific combinatorial presheaves that can be endowed with a commutative structure. The main examples are graphs, already studied in [Whi32], marked graphs (see Section 2.3.3), set partitions (see Section 2.3.4), simplicial complexes and posets.

2.3.1 Relations in general connected associative presheaves

Consider a connected associative presheaf $(h, *, 1)$. In this section we will not assume that h is commutative. Recall that for objects $a \in h[I], b \in h[J]$, if $A \subseteq I \sqcup J$ then $(a * b)|_A = a|_{A\cap I} * a|_{A\cap J}$, as described in Observation 2.1.5. We recall as well that an object $t \in h[I]$ is called *irreducible* if $t \neq 1$ and if $t = a * b$ only has trivial solutions. We define as well an irreducible coinvariant in $\mathcal{G}(h)$.

Definition 2.3.1 (Set composition and set partition)**.** Let I be a finite set. A *set composition* of I is a list $(B_1, \dots B_k)$, that can also be written as $B_1|\dots|B_k$, of pairwise disjoint nonempty subsets of I, such that $I = \bigcup_i B_i$. We denote by Π_I the family of set compositions of I. A *set partition* of I is a family $\boldsymbol{\pi} = \{I_1, \dots, I_k\}$ of pairwise disjoint nonempty sets, such that $\bigcup_i I_i = I$. We write Σ_I for the family of set partitions of I.

If $\vec{\boldsymbol{\pi}} \in \Pi_I$ is a set composition, we can define its underlying set partition of I by disregarding the order of the list. We denote it by $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}})$.

Definition 2.3.2 (Factorization of objects and coinvariants)**.** Consider an associative presheaf h that is connected, and an object $o \in h[I]$. A factorization of o is a word $(x_1, \dots, x_k)$ of objects such that $x_1 * \dots * x_k = o$.

A factorization $(x_1, \dots, x_k)$ of o is said to be *into irreducibles* when each x_i is an irreducible object for $i = 1, \dots, l$.

A factorization of a coinvariant $a \in \mathcal{G}(h)$ is a decomposition of the form $a = s_1 \cdots s_k$. This factorization is said to be into *irreducibles* if each s_i is *irreducible*.

It is clear to see that an object is irreducible if and only if its coinvariant is irreducible.

To a factorization $(x_1, \dots, x_k)$ of $o \in h[I]$, it corresponds a set composition $\vec{\boldsymbol{\pi}} = (I_1, \dots, I_k) \models I$, where $x_i \in h[I_i]$. This is indeed a set composition of I by definition of $*$. This correspondence is injective, that is to any two distinct factorizations of o it corresponds distinct underlying set compositions. Indeed, assume otherwise, that to the factorizations $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ it corresponds the same set composition $(I_1, \dots, I_k)$, Then, for any $i = 1, \dots, k$ we have

$$\begin{aligned} o|_{A_i} &= x_1|_{A_1 \cap A_i} * \dots * x_l|_{A_l \cap A_i} \\ &= x_1|_{\emptyset} * \dots * x_i|_{A_i} * \dots * x_l|_{\emptyset} \\ &= 1 * \dots * x_i * \dots * 1 = x_i \,, \end{aligned}$$

and similarly we have that $o|_{A_i} = y_i$, so that $x_i = y_i$.

Conversely, not all set compositions yield a factorization, and the irreducible elements are precisely the ones where only the trivial set composition with one block yields a factorization.

Theorem 2.3.3. Let h be an associative presheaf and $o \in h[I]$ an object. If $\vec{\boldsymbol{\pi}}_1, \vec{\boldsymbol{\pi}}_2$ are factorizations into irreducibles of o, then their underlying set partitions $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}_1)$ and $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}_2)$ are the same.

In particular, the number of irreducible factors $j(o)$ and the multiset of irreducible factors $\mathfrak{fac}(o)$ of an object are well defined and do not depend on the factorization into irreducibles at hand.

Proof. Suppose that o has two distinct factorizations

$$o = l_1 * \ldots * l_k = r_1 * \ldots * r_s \,, \tag{2.11}$$

where $\vec{\boldsymbol{\pi}}_1 = A_1 | \ldots | A_k$ and $\vec{\boldsymbol{\pi}}_2 = B_1 | \ldots | B_s$ are set compositions of X such that $l_i = o|_{A_i}$ and $r_i = o|_{B_i}$. Note that

$$l_j = o|_{A_j} = r_1|_{B_1 \cap A_j} * \ldots * r_s|_{B_s \cap A_j} \,.$$

Because l_j is irreducible, for each j there is exactly one i such that $B_i \cap A_j \neq \emptyset$, so $A_j \subseteq B_i$, and $\boldsymbol{\tau}(\vec{\boldsymbol{\pi}}_1)$ is coarser than $\boldsymbol{\tau}(\vec{\boldsymbol{\pi}}_2)$. By a symmetrical argument, we obtain that $\boldsymbol{\tau}(\vec{\boldsymbol{\pi}}_1)$ is finer than $\boldsymbol{\tau}(\vec{\boldsymbol{\pi}}_2)$, so we conclude that $\boldsymbol{\tau}(\vec{\boldsymbol{\pi}}_1) = \boldsymbol{\tau}(\vec{\boldsymbol{\pi}}_2)$. It follows that the number of factors and the multiset of factors are well defined. □

This implies the following for factorizations on $\mathcal{G}(h)$:

Corollary 2.3.4. Consider an associative presheaf $(h, *, 1)$, together with the usual product in $\mathcal{G}(h)$, denoted by $\cdot$. Consider also $a \in \mathcal{G}(h)$. If $a = x_1 \cdots x_l = s_1 \cdots s_s$ are two factorizations into irreducibles, then the multisets $\{x_1, \ldots, x_l\}$ and $\{s_1, \ldots, s_t\}$ coincide.

In particular, the number of irreducible factors $j(a)$ and the multiset of irreducible factors $\mathfrak{fac}(a)$ of a coinvariant are well defined and do not depend on the factorization into irreducibles at hand.

The algebraic structure of $\text{span}(\mathcal{G}(h), \cdot)$ is determined by the set $\mathcal{I}(h)$ together with the relations between irreducible elements. Corollary 2.3.4 imposes restrictions on the possible relations. This is different from the case with groups: these can be described by generators and relations, but these relations have more flexibility.

The following consequence is immediate:

Corollary 2.3.5. Let h be an associative presheaf. Then, the n-th component of its coradical filtration is precisely $\mathrm{span}\{\mathbf{p}_a \,|j(a) \leq n\}$.

Given an alphabet Ω, denote the set of words on Ω by $\mathcal{W}(\Omega)$.

Problem 2.3.6 (Factorization theorems in associative presheaves)**.** Given an associative presheaf $(h, *, 1)$, describe $\mathcal{E}(h)$, the collection of fibers of the map

$$\Pi : \mathcal{W}(\mathcal{I}(h)) \to \mathcal{G}(h)\,,$$

defined by taking the product of the letters of a word in $\mathcal{G}(h)$.

The fibers of this map, that is the sets of words $\Pi^{-1}(a)$, correspond to the different factorizations of an object $a \in \mathcal{G}(h)$. In general, according to Corollary 2.3.4, a fiber consists of a set of words of irreducible elements that result from one another by permuting its letters. Whenever h is a commutative presheaf, Corollary 2.3.8 tells us that the fibers are as big as possible, restricted to Corollary 2.3.4. This means that, in this case, each fiber is a set of words resulting from a permutation of a word in $\mathcal{W}(\mathcal{I}(h))$.

Take the example of the combinatorial presheaf $\texttt{Per}$, where no non-trivial rearrangement of the irreducible factors of a factorization of an object yields a distinct factorization of the same object. In this example, the fibers $\Pi^{-1}(a)$ are singletons. In the case of marked permutations, in Theorem 3.2.7 we show that only transpositions of specific irreducible marked permutations remain factorizations of the same coinvariant a.

Remark 2.3.7. It can be seen that the freeness proof in Theorem 2.3.9 depends solely on the corresponding unique factorization theorem, that is on $\mathcal{E}(h)$. That is also the case on the proof given in [Var14] for the presheaf on permutations, and the proof below for marked permutations.

This motivates Conjecture 1.6.4, as it seems that the freeness of the pattern Hopf algebra only depends on the description of the fibers $\mathcal{E}(h)$. In this way, for instance, we can immediately see that the pattern Hopf algebra $\mathcal{A}(\texttt{SComp})$, defined below, is free. This follows because it has a unique factorization theorem of the type of the one in the associative presheaf on permutations.

2.3.2 Proof of freeness on commutative presheaves

Recall that a commutative presheaf is an associative presheaf $(h, *, 1)$ such that $*_{A,B} = *_{B,A} \circ \mathrm{twist}_{A,B}$, see Definition 2.1.4. In the case of commutative presheaves, we have that any rearrangement of a factorization of an object o yields another factorization of o. For this reason, in the context of commutative presheaves, a set partition is also referred to as a factorization, and we have the following.

Corollary 2.3.8. Let $(h, *, 1)$ be a connected commutative presheaf, and $a \in \mathcal{G}(h)$ an object. Then, a has a unique factorization into irreducibles $l_1, \dots, l_{j(a)} \in \mathcal{I}(h)$ up to commutativity of factors. Equivalently, if $a \in h[X]$, there is a unique set partition that corresponds to a factorization of o into irreducibles.

Theorem 2.3.9 (Freeness of pattern algebras with commutative products)**.** Let $(h, *, 1)$ be a connected commutative presheaf. Consider $\mathcal{I}(h) \subseteq \mathcal{G}(h)$ the family of irreducible elements of h.

Then $\mathcal{A}(h)$ is free commutative, and $\{\mathbf{p}_\iota \,|\, \iota \in \mathcal{I}(h)\}$ is a set of free generators of $\mathcal{A}(h)$.

Proof. We will show that the family

$$\left\{ \prod_{\iota \in \mathcal{L}} \mathbf{p}_\iota \middle| \mathcal{L} \text{ multiset of elements in } \mathcal{I}(h) \right\}, \tag{2.12}$$

is a basis for $\mathcal{A}(h)$. The proof follows the strategy described in Section 2.1.3, by building an order $\leq$ in $\mathcal{G}(h)$ that is motivated in the unique factorization theorem in Theorem 2.3.3.

Define the following partial strict order $<_p$ in $\mathcal{G}(h)$: we say that $\alpha <_p \beta$ if:

- $|\alpha| < |\beta|$, or;
- $|\alpha| = |\beta|$ and $j(\alpha) < j(\beta)$.

In this way, $\leq_p$ is the order that we use to establish freeness.

Consider $\alpha \in \mathcal{G}(h)$ with $\alpha = \iota_1 \cdots \iota_{j(\alpha)}$ its unique factorization into irreducibles. Then we claim

$$\prod_{i=1}^{k} \mathbf{p}_{\iota_i} = \sum_{\beta \in \mathcal{G}(h)} \binom{\beta}{\iota_1, \dots, \iota_{j(\alpha)}} \mathbf{p}_\beta = \sum_{\beta \leq_p \alpha} c_\beta \, \mathbf{p}_\beta \,, \tag{2.13}$$

where $c_a \geq 1$. This concludes that (2.12) is a basis of $\mathcal{A}(h)$, and gives us the result.

Let us prove (2.13). Pick representatives l_i for ι_i such that $l_i \in h[A_i]$ and let also $a = l_1 * \cdots * l_{j(\alpha)}$ be an object. Observe that the coinvariant of a is α.

Observe that a is a quasi-shuffle of $l_1, \dots, l_{j(\alpha)}$ by considering the patterns $A_1, \dots, A_{j(\alpha)}$ (see Remark 2.1.6). Thus we have $c_\alpha \geq 1$.

To show that any term β in (2.13) with $c_\beta \neq 0$ has $\beta \leq_p \alpha$, consider the maximal $\beta \in \mathcal{G}(h)$ that is a quasi-shuffle of $l_1, \ldots, l_{j(\alpha)}$, and let b be a representative in β, such that $b \in h[Y]$. By maximality, we have that $\beta \geq \alpha$. Our goal is to show that $b \sim a$. Let $b = s_1 * \cdots * s_{j(b)}$ be the unique factorization of b into irreducibles, corresponding to the set partition $\{C_1, \ldots, C_{j(b)}\}$. Because b is a quasi-shuffle of $l_1, \ldots, l_{j(\alpha)}$, we can consider $B_1, \ldots, B_{j(\alpha)} \subseteq Y$ sets such that $\bigcup_i B_i = Y$ and

$$b|_{B_i} \sim l_i \quad \text{for } i = 1, \ldots, j(\alpha)\,.$$

In particular, we have that

$$l_i \sim b|_{B_i} = s_1|_{C_1 \cap B_i} * \cdots * s_{j(b)}|_{C_{j(b)} \cap B_i}\,,$$

for $i = 1, \ldots, j(\alpha)$.

By indecomposability of l_i, we have that, for each i, there is exactly one j such that $C_j \cap B_i \neq \emptyset$. From $\sqcup_j C_j = \bigcup_i B_i = Y$, we get that each B_i is contained in some C_j. So we can define a map $f : [j(\alpha)] \to [j(b)]$ such that $B_i \subseteq C_{f(i)}$. Thus, we have that $C_i = \bigcup_{k \in f^{-1}(i)} B_k$.

First observe that $|a| = \sum_i |A_i| = \sum_i |l_i| = \sum_i |B_i|$ and $|b| = \sum_j |C_j|$ but $|b| = |Y| \leq \sum_j \sum_{i \in f^{-1}(j)} |B_i| = |a|$. However, from $b \geq_p a$ we have that $|b| \geq |a|$ and thus we have $|b| = |a|$ and that the family $\{B_1, \ldots, B_{j(\alpha)}\}$ is disjoint. Further, f is a surjection, so we immediately have that $j(\alpha) \geq j(b)$ and because $b \geq_p a$, we must have an equality. Thus, f is a bijective map and we conclude that $C_{f(i)} = B_i$ for each $i = 1, \ldots, j(\alpha) = j(b)$.

We then conclude that

$$s_i = b|_{B_i} = b|_{C_{f(i)}} \sim l_{f(i)}\,,$$

and so, by commutativity,

$$b = s_1 * \cdots * s_{j(b)} = l_1 * \cdots * l_{j(\alpha)} = a\,,$$

as desired. □

2.3.3 Marked graphs

Definition 2.3.10 (Marked graphs and two products)**.** For a finite set I, a marked graph G^* on I is a graph on the vertex set $I \sqcup \{*\}$. This defines a combinatorial presheaf `MGr` via the usual notion of relabeling and induced subgraphs.

We can further endow the combinatorial presheaf MGr with two different associative presheaf structures.

First, the *joint union*, $\vee$, which is defined as follows: If $G_1^* \in \mathtt{MGr}[I], G_2^* \in \mathtt{MGr}[J]$ with $I \cap J = \emptyset$, then $G_1^* \vee_{I,J} G_2^*$ has no edges between I and J, and the marked vertices are merged.

The second product, the *inflation product* $\star$ is defined as follows: If $G_1^* \in \mathtt{MGr}[I], G_2^* \in \mathtt{MGr}[J]$ with $I \cap J = \emptyset$, then two vertices $i \in I, j \in J$ are connected in $G_1^* \star_{I,J} G_2^*$ if j and $*$ are connected in G_1^*. The unit of both products is the marked graph 1 with a unique vertex and no edges.

Remark 2.3.11. The graphs that are irreducible with respect to the $\vee$ product are the graphs G^* such that the graph resulting from removing the marked vertex and its incident edges is a connected graph. In this case, we say that G^* is $\vee$-connected.

In Fig. 2.1 we have an example of a $\vee$ -connected marked graph and a $\vee$-disconnected marked graph.

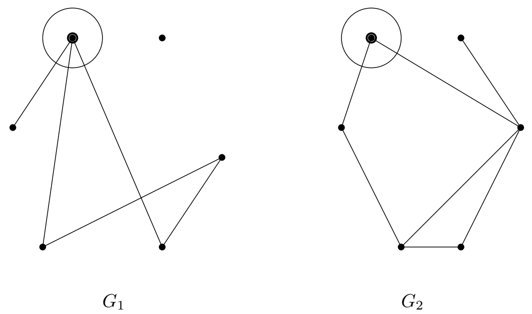


FIGURE 2.1: A marked graph G_1 with three $\vee$-connected components and a $\vee$-connected graph G_2.

Observe that $G_1^* \vee_{I,J} G_2^* = G_2^* \vee_{J,I} G_1^*$, so $\vee$ is a commutative operation, whereas $\star$ is not. It follows from Theorem 2.3.9 that $\mathcal{A}(\mathtt{MGr})$ is a free algebra. This is something that cannot be done directly with the inflation product of marked graphs. Indeed, a unique factorization theorem on this associative presheaf has not yet been found, and only small irreducible marked graphs can be constructed (see Fig. 2.2). It is worthwhile to observe that the fibers under the map Π described in Problem 2.3.6 are non-trivial, as an example of a non-trivial relation can be seen in Fig. 2.3

2.3.4 Set partitions

We define here associative presheaves on set partitions SPart and set compositions SComp, and show that *Sym*, the Hopf algebra of symmetric functions, is the pattern

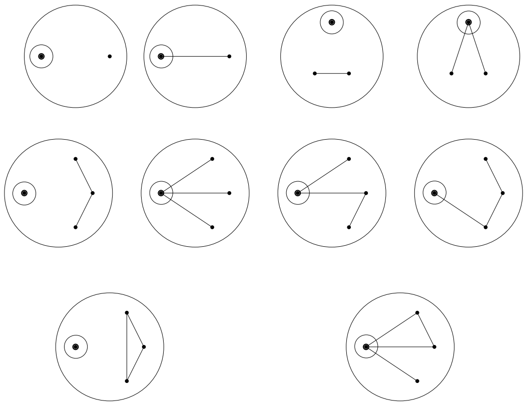


FIGURE 2.2: The ⋆-irreducible marked graphs of size up to three.

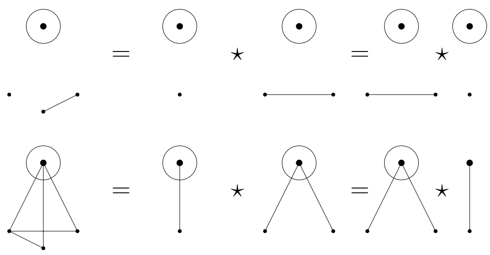


FIGURE 2.3: The first relations between irreducible marked graphs over ⋆.

Hopf algebra on set partitions. The functoriality of $\mathcal{A}$ also gives us a Hopf algebra morphism $Sym \to \mathcal{A}(\texttt{SComp})$.

Definition 2.3.12 (The presheaf on set partitions)**.** If $\boldsymbol{\pi}$ is a set partition of I and $J \subseteq I$, then $\boldsymbol{\pi}|_J = \{I_1 \cap J, \dots, I_k \cap J\}$ is a set partition of J, after disregarding the empty sets. This defines a presheaf structure $\texttt{SPart}$ with $\texttt{SPart}[I] = \Sigma_I$, the family of set partitions of I.

We further endow $\texttt{SPart}$ with an associative structure $\sqcup$ as follows: if $\boldsymbol{\pi} = \{I_1, \dots, I_q\}$, $\boldsymbol{\tau} = \{J_1, \dots, J_p\}$ are set partitions of the disjoint sets I, J, respectively, let $\boldsymbol{\pi} \sqcup \boldsymbol{\tau} = \{I_1, \dots, I_q, J_1, \dots, J_p\}$ be a set partition of $I \sqcup J$. It is straightforward to observe that $(\texttt{SPart}, \sqcup, \emptyset)$ is a commutative connected presheaf.

Note that by Theorem 2.3.9, the pattern Hopf algebra $\mathcal{A}(\texttt{SPart})$ is free and the generators correspond to the irreducible elements of $\mathcal{G}(\texttt{SPart})$. These correspond to the set partitions with only one block, and up to relabeling there is a unique such set partition of each size. We write $\mathcal{I}(\texttt{SPart}) = \{\{[n]\} | n \geq 1\}$.

Proposition 2.3.13. Let $\zeta : \mathcal{A}(\texttt{SPart}) \to Sym$ be the unique algebra morphism mapping $\zeta : \mathbf{p}_{\{[n]\}} \mapsto p_n$, where p_n is the power sum symmetric function. This defines a Hopf algebra isomorphism.

Proof. As ζ sends a free basis to a free basis, it is an isomorphism of algebras. Furthermore, we observe that both $\mathbf{p}_{\{[n]\}}$ and p_n are a primitive element in their respective Hopf algebras, so the described map is a bialgebra morphism. Because the antipode is unique, it must send the antipode of $\mathcal{A}(\texttt{SPart})$ to the one of Sym. Thus, this is a Hopf algebra isomorphism. □

Definition 2.3.14 (The presheaf on set compositions)**.** Let I be a finite set, and recall the definition of a set composition in Definition 2.3.1. If $J \subseteq I$ and $\vec{\boldsymbol{\pi}} = (I_1, \dots, I_k)$ is a set composition of I, then $\vec{\boldsymbol{\pi}}|_J = (I_1 \cap J, \dots, I_k \cap J)$ is a set composition of J, after disregarding the empty sets. This defines a presheaf structure $\texttt{SComp}$ with $\texttt{SComp}[I] = \Pi_I$, the family of set compositions of I.

We further endow $\texttt{SComp}$ with an associative structure $\sqcup$ as follows: if $\vec{\boldsymbol{\pi}} = (I_1, \dots, I_q)$, $\vec{\boldsymbol{\tau}} = (J_1, \dots, J_p)$ are set partitions of the disjoint sets I, J, respectively, let $\vec{\boldsymbol{\pi}} \sqcup \vec{\boldsymbol{\tau}} = (I_1, \dots, I_q, J_1, \dots, J_p)$ be a set composition of $I \sqcup J$.

It is straightforward to observe that $(\texttt{SComp}, \sqcup, \emptyset)$ is an associative connected presheaf. Further, we can also observe that the map $\boldsymbol{\lambda} : \texttt{SComp} \Rightarrow \texttt{SPart}$ is an associative presheaf morphism.

From the map $\boldsymbol{\lambda} : \texttt{SComp} \Rightarrow \texttt{SPart}$ we get a Hopf algebra morphism

$$\mathcal{A}(\boldsymbol{\lambda}) : Sym \to \mathcal{A}(\texttt{SComp}) .$$

Observe that $\mathcal{A}(\texttt{SComp})$ is a free algebra, because it has a unique factorization theorem of the same type of permutations under the $\oplus$ product, so according to Remark 2.3.7 the proof in [Var14] holds also in this associative presheaf. This is also the case for the well known Hopf algebra $QSym$, where it was established that it is free in [Haz01], and when we regard both Hopf algebras as filtered Hopf algebras, the enumeration of generators for each degree coincide with the number of Lyndon words of a given size.

Conjecture 2.3.15 ($QSym$ conjecture)**.** Consider the associative presheaf $(\texttt{SComp}, \sqcup, \emptyset)$ introduced in Definition 2.3.14. Then the pattern algebra $\mathcal{A}(\texttt{SComp})$ is isomorphic to $QSym$.

Chapter 3

Pattern Hopf algebra structure on marked permutations

This chapter is based on the article [Pen20], which is submitted for publication. The work in [Pen20] is split into this chapter and Chapter 2.

In this section we consider the algebra structure of $\mathcal{A}(\texttt{MPer})$, and show that this pattern algebra on marked permutations is freely generated. This will be done using a factorization theorem on marked permutations on the inflation product.

Our strategy is as follows: we describe a unique factorization of marked permutations with the inflation product, in Corollary 3.2.11. This unique factorization theorem describes all possible factorizations of a marked permutation into irreducibles.

We further consider the Lyndon words on the alphabet of irreducible marked permutations, as introduced in [CFL58]. This leads us to a notion of stable Lyndon marked permutations $\mathcal{L}_{SL}$, in Definition 3.3.7. Finally, we prove the following result, which is the main theorem of this section, as a corollary of Theorem 3.4.1:

Theorem 3.0.1. The algebra $\mathcal{A}(\texttt{MPer})$ is freely generated by $\{\mathbf{p}_{\iota^*} | \iota^* \in \mathcal{L}_{SL}\}$.

In the end of this section we compute the dimension of the space of primitive elements of the pattern Hopf algebra on marked permutations. This, according to Proposition 2.2.10, can be done by enumerating the irreducible elements in $\mathcal{G}(\texttt{MPer})$.

This chapter is organized as follows: we start in Section 3.1 by describing the presheaf structures on permutations and marked permutations. In Section 3.2 and in Section 3.3 we establish a unique factorization theorem in marked permutations. In Section 3.4 we state and prove the main theorem of this section. The proofs of technical lemmas used in these sections are left to Sections 3.5 and 3.6. Finally, in Section 3.7, we enumerate the irreducible marked permutations.

3.1 Non-commutative presheaves

We return to the presheaf on permutations. This is an example of a non-commutative associative presheaf.

Example 3.1.1 (Permutations and their pattern Hopf algebra)**.** To the presheaf $\texttt{Per}$ it corresponds a pattern algebra $\mathcal{A}(\texttt{Per})$ as discussed above. We can further consider $\texttt{Per}$ with a monoid structure via the direct sum of permutations $\oplus$, defined as follows: Suppose that $\pi \in \texttt{Per}[I], \tau \in \texttt{Per}[J]$ are two permutations based on the disjoint sets I, J, respectively. The permutation $\pi \oplus \tau \in \texttt{Per}[I \sqcup J]$ is the pair of total orders $(\leq_P^{\oplus}, \leq_V^{\oplus})$ extending both of the respective orders from π, τ to $I \sqcup J$ by forcing that $i \leq_P^{\oplus} j$ and $i \leq_V^{\oplus} j$ for any $i \in I, j \in J$. Correspondingly, the diagram of $\pi \oplus \tau$ results from the ones from π, τ as follows"

$$\pi \oplus \tau = \begin{array}{|c|c|}\hline & \tau \\ \hline \pi & \\ \hline \end{array}\ .$$

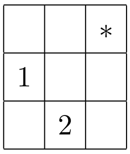
We note that this is not a commutative presheaf: in general, $\pi \oplus \tau$ is a different permutation than $\tau \oplus \pi$.

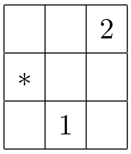
As mentioned above, the pattern Hopf algebra on permutations is the one discussed by Vargas in [Var14], where it is shown that it is free. There, free generators were constructed. These generators correspond to Lyndon words of $\oplus$-indecomposable permutations, see [CFL58] for an introduction to combinatorics of Lyndon words.

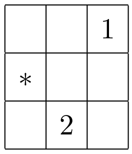
In this chapter we explore other associative presheaves that are non-commutative. Taking the presheaf on permutations as our starting point, we wish to study monoidal structures that are more complex than the $\oplus$ product, but still allow to establish the freeness property. We suggest the presheaf on marked permutations, which is equipped with the inflation product. This product is motivated by the inflation procedure on permutations described in [AAK03].

In this way, the presheaf on marked permutations introduced below will be one main focus of this chapter: Chapter 3 will be dedicated to the freeness problem on this presheaf, as well as enumerating the dimension of the primitive space of the pattern Hopf algebra.

Example 3.1.2 (Marked permutations and their pattern Hopf algebra)**.** A marked permutation π^* on I is a pair of orders $(\leq_P, \leq_V)$ on the set $I \sqcup \{*\}$. Intuitively, this gives us a rearrangement of the elements of $I \sqcup \{*\}$, where one element is special and marked. The relabelings and restriction maps are the natural ones borrowed from orders, giving us a combinatorial presheaf, that we call MPer. We can represent a marked permutation in a diagram, as we do for permutations. Note that in this case the marked element $*$ never changes position after relabelings. Take for instance the marked permutations $\pi^* = (1 <_P 2 <_P *, 2 <_V 1 <_V *)$, $\tau^* = (* <_P 1 <_P 2, 1 <_V * <_V 2)$, and $\sigma^* = (* <_P 2 <_P 1, 2 <_V * <_V 1)$. Observe that there is no isomorphism between π^* and τ^*, whereas there is one between τ^* and σ^*, via the relabeling $1 \mapsto 2, 2 \mapsto 1$.

$\pi^* =$ 

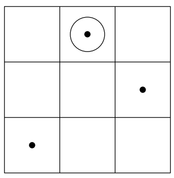
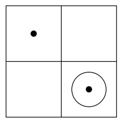
$\tau^* =$ 

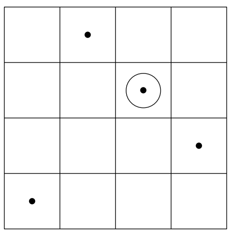
$\sigma^* =$ 

.

In this way, there are $((n+1)!)^2$ many elements in $\texttt{MPer}[n]$. Up to relabeling, we can represent a marked permutation as a diagram with one dot in each column and row, where a particular dot is the distinguished element $*$. Therefore, $\mathcal{G}(\texttt{MPer})$ has $(n+1) \times (n+1)!$ many isomorphism classes of marked permutations of size n.

Definition 3.1.3 (Inflation product)**.** The inflation product $\star$ in marked permutations is defined as follows: Given two marked permutations $\tau^* \in \texttt{MPer}[I]$ and $\pi^* \in \texttt{MPer}[J]$ with I, J disjoint sets, the inflation product $\tau^* \star \pi^* \in \texttt{MPer}[I \sqcup J]$ is a marked permutation resulting from replacing in the diagram of τ^* the marked element with the diagram of π^*. Here is an example:

$\tau^* =$ 
$\pi^* =$ 

$\tau^* \star \pi^* =$ 
.

Remark 3.1.4. Note that if $\pi^* \star \tau^* = \pi^* \star \sigma^*$, then $\tau^* = \sigma^*$. Similarly, if $\tau^* \star \pi = \sigma^* \star \pi^*$, then $\tau^* = \sigma^*$.

It is straightforward to observe that this is an associative presheaf with $\texttt{MPer}[\emptyset]$, where the unit is the unique marked permutation on $\emptyset$, denoted by $\bar{1}$. We call it the *presheaf* *of marked permutations*. Hence, from Theorem 3.1.5, the algebra $\mathcal{A}(\texttt{MPat})$ is indeed a Hopf algebra. This is not a commutative presheaf, so Theorem 1.6.2 does not apply. However, we still have the following:

Theorem 3.1.5 (Freeness of $\mathcal{A}(\texttt{MPat})$)**.** The pattern algebra $\mathcal{A}(\texttt{MPat})$ is free.

To establish the freeness of $\mathcal{A}(\texttt{MPat})$ we present a unique factorization theorem on marked permutations with the inflation product in Corollary 3.2.11. This is an analogue of [AAK03, Theorem 1] for the inflation product on marked permutations. With it, we find generators of the algebra on marked permutations, and use tools from word combinatorics, specifically the Lyndon factorization of words in [CFL58], to show that these generators are free generators.

In fact, Lyndon words are commonly used to establish the freeness of algebras. Examples are the shuffle algebra in [Rad79] (see also [GR14, Chapter 6]), the algebra of quasisymmetric functions in [Haz01, Theorem 8.1], and the algebra on word quasisymmetric functions in non-commutative variables, in [BZ09].

In Chapter 3 we also enumerate the dimension of the primitive space of the pattern Hopf algebra $\mathcal{A}(\texttt{MPer})$, which corresponds to enumerating the marked permutations that are irreducible with respect to the inflation product.

3.2 Unique factorizations

We work on the combinatorial presheaf of permutations $(\texttt{Per}, \oplus, \emptyset)$ and on the combinatorial presheaf of marked permutations $(\texttt{MPer}, \star, \bar{1})$, introduced in Example 1.5.2 and Definition 2.1.9.

Definition 3.2.1 (Decomposability on the operations $\oplus$ and $\ominus$)**.** We say that a permutation is $\oplus$-indecomposable if it has no non-trivial decomposition of the form $\tau_1 \oplus \tau_2$, and $\oplus$-decomposable otherwise. We say that a marked permutation is $\oplus$-indecomposable if it has no decomposition of the form $\tau \oplus \pi^*$ or $\pi^* \oplus \tau$, where π^* is a marked permutation and τ is a non-trivial permutation, and $\oplus$-decomposable otherwise. Similar definitions hold for $\ominus$. A permutation (resp. a marked permutation) is said to be *indecomposable* if it is both $\oplus$ and $\ominus$-indecomposable, and is said to be *decomposable* otherwise.

We remark that a permutation is $\oplus$-indecomposable whenever it is irreducible on the associative presheaf $(\texttt{Per}, \oplus, \emptyset)$ according to Definition 2.2.9. For marked permutations, Definition 2.2.9 specializes to the definition of irreducible marked permutations as follows:

Definition 3.2.2 (Irreducible marked permutations)**.** A marked permutation π^* is called *irreducible* if any factorization $\pi^* = \tau_1^* \star \tau_2^*$ has either $\tau_1^* = \bar{1}$ or $\tau_2^* = \bar{1}$.

Example 3.2.3. Examples of irreducible marked permutations include $\bar{1}423, 23\bar{1}$ and $31\bar{4}2$, see Fig. 3.1. These marked permutations are respectively an $\oplus$-decomposable, $\ominus$-decomposable and an indecomposable irreducible marked permutation.

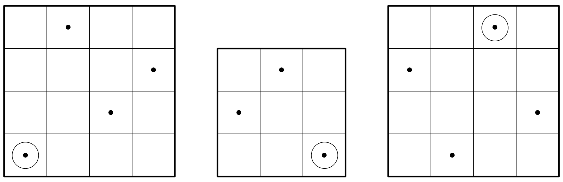


FIGURE 3.1: Irreducible marked permutations

Remark 3.2.4 (decomposable irreducible marked permutations)**.** If π is a permutation that is $\oplus$-indecomposable, then $\bar{1} \oplus \pi$ and $\pi \oplus \bar{1}$ are irreducible marked permutations. Similarly, if τ is an $\ominus$-indecomposable permutation, then $\bar{1} \ominus \tau$ and $\tau \ominus \bar{1}$ are irreducible marked permutations.

These are precisely the *decomposable* (resp. $\oplus$*-decomposable*, $\ominus$*-decomposable*) *irreducible marked permutations.*

These decomposable irreducible marked permutations play an important role in the description of a free basis of $\mathcal{A}(\texttt{MPer})$, because they are the only ones that get in the way of a unique factorization theorem for the inflation product. In the following we carefully unravel all these issues.

Remark 3.2.5 ($\oplus$-relations and $\ominus$-relations)**.** Consider τ_1, τ_2 $\oplus$-indecomposable permutations. Then we have the following relations, called $\oplus$*-relations*

$$(\bar{1} \oplus \tau_1) \star (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) \star (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 \,.$$

Consider now π_1, π_2 permutations that are $\ominus$-indecomposable. Then we have the following relation, called $\ominus$*-relations*

$$(\bar{1} \ominus \pi_1) \star (\pi_2 \ominus \bar{1}) = (\pi_2 \ominus \bar{1}) \star (\bar{1} \ominus \pi_1) = \pi_2 \ominus \bar{1} \ominus \pi_1 \,.$$

We wish to establish in Theorem 3.2.7 that these generate all the inflation relations between irreducible marked permutations.

We define the alphabet $\Omega :=$ {irreducible marked permutations}, and consider the set $\mathcal{W}(\Omega)$ of words on Ω. This set forms a monoid under the usual concatenation of words (we denote the concatenation of two words w_1, w_2 as $w_1 \cdot w_2$). When $w \in \mathcal{W}(\Omega)$, we write w^* for the consecutive inflation of its letters. So for instance $(\bar{1}2, \bar{2}1)^* = \bar{1}2 \star \bar{2}1 = \bar{2}13$. We use the convention that the inflation of the empty word on Ω is $\bar{1}$. This defines the *star map*, a morphism of monoids $\star : \mathcal{W}(\Omega) \to \mathcal{G}(\texttt{MPer})$.

For the sake of clarity, we avoid any ambiguity on the notation a^* by using Greek letters for marked permutations, lowercase Latin letters to represent words on any alphabet, and upper case Latin letters to represent sets with an added marked element (see Definition 2.1.9).

Definition 3.2.6 (Monoidal equivalence relation on $\mathcal{W}(\Omega)$)**.** We now define an equivalence relation on $\mathcal{W}(\Omega)$. For a word $w \in \mathcal{W}(\Omega)$, if $w = (\xi_1^*, \dots, \xi_k^*)$ is such that $\xi_i^* \star \xi_{i+1}^* = \xi_{i+1}^* \star \xi_i^*$ is an $\oplus$-relation or an $\ominus$-relation, we say that $w \sim (\xi_1^*, \dots \xi_{i-1}^*, \xi_{i+1}^*, \xi_i^*, \dots, \xi_k^*)$.

We further take the transitive and reflexive closure to obtain an equivalence relation on $\mathcal{W}(\Omega)$.

We trivially have that if $w_1 \sim w_2$ and $z_1 \sim z_2$, then $w_1 \cdot z_1 \sim w_2 \cdot z_2$. This means that the quotient $\mathcal{W}(\Omega)/_\sim$ is a monoid. Further, because of Remark 3.2.5, the star map factors to a monoid morphism $\mathcal{W}(\Omega)/_\sim \to \mathcal{G}(\texttt{MPer})$.

Theorem 3.2.7. The star map $\star : w \mapsto w^*$ defines an isomorphism from $\mathcal{W}(\Omega)/_\sim$ to the monoid of marked permutations with the inflation product.

We postpone the proof of Theorem 3.2.7 to Section 3.5, and explore its consequences here.

Informally, this theorem states that any two factorizations of a marked permutation π^* into irreducible marked permutations are related by $\sim$. As a consequence, we recover the following corollary, which was already obtained in Corollary 2.3.4 in a more general context:

Corollary 3.2.8. Consider α^* a marked permutation, together with factorizations $\alpha^* = \xi_1^* \star \cdots \star \xi_k^* = \rho_1^* \star \cdots \star \rho_j^*$ into irreducible marked permutations. Then $k = j$ and $\{\xi_1^*, \dots, \xi_k^*\} = \{\rho_1^*, \dots, \rho_j^*\}$ as multisets.

We was done in Section 2.3, we define $j(\alpha^*)$ to be the number of irreducible factors in any factorization of α^* into irreducible factors, and define $\mathfrak{fac}(\alpha^*)$ as the multiset of irreducible factors of α^* in $\mathcal{G}(\texttt{MPer})$. These are well defined as a consequence of Corollary 3.2.8.

Definition 3.2.9 (Stability conditions)**.** A factorization of a marked permutation α^* into irreducible marked permutations,

$$\alpha^* = \xi_1^* \star \cdots \star \xi_j^* \,,$$

or the corresponding word $(\xi_1^*, \ldots, \xi_j^*)$ in $\mathcal{W}(\Omega)$, is said to be

- ***i*-⊕-stable** if there is no π, τ $\oplus$-indecomposable permutations such that

$$\xi_i^* = \bar{1} \oplus \pi \;\text{ and }\; \xi_{i+1}^* = \tau \oplus \bar{1}\,;$$

- ***i*-⊖-stable** if there is no π, τ $\ominus$-indecomposable permutations such that

$$\xi_i^* = \tau \ominus \bar{1} \text{ and } \xi_{i+1}^* = \bar{1} \ominus \pi\,.$$

Such factorization or word is said to be **⊕-stable** (resp. **⊖-stable**) if it is i-⊕-stable (resp. i-⊖-stable) for any $i = 1, \ldots, j-1$. Finally, such a factorization or word is said to be **stable** if it is both ⊕-stable and ⊖-stable.

Remark 3.2.10 (Stability reduction)**.** If we are given a factorization of $\alpha^* = \xi_1^* \star \cdots \star \xi_j^*$ that is not i-⊕-stable or i-⊖-stable for some $i = 1, \ldots, j-1$, we can perform an *i-reduction*, that maps $(\xi_1^*, \ldots, \xi_i^*, \xi_{i+1}^*, \cdots \star \xi_j^*)$ to $(\xi_1^*, \ldots, \xi_{i+1}^*, \xi_i^*, \cdots \star \xi_j^*)$.

It is immediate to see that this procedure of finding some i and applying an i-reduction always terminates. The final word is stable and is independent of the order in which we apply the reductions.

Because of the above, any equivalence class in $\mathcal{W}(\Omega)/_\sim$ admits a unique stable word, and a consequence of Theorem 3.2.7 is the following.

Corollary 3.2.11 (Unique stable factorization)**.** Let α^* be a marked permutation. Then, α^* has a unique stable factorization into irreducible marked permutations. We refer to it as *the stable factorization of* α^*.

3.3 Lyndon factorization on marked permutation

We introduce an order on permutations, two orders on marked permutations and an order in $\mathcal{W}(\Omega)$.

Definition 3.3.1 (Orders on marked permutations)**.** The *lexicographic order on permutations* is the lexicographic order when reading the one-line notation of permutations, and is written $\pi \leq_{per} \tau$.

Recall that, for a marked permutation $\alpha^* = (\leq_P, \leq_V)$ in I, we define its rank $\mathfrak{rk}(\alpha^*)$ as the rank of $*$ in $I \sqcup \{*\}$ with respect to the order $\leq_P$. We also write α for referring to the corresponding permutation in the set $I \sqcup \{*\}$. We define the *lexicographic order on marked permutations*, also denoted $\leq_{per}$, as follows: we say that $\pi^* \leq_{per} \tau^*$ if $\pi <_{per} \tau$ or if $\pi = \tau$ and $\mathfrak{rk}(\pi^*) \leq \mathfrak{rk}(\tau^*)$.

This in particular endows our alphabet Ω of irreducible marked permutations with an order. When we compare words on $\mathcal{W}(\Omega)$ we order them lexicographically according to $\leq_{per}$, and denote it simply as $\leq$.

We define the *factorization order* $\leq_{fac}$ on marked permutations as follows: Let $\pi^* = \xi_1^* \star \cdots \star \xi_k^*$ and $\tau^* = \tau_1^* \star \cdots \star \tau_j^*$ be the respective unique stable factorizations of π^* and τ^*. Then, we say that $\pi^* \leq_{fac} \tau^*$ if $(\xi_1^*, \ldots, \xi_k^*) \leq (\tau_1^*, \ldots, \tau_j^*)$ in $\mathcal{W}(\Omega)$.

Example 3.3.2. On permutations we have $12345 \leq_{per} 132 \leq_{per} 231 \leq_{per} 4123$. Observe that the empty permutation is the smallest permutation.

On marked permutations, we have $1\bar{3}2 \leq_{per} 13\bar{2} \leq_{per} \bar{2}31 \leq_{per} 412\bar{3}$. Observe that the trivial marked permutation $\bar{1}$ is the smallest marked permutation.

On words, we have the following examples: $(24\bar{1}3, 31\bar{4}2) \leq (31\bar{4}2, 24\bar{1}3)$, $(\bar{1}32, 21\bar{3}) \leq (\bar{1}432)$ and $(2\bar{4}13, \bar{1}423, 2\bar{4}13) \leq (2\bar{4}13, 2\bar{4}13, \bar{1}423)$.

On marked permutations, because $(24\bar{1}3, 31\bar{4}2)$ and $(31\bar{4}2, 24\bar{1}3)$ are stable, we have that $24\bar{1}3 \star 31\bar{4}2 \leq_{fac} 31\bar{4}2 \star 24\bar{1}3$. Notice however, that $(\bar{1}32, 21\bar{3})$ is not a stable word, as it does not satisfy the 1-$\oplus$-stability condition. Instead, we have $(\bar{1}32, 21\bar{3})^* = 21\bar{3} \star \bar{1}32 \geq_{fac} \bar{1}432$.

Sometimes the orders $\leq_{per}$ and $\leq_{fac}$ on marked permutations do not agree, as exemplified in Fig. 3.2.

Remark 3.3.3. If $w_1 \geq w_2$ are stable words in $\mathcal{W}(\Omega)$, then these correspond precisely to the stable irreducible factorizations of w_1^*, w_2^*. Thus, $w_1^* \geq_{fac} w_2^*$.

Remark 3.3.4. If a word $w = (\rho_1^*, \ldots, \rho_k^*)$ in $\mathcal{W}(\Omega)$ is not i-$\oplus$-stable or i-$\ominus$-stable, then $\rho_i^* <_{per} \rho_{i+1}^*$. Thus, from Theorem 3.2.7 and Remark 3.2.10, for a fixed marked permutation α^*, among all words w on irreducible marked permutations such that $w^* = \alpha^*$, the stable factorization is the biggest one in the lexicographical order.

We start the discussion on the topic of Lyndon words. This is useful because the unique factorization theorem obtained in Corollary 3.2.11 for marked permutations is

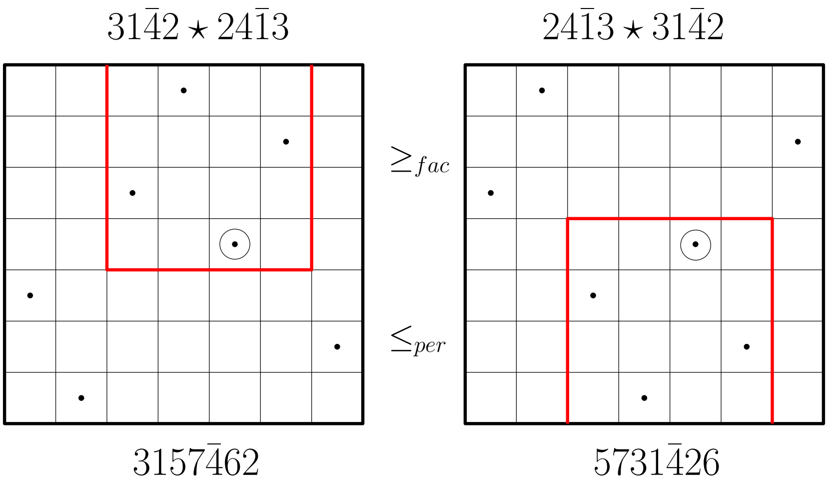


FIGURE 3.2: Two marked permutations and their order relations.

not enough to establish the freeness of $\mathcal{A}(\texttt{MPer})$ and, as in [Var14], Lyndon words are the tool that allows us to describe an improved unique factorization theorem in Theorem 3.3.9.

Definition 3.3.5 (Lyndon words)**.** Given an alphabet A with a total order, a word $l \in \mathcal{W}(A)$ is said to be a *Lyndon word* if, for any concatenation factorization $l = a_1 \cdot a_2$ into non-empty words, we have that $a_2 \geq l$.

Example 3.3.6 (Examples of Lyndon words)**.** Consider the (usual) alphabet

$$\Omega = \{\bar{1} <_{per} \bar{1}2 <_{per} \bar{1}32 <_{per} \cdots <_{per} 23\bar{1} <_{per} 24\bar{1}3 <_{per} \dots\},$$

then $(\bar{1}2, \bar{1}32, \bar{1}2, 24\bar{1}3)$ is a Lyndon word in this alphabet. Meanwhile, $(\bar{1}, \bar{1})$ is **not** a Lyndon word.

Definition 3.3.7 (Stable Lyndon marked permutations)**.** A word on irreducible marked permutations $w = (\xi_1^*, \dots, \xi_j^*) \in \mathcal{W}(\Omega)$ is called *stable Lyndon*, or SL for short, if it is a Lyndon word and satisfies the *stability* conditions introduced in Definition 3.2.9.

A marked permutation π^* is called *stable Lyndon*, or SL for short, if there exists an SL word $l = (\xi_1^*, \dots, \xi_j^*)$ such that $l^* = \pi^*$. We write $\mathcal{L}_{SL}$ for the set of SL marked permutations. Observe that, from Corollary 3.2.11, if such an SL word exists it is unique.

We see latter in Theorem 3.4.1 that $\mathcal{L}_{SL}$ is precisely the set that indexes a free basis of $\mathcal{A}(\texttt{MPer})$. To establish the unique factorization theorem in the context of marked permutations, we first recall a classical fact in Lyndon words.

Theorem 3.3.8 (Unique Lyndon factorization theorem, [CFL58])**.** Consider a finite alphabet A with a total order. Then any word has a unique factorization, in the concatenation product, into Lyndon words $l_1, \dots, l_k$ such that $l_1 \geq \cdots \geq l_k$ for the lexicographical order in $\mathcal{W}(A)$.

This theorem is central in establishing the freeness of the shuffle algebra in [Rad79]. So is the unique factorization into Lyndon marked permutations below, in establishing the freeness of $\mathcal{A}(\texttt{MPer})$.

Theorem 3.3.9 (Unique stable Lyndon factorization theorem)**.** Let α^* be a marked permutation. Then there is a unique sequence of SL words on Ω, $l_1, \dots, l_k$ such that $l_i \geq l_{i+1}$ and $\alpha^* = l_1^* \star \cdots \star l_k^*$.

To such a sequence of words, we call the *SL factorization* of α^*.

Proof. The existence follows from Corollary 3.2.11 and Theorem 3.3.8. Indeed, for any marked permutation α^*, there is a unique stable factorization $\xi_1^*, \dots, \xi_{j(\alpha^*)}^*$, and from the Lyndon factorization theorem, the word $(\xi_1^*, \dots, \xi_{j(\alpha^*)}^*) \in \mathcal{W}(\Omega)$ admits a factorization into Lyndon words $l_1, \dots, l_k$ such that $l_i \geq l_{i+1}$. These words are stable because $(\xi_1^*, \dots, \xi_{j(\alpha^*)}^*)$ is stable, and from $(\xi_1^*, \dots, \xi_{j(\alpha^*)}^*) = l_1 \cdots l_k$ we have that $\alpha^* = l_1^* \star \cdots \star l_k^*$. We then obtain the desired sequence of SL words $l_1, \dots, l_k$.

For the uniqueness of such a factorization, suppose we have SL words $m_1 \geq \cdots \geq m_{k'}$ that form an SL factorization of α^*. We wish to show that this is precisely the sequence $l_1, \dots, l_k$ constructed above. Write $m_k = (\rho_{k,1}^*, \dots, \rho_{k,z_k}^*)$ for $v = 1, \dots, k' - 1$, where $z_v = |m_v|$ and, for readability purposes, consider as well the re-indexing $m_1 \cdots m_{k'} = (\rho_1^*, \dots, \rho_z^*)$.

First observe that from $\alpha^* = m_1^* \star \cdots \star m_{k'}^*$ we get that $\rho_1^* \star \cdots \star \rho_z^*$ is a factorization of α^* into irreducibles. Further, because each m_j is stable, the i-$\oplus$-stability and i-$\ominus$-stability of this factorization is given for any i that is not of the form $z_1 + \cdots + z_v$, for some $v = 1, \dots, k' - 1$. On the other hand, it follows from the Lyndon property that $\rho_{k,z_k}^* >_{per} \rho_{k,1}^*$; further, because $m_k \geq m_{k+1}$, we have that $\rho_{k,1}^* \geq_{per} \rho_{k+1,1}^*$. We conclude that $\rho_{k,z_k}^* >_{per} \rho_{k+1,1}^*$. Comparing with Remark 3.3.4, we have the i-$\oplus$-stability and i-$\ominus$-stability condition for any i that is of the form $z_1 + \cdots + z_v$, for some $v = 1, \dots, k' - 1$.

Thus, $\rho_1^* \star \cdots \star \rho_z^*$ is the stable factorization of α^*, so that $(\rho_1^*, \dots, \rho_z^*) = (\xi_1^*, \dots, \xi_k^*)$. Further, the sequence $m_1 \geq \cdots \geq m_{k'}$ is the Lyndon factorization of $(\rho_1^*, \dots, \rho_z^*)$, so $(m_1, \dots, m_{k'}) = (l_1, \dots, l_k)$ by the uniqueness in Theorem 3.3.8. □

We define $k(\alpha^*)$ to be the number of factors in the stable Lyndon factorization of α^*. Note that for any marked permutation α^*, $k(\alpha^*) \leq j(\alpha^*)$ where we recall that $j(\alpha^*)$ is the number of irreducible factors in a factorization of α^* into irreducible marked permutations.

Definition 3.3.10 (Word shuffle)**.** Consider Ω an alphabet, and let $w, l_1, \dots, l_k \in \mathcal{W}(\Omega)$. We say that $w = (w_1, \dots, w_j)$ is a *word shuffle* of $l_1, \dots, l_k$ if $[j]$ can be partitioned into k many disjoint sets $\{q_1^{(i)} < \cdots < q_{|l_i|}^{(i)}\}$, where i runs over $i = 1, \dots, k$, such that

$$l_i = (w_{q_1^{(i)}}, \dots, w_{q_{|l_i|}^{(i)}}),$$

for all $i = 1, \dots, k$.

The following theorem is proven in [Rad79, Theorem 2.2.2] and is the main property of Lyndon factorizations on words that we wish to use here. This is also the main ingredient in showing that the shuffle algebra is free.

Theorem 3.3.11 (Lyndon shuffles - [Rad79])**.** Take Ω an ordered alphabet, and $l_1 \geq \cdots \geq l_k$ Lyndon words on $\mathcal{W}(\Omega)$. Consider $w \in \mathcal{W}(\Omega)$, and assume that w is a word shuffle of $l_1, \dots, l_k$. Then $w \leq l_1 \cdots l_k$.

We remark that there are substantial notation differences: particularly, in [Rad79] a *prime factorization* is what we call here the *Lyndon factorization*. Our notation follows [GR14].

3.4 Freeness of the pattern algebra in marked permutations

In this section we state the main steps of the proof of Theorem 3.0.1. The proof of the technical lemmas is postponed to Section 3.6.

We consider the set of SL marked permutations $\mathcal{L}_{SL}$, which play the role of free generators, and consider a multiset of SL marked permutations $\{\iota_1^* \geq_{fac} \cdots \geq_{fac} \iota_k^*\}$ and the marked permutation $\alpha^* = \iota_1^* \star \cdots \star \iota_k^*$.

Then, all the terms that occur in the right hand side of

$$\prod_{i=1}^{k} \mathbf{p}_{\iota_i^*} = \sum_{\beta^*} \binom{\beta^*}{\iota_1^*, \dots, \iota_k^*} \mathbf{p}_{\beta^*}, \tag{3.1}$$

correspond to *quasi-shuffles* of $\iota_1^*, \dots, \iota_k^*$. Below we establish that the marked permutation $\alpha^* = \iota_1^* \star \cdots \star \iota_k^*$ is the biggest such marked permutation occurring on the right hand side of (3.1), with respect to a suitable total order related to $\leq_{fac}$.

Theorem 3.4.1. Let α^* be a marked permutation, and suppose that $\iota_1^*, \ldots, \iota_k^*$ is its Lyndon factorization. Then there are coefficients $c_{\beta^*} \geq 0$ such that

$$\prod_{i=1}^{k} \mathbf{p}_{\iota_i^*} = \sum_{|\beta^*|<|\alpha^*|} c_{\beta^*}\, \mathbf{p}_{\beta^*} + \sum_{\substack{|\beta^*|=|\alpha^*| \\ j(\beta^*)<j(\alpha^*)}} c_{\beta^*}\, \mathbf{p}_{\beta^*} + \sum_{\substack{|\beta^*|=|\alpha^*| \\ j(\beta^*)=j(\alpha^*) \\ \beta^* \leq_{fac} \alpha^*}} c_{\beta^*}\, \mathbf{p}_{\beta^*} \,. \tag{3.2}$$

Furthermore, $c_{\alpha^*} \geq 1$.

With this, the linear independence of all products of the form $\prod_{i=1}^{k} \mathbf{p}_{\iota_i^*}$ follows from the linear independence of $\{\mathbf{p}_{\alpha^*} \,|\alpha^* \in \mathcal{G}(\texttt{MPer})\}$, established earlier in Remark 2.2.2. In other terms, Theorem 3.4.1 implies Theorem 3.0.1. The technical lemmas necessary to prove Theorem 3.4.1 are the following:

Lemma 3.4.2. Let β^* be a quasi-shuffle of the marked permutations $\iota_1^*, \ldots, \iota_k^*$. Then, $|\beta^*| \leq |\alpha^*|$. Further, if $|\beta^*| = |\alpha^*|$, then $j(\beta^*) \leq j(\alpha^*)$.

Lemma 3.4.3 (Factor breaking lemma)**.** Let β^* be a quasi-shuffle of the marked permutations $\iota_1^*, \ldots, \iota_k^*$, such that $|\beta^*| = |\alpha^*|$ and $j(\beta^*) = j(\alpha^*)$. Then $\mathfrak{fac}(\beta^*) = \mathfrak{fac}(\alpha^*)$.

Furthermore, if for each $i = 1, \ldots, k$,

$$\iota_i^* = \zeta_{1,i}^* \star \cdots \star \zeta_{j(\iota_i^*),i}^*$$

is the stable factorization of ι_i^*, then there is a marked permutation γ^* with a factorization into irreducibles given by $\gamma^* = \tau_1^* \star \cdots \star \tau_{j(\gamma^*)}^*$ such that

- we have $\mathfrak{fac}(\gamma^*) = \mathfrak{fac}(\beta^*)$. In particular, $|\gamma^*| = |\beta^*| = |\alpha^*|$ and $j(\gamma^*) = j(\beta^*) = j(\alpha^*)$;
- we have $\gamma^* \geq_{fac} \beta^*$;
- the word $(\tau_1^*, \ldots, \tau_{j(\gamma^*)}^*)$ is a word shuffle of the words $\{(\zeta_{1,i}^*, \ldots, \zeta_{j(\iota_i^*),i}^*)\}_{i=1,\ldots,k}$.

These lemmas will be proven below, in Section 3.6. We remark that, here, the chosen ordering $\leq_{per}$ on the irreducible marked permutations plays a role. That is, in proving that $\gamma^* \geq_{fac} \beta^*$ we use properties of the order $\leq_{per}$ introduced above, like Remark 3.3.4. This is unlike the work in [Var14], where any order in the $\oplus$-indecomposable permutations gives rise to a set of free generators.

Lemma 3.4.4 (Factor preserving lemma)**.** Let γ^* be a marked permutation with a factorization into irreducibles given by $\gamma^* = \tau_1^* \star \cdots \star \tau_{j(\gamma^*)}^*$, and let $l_1, \ldots, l_k$ be SL words, such that

- $l_i \geq l_{i+1}$ for each $i = 1, \dots, k-1$ and;
- The word $(\tau_1^*, \dots, \tau_{j(\gamma^*)}^*)$ is a word shuffle of the words $\{l_i\}_{i=1,\dots,k}$.

Then, $\gamma^* \leq_{fac} l_1^* \star \cdots \star l_k^*$.

In the remaining of the section we assume these lemmas and conclude the proof of the freeness of the pattern algebra, by establishing Theorem 3.4.1.

Proof of Theorem 3.4.1. Because a product is a quasi-shuffle of its factors, it follows from Remark 2.1.6 and from $\alpha^* = \iota_1^* \star \cdots \star \iota_k^*$ that $c_{\alpha^*} \geq 1$. We conclude the proof if we show that any β^* that is a *quasi-shuffle* of $\iota_1^* \dots, \iota_k^*$ satisfies either

- $|\beta^*| < |\alpha^*|$;
- or $|\beta^*| = |\alpha^*|$ and $j(\beta^*) < j(\alpha^*)$;
- or $|\beta^*| = |\alpha^*|$, $j(\beta^*) = j(\alpha^*)$ and $\beta^* \leq_{fac} \alpha^*$.

Suppose that β^* is a quasi-shuffle of $\iota_1^*, \dots, \iota_k^*$. For each $i = 1, \dots k$, let l_i be the SL word corresponding to the SL marked permutation ι_i^*. From Lemma 3.4.2 we only need to consider the case where $|\beta^*| = |\alpha^*|$ and $j(\beta^*) = j(\alpha^*)$.

From Lemma 3.4.3 we have that in this case, $\mathfrak{fac}(\beta^*) = \mathfrak{fac}(\alpha^*)$, and there is a marked permutation γ^* that satisfies $\mathfrak{fac}(\gamma^*) = \mathfrak{fac}(\beta^*)$, $\gamma^* \geq_{fac} \beta^*$, and also has a factorization into irreducibles that is a word shuffle of $l_1, \dots, l_k$. Thus, from Lemma 3.4.4 we have that $\gamma^* \leq_{fac} l_1^* \star \cdots \star l_k^* = \alpha^*$. It follows that $\beta^* \leq_{fac} \gamma^* \leq_{fac} \alpha^*$, as desired. □

3.5 Proof of unique factorization theorem

We start this section with the concept of *DC intervals* and *DC chains* of a marked permutation α^*. We will see that these are closely related to factorizations of α^*, and we will exploit this correspondence to prove the unique factorization theorem Theorem 3.2.7.

Recall that if β^* is a marked permutation in X, and $I \subseteq X$, we denote $I^* := I \sqcup \{*\}$ for simplicity of notation. We also write $\mathbb{X}(\beta^*) = X$.

Definition 3.5.1. Let β^* be a marked permutation on the set X, *i.e.*, $\beta^* = (\leq_P, \leq_V)$ is a pair of orders on the set X^*. A *doubly connected interval* on β^*, or a *DC interval* for short, is a set $I \subseteq X$ such that I^* is an interval on both orders $\leq_P, \leq_V$. A *proper* DC interval is a DC interval I such that $I \neq X$.

Note that X and $\emptyset$ are always DC intervals. Note as well that if I_1, I_2 are DC intervals, then $I_1 \cup I_2$ is also a DC interval.

Remark 3.5.2. Suppose that $\beta^* = (\leq_P, \leq_V)$ is a marked permutation, and I is a DC interval of β^*. Consider $m_P, M_P \in X^*$ the minimal, respectively maximal, element for the order $\leq_P$. If $m_P, M_P \in I^*$, then $I = X$.

Symmetrically, if $m_V, M_V \in X^*$ are the minimal, respectively maximal, elements for the order $\leq_V$, and $m_V, M_V \in I^*$, then $I = X$.

For a marked permutation β^*, we use the notation m_P, M_P, m_V, M_V in the remaining of the section to refer to its respective extremes, as in Remark 3.5.2.

Definition 3.5.3. For a DC interval I of a marked permutation $\alpha^* = (\leq_P, \leq_V)$, we define the permutation $\alpha^* \backslash_{I^*}$ in the set I^c resulting from the restriction of the orders $\leq_P, \leq_V$ to the set I^c. Alternatively, this is the permutation resulting from the removal of the marked element in $\alpha^*|_{I^c}$.

Remark 3.5.4. If $\alpha^* = a_1^* \star a_2^*$, then $\mathbb{X}(a_2^*)$ is a DC interval in α^*. On the other hand, if I is a DC interval of α^*, then $\alpha^* = \alpha^*|_{I^c} \star \alpha^*|_I$, so right factors of α^* are in bijection with DC intervals of α^*. Furthermore, I is a maximal proper DC interval of α^* if and only if $\alpha^*|_{I^c}$ is irreducible.

To a factorization $\beta^* = \rho_1^* \star \cdots \star \rho_j^*$ of a marked permutation $\beta^* \in \texttt{MPer}[X]$ it corresponds a chain of DC intervals

$$\emptyset = J_{j+1} \subsetneq J_j \subsetneq \cdots \subsetneq J_1 = X \,.$$

The chain is defined by $J_k = \mathbb{X}(\rho_k^* \star \cdots \star \rho_j^*)$ and satisfies $\beta^*|_{J_k \backslash J_{k+1}} = \rho_k^*$ for any $k = 1, \ldots, j$. This chain of DC intervals is maximal if and only if the original factorization has only irreducible factors.

Remark 3.5.5. The marked permutation α^* is $\oplus$-decomposable if and only if there is a DC interval I of α^* such that I^* contains both m_P, m_V, or contains both M_P, M_V.

In the first case, α^* factors as $\beta_1^* \oplus \beta_2$, where $\beta_1^* = \alpha^*|_I$ and $\beta_2 = \alpha^* \backslash_{I^*}$. In the second case, α^* factors as $\beta_1 \oplus \beta_2^*$, where $\beta_1 = \alpha^* \backslash_{I^*}$ and $\beta_2^* = \alpha^*|_I$.

Similarly, α^* is $\ominus$-decomposable if and only if there is a DC interval I of α^* such that I^* contains both M_P, m_V, or contains both m_P, M_V.

In the first case, α^* factors as $\beta_1^* \ominus \beta_2$, where $\beta_1^* = \alpha^*|_I$ and $\beta_2 = \alpha^* \backslash_{I^*}$. In the second case, α^* factors as $\beta_1 \ominus \beta_2^*$, where $\beta_1 = \alpha^* \backslash_{I^*}$ and $\beta_2^* = \alpha^*|_I$.

This characterization of $\oplus$-decomposable marked permutations will be useful in the proof of the classification of the factorizations below, specifically in Lemma 3.5.10. It is also useful to characterize all $\oplus$-decomposable marked permutations that are irreducible in Section 3.7.

Observation 3.5.6 ($\oplus$-factorization in marked permutations)**.** Let α^* be a marked permutation. Then there are unique $\oplus$-indecomposable permutations $\epsilon_1, \dots, \epsilon_k$, $\lambda_1, \dots, \lambda_j$ and β^* an $\oplus$-indecomposable marked permutation such that

$$\alpha^* = \epsilon_1 \oplus \cdots \oplus \epsilon_k \oplus \beta^* \oplus \lambda_j \oplus \cdots \oplus \lambda_1 \,.$$

In this case, we say that α^* is q-$\oplus$-decomposable, where $q = k + j$.

Example 3.5.7 (q-$\oplus$-decomposition)**.** Consider the marked permutation $21\bar{3}54$. This is a 2-decomposable marked permutation, as it admits the $\oplus$-factorization

$$21\bar{3}54 = 21 \oplus \bar{1} \oplus 21 \,.$$

A marked permutation is $\oplus$-indecomposable if and only if it is 0-$\oplus$-decomposable.

Lemma 3.5.8 (Irreducible $\oplus$-factor lemma)**.** Suppose that α^* is a marked permutation that has the following $\oplus$-factorization

$$\alpha^* = \epsilon_1 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_1 \,,$$

where $u + v > 0$. Consider any factorization $\alpha^* = \sigma^* \star \gamma^*$, where σ^* is an irreducible marked permutation.

- If both $u > 0$ and $v > 0$, then either $\sigma^* = \bar{1} \oplus \lambda_1$ or $\sigma^* = \epsilon_1 \oplus \bar{1}$.
- If $u = 0$, then $\sigma^* = \bar{1} \oplus \lambda_1$.
- If $v = 0$, then $\sigma^* = \epsilon_1 \oplus \bar{1}$.

Proof. Let us deal with the case where both $u > 0$ and $v > 0$ first. Assume that neither $\sigma^* = \bar{1} \oplus \lambda_1$ nor $\sigma^* = \epsilon_1 \oplus \bar{1}$. Consider the following DC intervals, that are all distinct

$$Y_1 = \mathbb{X}(\epsilon_2 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_1) \,,$$

$$Y_2 = \mathbb{X}(\epsilon_1 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_2) \,,$$

$$Y = \mathbb{X}(\gamma^*) \,.$$

Note that $m_P \in Y_2^* \backslash Y_1^*$ and $M_P \in Y_1^* \backslash Y_2^*$ by construction. Note also that $\alpha^*|_{Y_1^c} = \epsilon_1 \oplus \bar{1}$ and $\alpha^*|_{Y_2^c} = \bar{1} \oplus \lambda_1$ are irreducible marked permutations, so both Y_1, Y_2 are maximal proper DC intervals. Furthermore, Y is also a maximal proper DC interval. This maximality gives us that $Y \cup Y_1 = X = Y \cup Y_2$.

From $Y \cup Y_1 = X$ and $m_P \notin Y_1^*$, we have that $m_P \in Y^*$, and from $Y \cup Y_2 = X$ and $M_P \notin Y_2^*$ we get that $M_P \in Y^*$. From Remark 3.5.2, this is a contradiction with the fact that Y is a proper DC interval, thus contradicting the assumption that neither $\sigma^* = \bar{1} \oplus \lambda_1$ nor $\sigma^* = \epsilon_1 \oplus \bar{1}$, as desired.

Now suppose that $u = 0$, and $v > 0$. Assume for the sake of contradiction that $\sigma^* \neq \bar{1} \oplus \lambda_1$, and define the following distinct DC intervals

$$Y_2 = \mathbb{X}(\beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_2) \text{ and } Y = \mathbb{X}(\gamma^*)\,.$$

Because both σ^* and $\bar{1} \oplus \lambda_1$ are irreducible, the DC intervals Y_2, Y are both maximal proper, so $X = Y \cup Y_2$. Further, $m_P \in Y_2^*$ by construction, so $M_P \in Y^*$ by Remark 3.5.2. Consider the DC interval $I = \mathbb{X}(\beta^*)$, and notice that $m_P \in I^*$ by construction. Thus we have that $I^* \cup Y^* = X^*$, and $Y^c \subseteq I$.

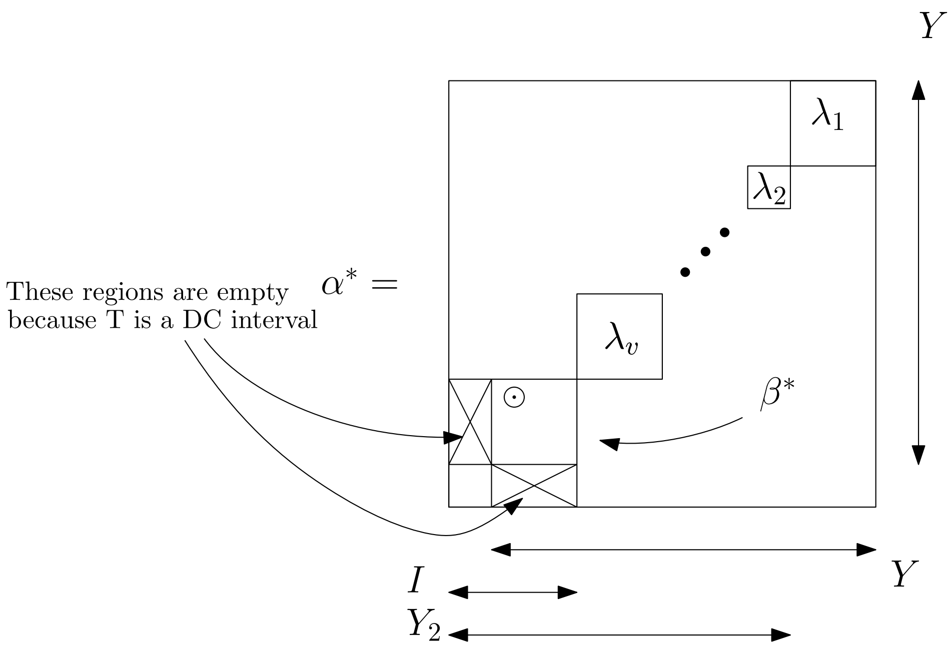


FIGURE 3.3: A decomposition of β^*, whenever the DC interval Y is proper

We have the following decomposition, depicted in Fig. 3.3:

$$\beta^* = \alpha^* \setminus_{Y^*} \oplus \alpha^*|_{I \cap Y}\,.$$

Notice that $\alpha^*\backslash_{Y^*}$ is not the empty permutation, because Y is a proper DC interval. This is a contradiction with the fact that β^* is $\oplus$-indecomposable, so $\sigma^* = \bar{1} \oplus \lambda_1$.

The case where $u > 0$ and $v = 0$ is done in a similar way, so the result follows. $\square$

Corollary 3.5.9. Consider a marked permutation α^* that is q-$\oplus$-decomposable, so that $\alpha^* = \epsilon_1 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_1$ is the $\oplus$-decomposition of α^*. If $\sigma_1^* \star \cdots \star \sigma_j^*$ is a factorization of α^* into irreducibles, then $j \geq q$ and

1. $\sigma_{q+1}^* \star \cdots \star \sigma_j^* = \beta^*$.
2. By applying $\oplus$-relations to $(\sigma_1^*, \ldots, \sigma_q^*)$ we can obtain

$$(\bar{1} \oplus \lambda_1, \ldots, \bar{1} \oplus \lambda_v, \epsilon_1 \oplus \bar{1}, \ldots, \epsilon_u \oplus \bar{1})\,.$$

Proof. We use induction on q. The base case is $q = 0$, where there is nothing to establish in 2. and we need only to show that

$$\sigma_1^* \star \cdots \star \sigma_j^* = \beta^*\,,$$

which follows because $\sigma_1^* \star \cdots \star \sigma_j^* = \alpha^* = \beta^*$.

Now for the induction step, we assume that $q \geq 1$. From Lemma 3.5.8, σ_1^* is either $\epsilon_1 \oplus \bar{1}$ or $\bar{1} \oplus \lambda_1$, thus according to Remark 3.1.4 $\zeta^* := \sigma_2^* \star \cdots \star \sigma_k^*$ is either

$$\epsilon_2 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_1\,,$$

or

$$\epsilon_1 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_2.$$

Without loss of generality, assume the first case. Then ζ^* is $(q-1)$-$\oplus$-decomposable, so the induction hypothesis applies to the factorization of ζ^*. That is, we have that

$$\sigma_{q+1}^* \star \cdots \star \sigma_j^* = \beta^*\,.$$

Furthermore, the induction case gives us that $(\bar{1} \oplus \lambda_1, \ldots, \bar{1} \oplus \lambda_v, \epsilon_2 \oplus \bar{1}, \ldots, \epsilon_u \oplus \bar{1})$ can be obtained from $(\sigma_2^*, \ldots, \sigma_q^*)$ by a series of $\oplus$-relations.

So, by only using $\oplus$-relations,

$$\begin{aligned} & (\bar{1} \oplus \lambda_1, \dots, \bar{1} \oplus \lambda_v, \epsilon_1 \oplus \bar{1}, \epsilon_2 \oplus \bar{1}, \dots, \epsilon_u \oplus \bar{1}) \\ \sim \quad & (\epsilon_1 \oplus \bar{1}, \bar{1} \oplus \lambda_1, \dots, \bar{1} \oplus \lambda_v, \epsilon_2 \oplus \bar{1}, \dots, \epsilon_u \oplus \bar{1}) \\ \sim_{ind.hip.} & (\sigma_1^*, \sigma_2^*, \dots, \sigma_q^*) \end{aligned} \tag{3.3}$$

concluding the the proof. □

We remark that Observation 3.5.6, Lemma 3.5.8, and Corollary 3.5.9 have counterparts for the $\ominus$ operation.

Lemma 3.5.10 (Indecomposable irreducible factor lemma)**.** Suppose that α^* is a non-trivial marked permutation that is both $\oplus$-indecomposable and $\ominus$-indecomposable.

Consider two factorizations of α^*, $\sigma_1^* \star \beta_1^*$ and $\sigma_2^* \star \beta_2^*$, where σ_1^*, σ_2^* are irreducible marked permutations. Then $\sigma_1^* = \sigma_2^*$ and $\beta_1^* = \beta_2^*$.

Proof. According to Remark 3.5.4, it suffices to see that α^* only has one maximal proper DC interval. Suppose otherwise, that there are Y_1, Y_2 distinct proper maximal DC intervals, for the sake of contradiction. Further, consider m_P, M_P, m_V, M_V the usual maximal and minimal elements in $\mathbb{X}(\alpha^*)^*$.

We have that $Y_1 \cup Y_2 = X$, so $m_P, M_P, m_V, M_V \in Y_1^* \cup Y_2^*$. Without loss of generality suppose that $m_P \in Y_1^*$. We know that $\{m_V, M_V\} \not\subseteq Y_2^*$, so there are only two cases to consider:

- We have that $m_P, m_V \in Y_1^*$. Then α^* is $\oplus$-decomposable, according to Remark 3.5.5.
- We have that $m_P, M_V \in Y_1^*$. Then α^* is $\ominus$-decomposable, according to Remark 3.5.5.

In either case, we reach a contradiction. It follows that $\beta_1^* = \beta_2^*$ and $\sigma_1^* = \sigma_2^*$. □

Proof of Theorem 3.2.7. Consider a marked permutation α^*, and take words $w_1 = (\xi_1^*, \dots, \xi_k^*)$ and $w_2 = (\rho_1^*, \dots, \rho_j^*)$ in $\mathcal{W}(\Omega)$ such that

$$\xi_1^* \star \cdots \star \xi_k^* = \rho_1^* \star \cdots \star \rho_j^* =: \alpha^* , \tag{3.4}$$

and assume for the sake of contradiction that these can be chosen with $w_1 \not\sim w_2$. Further choose such words minimizing $k + j = |w_1| + |w_2|$. We need only to consider four cases:

The marked permutation $\alpha^* = \bar{1}$ is trivial: then by a size argument we have that both words w_1 and w_2 are empty, thus $w_1 = w_2$.

The marked permutation α^* is indecomposable: in this case, from Lemma 3.5.10 we know that $\xi_1^* = \rho_1^*$ and that $\xi_2^* \star \cdots \star \xi_k^* = \rho_2^* \star \cdots \star \rho_j^*$. Thus, by minimality, $(\xi_2^*, \dots, \xi_k^*) \sim (\rho_2^*, \dots, \rho_j^*)$, which implies $(\xi_1^*, \dots, \xi_k^*) \sim (\rho_1^*, \dots, \rho_j^*)$ and that is a contradiction.

The marked permutation α^* is $\oplus$-decomposable: Assume now that α^* is q-$\oplus$-decomposable, for some $q > 0$. That is, there are $\oplus$-indecomposables such that

$$\alpha^* = \epsilon_1 \oplus \cdots \oplus \epsilon_u \oplus \beta^* \oplus \lambda_v \oplus \cdots \oplus \lambda_1, \quad u + v = q\,.$$

Then Corollary 3.5.9 tells us that $q \leq k, j$ and that

$$\xi_{q+1}^* \star \cdots \star \xi_k^* = \beta^* = \rho_{q+1}^* \star \cdots \star \rho_j^*\,. \tag{3.5}$$

Further, we also have that

$$(\xi_1^*, \dots, \xi_q^*) \sim (\bar{1} \oplus \lambda_1, \dots, \bar{1} \oplus \lambda_v, \epsilon_1 \oplus \bar{1}, \dots, \epsilon_u \oplus \bar{1}) \sim (\rho_1^*, \dots, \rho_q^*)\,. \tag{3.6}$$

Because $q > 0$, from (3.5) and the minimality of $(\xi_1^*, \dots, \xi_k^*), (\rho_1^*, \dots, \rho_j^*)$ we have that

$$(\xi_{q+1}^*, \dots, \xi_k^*) \sim (\rho_{q+1}^*, \dots, \rho_j^*)\,.$$

This, together with (3.6) gives us $(\xi_1^*, \dots, \xi_k^*) \sim (\rho_1^*, \dots, \rho_j^*)$ and that is a contradiction.

The marked permutation α^* is $\ominus$-decomposable: This case is similar to the previous one.

We conclude that a factorization of a marked permutation into irreducibles is unique up to the relations in Remark 3.2.5. □

3.6 Proofs of Lemmas 3.4.4 to 3.4.2

We start by fixing some notation. Consider a multiset $\{\iota_1^* \geq \cdots \geq \iota_k^*\}$ of SL marked permutations, and let α^* be the marked permutation with Lyndon factorization $(\iota_1^*, \dots, \iota_k^*)$ which exists and is unique by Theorem 3.3.9, and has $\alpha^* = \iota_1^* \star \cdots \star \iota_k^*$.

Note that $\mathcal{S} := \biguplus_{i=1}^k \mathfrak{fac}(\iota_i^*) = \mathfrak{fac}(\alpha^*)$ as multisets, from Corollary 3.2.8.

Proof of Lemma 3.4.2. Let β^* be a marked permutation in X. By Definition 2.2.5, there are sets $I_1, \dots, I_k$ such that:

$$I_1 \cup \dots \cup I_k = X \text{ and } \beta^*|_{I_i} \sim \iota_i^* \ \ \forall i = 1, \dots, k \,. \tag{3.7}$$

Suppose that β^* has stable factorization $\beta^* = \rho_1^* \star \dots \star \rho_j^*$, where $j = j(\beta^*)$, with a corresponding maximal DC interval chain

$$\emptyset = J_{j+1} \subsetneq J_j \subsetneq \dots \subsetneq J_1 = X \,,$$

as given in Remark 3.5.4, so that $\beta^*|_{J_p \setminus J_{p+1}} = \rho_p$ for $p = 1, \dots, j$.

Since β^* is a quasi-shuffle of $\iota_1^*, \dots, \iota_k^*$, the sets $I_1, \dots, I_k$ cover X and so

$$|\beta^*| = |X| \leq \sum_{i=1}^{k} |I_i| = \sum_{i=1}^{k} |\iota_i^*| = |\alpha^*| \,, \tag{3.8}$$

so we conclude that $|\beta^*| \leq |\alpha^*|$.

Now, assume that we have $|\beta^*| = |\alpha^*|$, so we have equality in (3.8), thus the sets I_i are mutually disjoint. Recall that $j(\alpha^*) = \sum_i j(\iota_i^*)$, as shown in Corollary 3.2.8. Therefore, we only need to establish that $\sum_i j(\iota_i^*) \geq j$. For each $i = 1, \dots, k$, the following is a DC interval chain of ι_i^*, which is not necessarily a maximal one:

$$\emptyset = J_{j+1} \cap I_i \subseteq J_j \cap I_i \subseteq \dots \subseteq J_1 \cap I_i = I_i \,, \tag{3.9}$$

so let us consider the set $U_p := \{i \in [k] | J_{p+1} \cap I_i \neq J_p \cap I_i\}$.

First it is clear that each U_p is non empty, as otherwise we would have

$$J_{p+1} = \bigcup_i J_{p+1} \cap I_i = \bigcup_i J_p \cap I_i = J_p \,.$$

On the other hand, the length of the DC chain is given precisely by the number of strict inequalities in (3.9), that is by $|\{p \in [j] | i \in U_p\}|$. From Corollary 3.2.8 and Remark 3.5.4, this is at most $j(\beta^*|_{I_i}) = j(\iota_i^*)$, so

$$\sum_i j(\iota_i^*) \geq \sum_i |\{p \in [j] | i \in U_p\}| = \sum_{p=1}^{j} |U_p| \geq j \,. \tag{3.10}$$

So we conclude that $j(\alpha^*) \geq j(\beta^*)$. $\square$

Proof of Lemma 3.4.3. Assume now that $|\beta^*| = |\alpha^*|$ and $j(\beta^*) = j(\alpha^*)$. As a corollary of the proof above, and using the same notation, we have that the sets I_i are pairwise disjoint, and we obtain equality all through (3.10), so that each U_p is in fact a singleton.

We wish to split the indexing set $[j(\gamma^*)]$ into k many disjoint increasing sequences $q_1^{(i)} < \cdots < q_{j(\iota_i^*)}^{(i)}$ for $i = 1, \ldots, k$ such that

$$\iota_i^* = \tau^*_{q_1^{(i)}} \star \cdots \star \tau^*_{q^{(i)}_{j(\iota_i^*)}} , \tag{3.11}$$

is precisely the stable factorization of each ι_i^*.

Define the map $\zeta : [j] \to [k]$ that sends p to the unique element in U_p. This map will give us the desired increasing sequences. It satisfies

$$|\zeta^{-1}(i)| = |\{p \in [j] | i \in U_p\}| = j(\iota_i^*) \text{ for } i = 1, \ldots, k\,.$$

Write $\zeta^{-1}(i) = \{q_1^{(i)} < \cdots < q^{(i)}_{j(\iota_i^*)}\} \subseteq [j]$ and set $q^{(i)}_{j(\iota_i^*)+1} = j+1$. In the following, we identify the marked permutations ι_i^* and $\beta^*|_{I_i}$ in order to find a factorization of ι^* into irreducibles. Specifically, we get that

$$\iota_i^* = \beta^*|_{I_i} = \rho_1^*|_{I_i \cap J_1 \setminus J_2} \star \cdots \star \rho_j^*|_{I_i \cap J_j \setminus J_{j+1}} \tag{3.12}$$

is a factorization of ι_i^*. Because each U_p is a singleton, $I_i \cap J_p \setminus J_{p+1}$ is either $J_p \setminus J_{p+1}$ or $\emptyset$, thus $\rho_p^*|_{I_i \cap J_p \setminus J_{p+1}}$ is either ρ_p^* of $\bar{1}$, and the factorization in (3.12), after removing the trivial terms, becomes the following factorization into irreducibles:

$$\iota_i = \rho^*_{q_1^{(i)}} \star \cdots \star \rho^*_{q^{(i)}_{j(\iota_i^*)}} . \tag{3.13}$$

Then, we indeed have that $\mathfrak{fac}(\beta^*) = \biguplus_{i=1}^k \mathfrak{fac}(\iota_i^*)$. We now construct the desired marked permutation γ^*.

According to Corollary 3.2.8, for each i, the unique stable factorization of ι_i^* results from $\iota_i^* = \rho^*_{q_1^{(i)}} \star \cdots \star \rho^*_{q^{(i)}_{j(\iota_i^*)}}$ by a permutation of the factors. Thus, for each i we can find indices $p_1^{(i)}, \ldots, p^{(i)}_{j(\iota_i^*)}$ such that $\{p_1^{(i)}, \ldots, p^{(i)}_{j(\iota_i^*)}\} = \{q_1^{(i)}, \ldots, q^{(i)}_{j(\iota_i^*)}\}$ and

$$\iota_i^* = \rho^*_{q_1^{(i)}} \star \cdots \star \rho^*_{q^{(i)}_{j(\iota_i^*)}} = \rho^*_{p_1^{(i)}} \star \cdots \star \rho^*_{p^{(i)}_{j(\iota_i^*)}}$$

is the stable factorization of ι_i^*.

For each $s \in [j]$, if $s = q_u^{(i)}$ for some integers i, u, define $\tau_s := \rho^*_{p_u^{(i)}}$ and finally define

$$\gamma^* = \tau_1^* \star \cdots \star \tau_j^* \,.$$

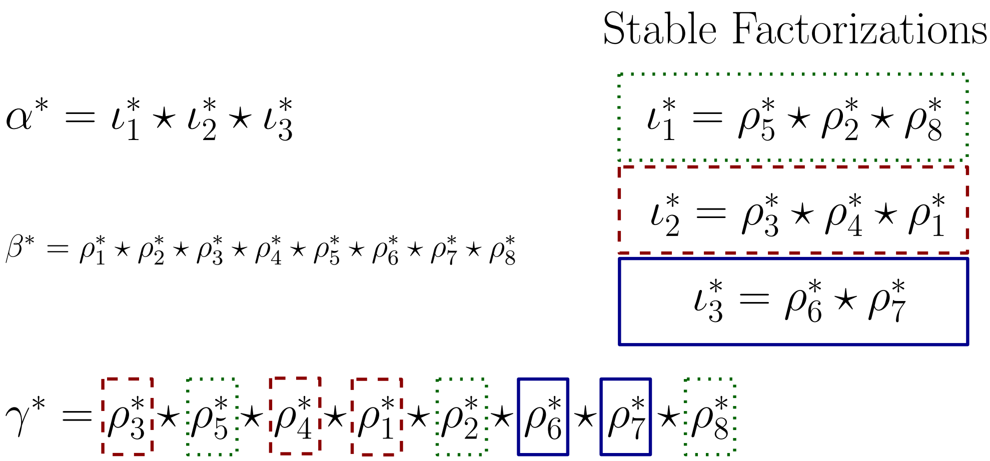


FIGURE 3.4: Example of construction of γ^* for $j = 8$.

A graphical description of this construction for $j = 8$ is given in Fig. 3.4. We claim that γ^* satisfies the three conditions described in the lemma. First, it is clear that $\mathfrak{fac}(\beta^*) = \biguplus_{i=1}^k \mathfrak{fac}(\iota_i^*) = \mathfrak{fac}(\gamma)$, and that the disjoint increasing sequences $q_1^{(i)} < \cdots < q_{j(\iota_i^*)}^{(i)}$ are such that

$$\tau^*_{q_1^{(i)}} \star \cdots \star \tau^*_{q_{j(\iota_i^*)}^{(i)}} = \rho^*_{p_1^{(i)}} \star \cdots \star \rho^*_{p_{j(\iota_i^*)}^{(i)}}$$

is precisely the stable factorization of ι_i^*. Thus, we need only to establish that $\beta^* \leq_{fac} \gamma^*$.

We claim that $(\tau_1^*, \dots, \tau_j^*) \geq (\rho_1^*, \dots, \rho_j^*)$. Indeed, $(\tau_1^*, \dots, \tau_j^*)$ is obtained from the word $(\rho_1^*, \dots, \rho_j^*)$ by the stabilization procedure in its subwords. From the stabilization procedure we obtain bigger words in the lexicographical order, according to Remark 3.3.4. This still holds true even if only applied to a subword, thus the resulting word $(\tau_1^*, \dots, \tau_j^*)$ is lexicographically bigger than $(\rho_1^*, \dots, \rho_j^*)$.

Now, if $\gamma^* = \epsilon_1^* \star \cdots \star \epsilon_j^*$ is the stable factorization of γ^*, because $\gamma^* = \tau_1^* \star \cdots \star \tau_j^*$ we have that from Remark 3.3.4 that

$$(\tau_1^*, \dots, \tau_j^*) \leq (\epsilon_1^*, \dots, \epsilon_j^*) \,,$$

Thus

$$(\rho_1^*, \dots, \rho_j^*) \leq (\epsilon_1^*, \dots, \epsilon_j^*) \,,$$

and so $\beta^* \leq_{fac} \gamma^*$, as desired. □

In the following proof, we will start by showing that the given factorization of γ^* can be assumed to be the stable factorization. Then, using Theorem 3.3.8 and the fact that the factorization of γ^* is a shuffle of Lyndon words to establish the desired inequality.

Proof of Lemma 3.4.4. Write $j = j(\gamma^*)$. We first see that if γ^* has some factorization that is a word shuffle of stable words $l_1, \dots, l_k$, then the stable factorization is also a word shuffle of these words.

Indeed, take a factorization $\tau_1^* \star \cdots \star \tau_j^*$ such that $(\tau_1^*, \dots, \tau_j^*)$ is a word quasi-shuffle of $l_1, \dots, l_k$, and say that there is some $u \in \{1, \dots, j-1\}$ such that this factorization is not u-$\oplus$-stable or is not u-$\ominus$-stable. According to Remark 3.2.10, to show that the stable factorization is also a word shuffle of $l_1, \dots, l_k$, it suffices to show that the word resulting from swapping τ_u^* and τ_{u+1}^* in $(\tau_1^*, \dots, \tau_j^*)$ is still a word quasi-shuffle of $l_1, \dots, l_k$.

When we apply a u-stability reduction (see Remark 3.2.10), we can still find a suitable partition of $[j]$ into k many disjoint increasing sequences. Say that $[j]$ is partitioned into the blocks $\{q_1^{(i)} < \cdots < q_{j(\iota_i^*)}^{(i)}\}$, then there are integers i_1, i_2, v_1, v_2 such that $u = q_{v_1}^{(i_1)}$ and $u + 1 = q_{v_2}^{(i_2)}$. Because $(\tau^*_{q_1^{(i)}}, \dots, \tau^*_{q^{(i)}_{j(\iota_i^*)}})$ is stable for each i, we cannot have $i_1 = i_2$. Therefore, by swapping the elements $u, u+1$ we obtain a new partition for the new factorization, thus showing that it is a word quasi-shuffle of $l_1, \dots, l_k$.

Now let $\rho_1^* \star \cdots \star \rho_j^*$ be the stable factorization of γ^*. Since $(\rho_1^*, \dots, \rho_j^*)$ is a word shuffle of $l_1, \dots, l_k$, which are Lyndon words in $\mathcal{W}(\Omega)$, we have from Theorem 3.3.11 that $(\rho_1^*, \dots, \rho_j^*) \leq l_1 \cdots l_k$, and so $\gamma^* \leq_{fac} l_1^* \star \cdots \star l_k^*$. □

3.7 Primitive elements, growth rates and asymptotic analysis

Recall that in Section 2.2.3 we define the space of primitive elements $P(H)$ of a Hopf algebra H is the subspace of H given by $\{a \in H | \Delta a = a \otimes 1 + 1 \otimes a\}$. In the particular case of the pattern Hopf algebra $\mathcal{A}(\texttt{MPer})$, its primitive space is spanned by $\{\mathbf{p}_{\pi^*} | \pi^* \text{ is irreducible marked permutation}\}$, according to Proposition 2.2.10. So, we are interested in enumerating the irreducible marked permutations. We consider some generating functions:

- The power series $P^*(x) = \sum_{\pi^* \text{ marked permutation}} x^{|\pi^*|} = \sum_{n \geq 1} n \cdot n!\, x^{n-1}$ counts marked permutations.
- The power series $P(x) = \sum_{\pi \text{ permutation}} x^{|\pi|} = \sum_{n \geq 0} n!\, x^n$ counts permutations.

- The power series $S^*(x) = \sum_{k\geq 0} s_k x^k = \sum_{\pi^* \text{ irreducible marked permutation}} x^{|\pi^*|}$ counts irreducible marked permutations. This is the main generating function that we aim to enumerate here.
- The power series $S^*_o(x) = \sum_{k\geq 0} so_k x^k = \sum_{\substack{\pi^* \text{ irreducible and indecomposable} \\ \text{marked permutation}}} x^{|\pi^*|}$ counts irreducible marked permutations that are indecomposable, that is marked permutations that are neither $\oplus$-decomposable nor $\ominus$-decomposable.
- The power series $P^{\oplus}(x) = \sum_{\pi \text{ is } \oplus- \text{ decomposable}} x^{|\pi|}$ counts permutations that are $\oplus$-indecomposable. This also counts the permutations that are $\ominus$-indecomposable.

Because we have a unique factorization theorem on permutations under the $\oplus$ product, the coefficients of $P^{\oplus}(x)$ can be easily extracted via the following power series relation

$$\frac{1}{1 - P^{\oplus}(x)} = P(x) = \sum_{n\geq 0} n! x^n \,, \text{ which implies } P^{\oplus}(x) = 1 - P(x)^{-1} \,.$$

Observation 3.7.1. Any $\oplus$-decomposable irreducible marked permutation is either of the form $\bar{1} \oplus \pi$ or of the form $\pi \oplus \bar{1}$ for π an $\oplus$-indecomposable permutation, and symmetrically for $\ominus$-decomposable irreducible marked permutations.

Thus, we have

$$S^*(x) - S^*_o(x) = 4P^{\oplus}(x) \,. \tag{3.14}$$

The following proposition allows us to enumerate easily the irreducible marked permutations and the irreducible indecomposable marked permutations, as done in Table 3.1.

Proposition 3.7.2 (Power series of irreducible marked permutations)**.**

$$S^*(x) = 3 + \frac{2}{P(x)^2} - \frac{1}{P'(x)} - \frac{4}{P(x)} \,.$$

$$S^*_o(x) = -1 + \frac{2}{P(x)^2} - \frac{1}{P'(x)}$$

where P' is the formal differential of the power series P.

We compare this result with the enumeration of simple permutations in [AAK03, Equation 1], where the power series enumerating simple permutations is given as the solution of a functional equation that is not rational. Thus, we expect it to be simpler to compute the coefficients explicitly.

Lemma 3.7.3. Let π^* be a marked permutation. Then, there are four cases

- There are unique marked permutations σ^* and α^* such that σ^* is indecomposable and irreducible, and $\pi^* = \sigma^* * \alpha^*$.

n	0	1	2	3	4	5	6	7	8	9
so_n	0	0	0	8	78	756	7782	85904	1016626	12865852
s_n	0	4	4	20	130	1040	9626	99692	1132998	13959224

Table 3.1: First elements of the sequences so_n and s_n.

- The marked permutation π^* is $\oplus$-decomposable.
- The marked permutation π^* is $\ominus$-decomposable.
- $\pi^* = \bar{1}$.

In particular, we have the following equation

$$P^*(x) = S_o^*(x)P^*(x) + 2(P(x) - P^{\oplus}(x))' + 1\,. \tag{3.15}$$

Proof. If π^* is indecomposable, then, either $\pi^* = \bar{1}$, or there are unique marked permutations σ^* and α^* such that σ^* is indecomposable and irreducible, and $\pi^* = \sigma^* * \alpha^*$, according to Lemma 3.5.10. This concludes the first part of the lemma.

Further, we observe that

$$(P(x) - P^{\oplus}(x))'$$

counts the marked permutations that are $\oplus$-decomposable (and by symmetry, the ones that are $\ominus$-decomposable). $\square$

Proof of Proposition 3.7.2. From (3.15) along with the fact that $P^*(x) = P'(x)$, we have that

$$S_o^*(x) = -1 + 2P(x)^{-2} - P'(x)^{-1}\,,$$

$$S^*(x) = 3 + 2P(x)^{-2} - P'(x)^{-1} - 4P(x)^{-1}\,,$$

as desired. $\square$

Chapter 4

The kernel of chromatic quasi-symmetric functions on graphs and hypergraphic polytopes

This chapter is a work based on the article [Pen18], to be published in *Journal of Combinatorial Theory A*. A short version was published in the proceedings of *Formal Power Series and Algebraic Combinatorics* (talk presentation). The article [Pen18] is split into this chapter and Chapter 5. The work in Section 4.5 is original to this thesis.

4.1 Introduction

Chromatic function on graphs

For a graph G with vertex set $V(G)$, a coloring f of the graph G is a function $f : V(G) \to \mathbb{N}$. A coloring is *proper* in G if no edge is monochromatic.

We denote by $\mathbf{G}$ the graph Hopf algebra, which is a vector space freely generated by the graphs whose vertex sets are of the form $[n]$ for some $n \geq 0$. This can be endowed with a Hopf algebra structure, as described by Schmitt in [Sch94, Chapter 12], and also presented below in Section 4.2.2.

Stanley defines in [Sta95] the *chromatic symmetric function* of G in commuting variables $\{x_i\}_{i\geq 1}$ as

$$\Psi_{\mathbf{G}}(G) = \sum\nolimits_f x_f \,, \tag{4.1}$$

where we write $x_f = \prod_{v \in V(G)} x_{f(v)}$, and the sum runs over proper colorings of G. Note that $\Psi_{\mathbf{G}}(G)$ is in the ring *Sym* of symmetric functions. The ring *Sym* is a Hopf subalgebra of *QSym*, the ring of quasi-symmetric functions introduced by Gessel in [Ges84]. A long standing conjecture in this subject, commonly referred to as the *tree conjecture*, is that if two trees T_1, T_2 are not isomorphic, then $\Psi_{\mathbf{G}}(T_1) \neq \Psi_{\mathbf{G}}(T_2)$.

When $V(G) = [n]$, the natural ordering on the vertices allows us to consider a non-commutative analogue of $\Psi_{\mathbf{G}}$, as done by Gebhard and Sagan in [GS01]. They define the chromatic symmetric function on non-commutative variables $\{\mathbf{a}_i\}_{i\geq 1}$ as

$$\boldsymbol{\Upsilon}_{\mathbf{G}}(G) = \sum\nolimits_f \mathbf{a}_f \,,$$

where we write $\mathbf{a}_f = \mathbf{a}_{f(1)} \dots \mathbf{a}_{f(n)}$, and we sum over the proper colorings f of G.

Note that $\boldsymbol{\Upsilon}_{\mathbf{G}}(G)$ is homogeneous and symmetric in the variables $\{\mathbf{a}_i\}_{i\geq 1}$. Such power series are called *word symmetric functions*. The ring of word symmetric functions, **WSym** for short, was introduced in [RS06], and is sometimes called the ring of symmetric functions in non-commutative variables, or **NCSym**, for instance in [BZ09]. Here we adopt the former name to avoid confusion with the ring of non-commutative symmetric functions.

In this chapter we describe generators for $\ker \Psi_{\mathbf{G}}$ and $\ker \boldsymbol{\Upsilon}_{\mathbf{G}}$. A similar problem was already considered for posets. In [Fér15], Féray studies $\Psi_{\mathbf{Pos}}$, the Gessel quasi-symmetric function defined on the poset Hopf algebra, and describes a set of generators of its kernel.

Some elements of the kernel of $\Psi_{\mathbf{G}}$ have already been constructed in [GP13] by Guay-Paquet and independently in [OS14] by Orellana and Scott. These relations, called *modular relations*, extend naturally to the non-commutative case. We introduce them now.

Given a graph G and an edge set E that is disjoint from $E(G)$, let $G \cup E$ denote the graph G with the edges in E added. If we have edges $e_3 \in G$ and $e_1, e_2 \notin G$ such that $\{e_1, e_2, e_3\}$ forms a triangle, then we also have

$$\boldsymbol{\Upsilon}_{\mathbf{G}}(G) - \boldsymbol{\Upsilon}_{\mathbf{G}}(G \cup \{e_1\}) - \boldsymbol{\Upsilon}_{\mathbf{G}}(G \cup \{e_2\}) + \boldsymbol{\Upsilon}_{\mathbf{G}}(G \cup \{e_1, e_2\}) = 0\,. \tag{4.2}$$

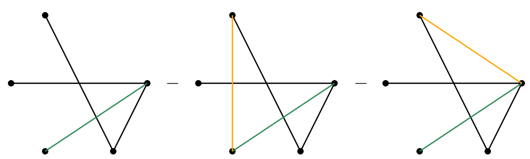


FIGURE 4.1: Example of a modular relation.

We call the formal sum $G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ in $\mathbf{G}$ a *modular relation on graphs*. An example is given in Fig. 4.1. Our first result is that these modular relations span the kernel of the chromatic symmetric function in non-commuting variables. The structure of the proof also allows us to compute the image of the map.

Theorem 4.1.1 (Kernel and image of $\mathbf{\Upsilon_G} : \mathbf{G} \to \mathbf{WSym}$)**.** The modular relations span $\ker \mathbf{\Upsilon_G}$. The image of $\mathbf{\Upsilon_G}$ is $\mathbf{WSym}$.

Two graphs G_1, G_2 are said to be isomorphic if there is a bijection between the vertices that preserves edges. For the commutative version of the chromatic symmetric function, if two isomorphic graphs G_1, G_2 are given, it holds that $\Psi_{\mathbf{G}}(G_1)$ and $\Psi_{\mathbf{G}}(G_2)$ are the same. The formal sum in $\mathbf{G}$ given by $G_1 - G_2$ is called an *isomorphism relation on graphs*.

Theorem 4.1.2 (Kernel and image of $\Psi_{\mathbf{G}} : \mathbf{G} \to Sym$)**.** The modular relations and the isomorphism relations generate the kernel of the commutative chromatic symmetric function $\Psi_{\mathbf{G}}$. The image of $\Psi_{\mathbf{G}}$ is *Sym*.

It was already noticed that $\Psi_{\mathbf{G}}$ is surjective. For instance, in [CvW15], several bases of Sym_n are constructed, which are the chromatic symmetric function of graphs, namely are of the form $\{\Psi_{\mathbf{G}}(G_\lambda) | \lambda \vdash n\}$ for suitable graphs G_λ on n vertices. Here we present a new such family of graphs.

At the end of Section 4.3 we introduce a new graph invariant $\tilde{\Psi}$, called the *augmented chromatic invariant*. We observe that modular relations on graphs are in the kernel of the augmented chromatic invariant. It follows from Theorem 4.1.2 that $\ker \Psi_G = \ker \tilde{\Psi}$. This reduces the tree conjecture in $\Psi_{\mathbf{G}}$ to a similar conjecture on this new invariant $\tilde{\Psi}$, which contains seemingly more information.

Generalized Permutahedra

Another goal of this chapter is to look at other kernel problems of chromatic flavor. In particular, we establish similar results to Theorems 4.1.1 and 4.1.2 in the combinatorial Hopf algebra of hypergraphic polytopes, which is a Hopf subalgebra of generalized permutahedra.

Generalized permutahedra form a family of polytopes that include permutahedra, associahedra and graph zonotopes. This family has been studied, for instance, in [PRW08], and we introduce it now.

Recall that the Minkowski sum of two polytopes $\mathfrak{a}, \mathfrak{b}$ is set as $\mathfrak{a}+\mathfrak{b} = \{a+b | \, a \in \mathfrak{a}, \, b \in \mathfrak{b}\}$. The Minkowski difference $\mathfrak{a} - \mathfrak{b}$ is only sometimes defined: it is the unique polytope $\mathfrak{c}$ that satisfies $\mathfrak{b} + \mathfrak{c} = \mathfrak{a}$, if it exists. We denote as $\sum_i \mathfrak{a}_i$ the Minkowski sum of several polytopes.

If we let $\{e_i | i \in I\}$ be the canonical basis of $\mathbb{R}^I$, a *simplex* is a polytope of the form $\mathfrak{s}_J = \text{conv}\{e_j | j \in J\}$ for non-empty $J \subseteq I$. A generalized permutahedron in $\mathbb{R}^I$ is a polytope given by real numbers $\{a_J\}_{\emptyset \neq J \subseteq I}$ as follows: Let $A_+ = \{J | a_J > 0\}$ and $A_- = \{J | a_J < 0\}$. Then, the corresponding generalized permutahedron is

$$\mathfrak{q} = \left(\sum_{J \in A_+} a_J \mathfrak{s}_J \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_J \right) , \tag{4.3}$$

if the Minkowski difference exists. We identify a generalized permutahedron $\mathfrak{q}$ with the list $\{a_J\}_{\emptyset \neq J \subseteq I}$. Note that not every list of real numbers will give us a generalized permutahedron, since the Minkowski difference is not always defined.

In [Pos09], generalized permutahedra are introduced in a different manner. A polytope is said to be a generalized permutahedron if it can be described as

$$\mathfrak{q} = \left\{ x \in \mathbb{R}^n | \sum_{i \in I} x_i \leq z_I \text{ for } I \subsetneq [n] \text{ non-empty } ; \sum_{i \in [n]} x_i = z_{[n]} \right\} ,$$

for reals $\{z_J\}_{\emptyset \neq J \subseteq I}$.

A third definition of generalized permutahedra is present in [AA17]. Here, a generalized permutahedron is a polytope whose normal fan coarsens the one of the permutahedron. These three definitions are equivalent, and a discussion regarding this can be seen in Section 4.2.4.

A *hypergraphic polytope* is a generalized permutahedron where the coefficients a_J in (4.3) are non-negative. For a hypergraphic polytope $\mathfrak{q}$, we denote by $\mathcal{F}(\mathfrak{q}) \subseteq 2^I \setminus \{\emptyset\}$ the family of sets $J \subseteq I$ such that $a_J > 0$. A *fundamental hypergraphic polytope* on $\mathbb{R}^I$ is a hypergraphic polytope $\sum_{\emptyset \neq J \subseteq I} a_J \mathfrak{s}_J$ such that $a_J \in \{0, 1\}$. Finally, for a set $A \subseteq 2^I \setminus \{\emptyset\}$, we write $\mathcal{F}^{-1}(A)$ for the hypergraphic polytopes $\mathfrak{q} = \sum_{J \in A} \mathfrak{s}_J$. Note that a fundamental hypergraphic polytope is of the form $\mathcal{F}^{-1}(A)$ for some family $A \subseteq 2^I \setminus \{\emptyset\}$.

One can easily note that the hypergraphic polytope $\mathfrak{q}$ and $\mathcal{F}^{-1}(\mathcal{F}(\mathfrak{q}))$ are, in general, distinct, so some care will come with this notation. However, the face structure is the same, and we give an explicit combinatorial equivalence in Proposition 4.4.9. If $\mathfrak{q}$ is a hypergraphic polytope such that $\mathcal{F}(\mathfrak{q})$ is a building set, then $\mathfrak{q}$ is called a *nestohedron*, see

[Pil17] and [AA17]. Hypergraphic polytopes and its subfamilies are studied in [AA17, Part 4], where they are also called y-positive generalized permutahedra.

In [AA17], Aguiar and Ardila define **GP**, a Hopf algebra structure on the linear space generated by generalized permutahedra in $\mathbb{R}^n$ for $n \geq 0$. The Hopf subalgebra **HGP** is the linear subspace generated by hypergraphic polytopes. We warn the reader of the use of the same notation for Minkowski operations (Minkowski sum and dilations) and for algebraic operations in **GP**. However, the distinction should be clear from the context. In [Dok11], generalized permutahedra are also debated.

In [Gru16], Grujić introduced a quasi-symmetric map in generalized permutahedra $\Psi_{\mathbf{GP}} : \mathbf{GP} \to QSym$ that was extended to a weighted version in [GPS19]. For a polytope $\mathfrak{q} \subseteq \mathbb{R}^I$, Grujić defines a function $f : I \to \mathbb{N}$ to be $\mathfrak{q}$-generic if the face of $\mathfrak{q}$ that minimizes $\sum_{i\in I} f(i)x_i$, denoted $\mathfrak{q}_f$, is a point. Equivalently, f is $\mathfrak{q}$-generic if it lies in the interior of the normal cone of some vertex of $\mathfrak{q}$. Then, Grujić defines for a set $\{x_i\}_{i\geq 1}$ of commutative variables, the quasi-symmetric function:

$$\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} x_f \, . \tag{4.4}$$

This quasi-symmetric function is called the *chromatic quasi-symmetric function on generalized permutahedra*, or simply *chromatic quasi-symmetric function*.

We discuss now a non-commutative version of $\Psi_{\mathbf{GP}}$, where we establish an analogue of Theorem 4.1.1 for hypergraphic polytopes. For that, consider the Hopf algebra of word quasi-symmetric functions **WQSym**, an analogue of $QSym$ in non-commutative variables introduced in [NT06] that is also called non-commutative quasi-symmetric functions, or **NCQSym**, for instance in [BZ09]. For a generalized permutahedron $\mathfrak{q}$ and non-commutative variables $\{\mathbf{a}_i\}_{i\geq 1}$, let $\mathbf{a}_f = \mathbf{a}_{f(1)} \cdots \mathbf{a}_{f(n)}$ and define

$$\Upsilon_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} \mathbf{a}_f \, .$$

We see from Proposition 4.2.11 that $\Upsilon_{\mathbf{GP}}(\mathfrak{q})$ is a word quasi-symmetric function. Moreover, a straightforward computation shows that $\Upsilon_{\mathbf{GP}}$ defines a Hopf algebra morphism between **GP** and **WQSym**. Let us call $\Psi_{\mathbf{HGP}}$ and $\Upsilon_{\mathbf{HGP}}$ the restrictions of $\Psi_{\mathbf{GP}}$ and $\Upsilon_{\mathbf{GP}}$ to **HGP**, respectively.

Our next theorems describe the kernel of the maps $\Psi_{\mathbf{HGP}}$ and $\Upsilon_{\mathbf{HGP}}$, using two types of relations:

- the *simple relations*, which are presented in Proposition 4.4.9, and convey that $\Upsilon_{\mathbf{GP}}(\mathfrak{q})$ only depends on which coefficients a_I are positive;
- the *modular relations*, which are exhibited in Theorem 4.4.10. We note for future reference that these generalize the ones for graphs: some of the modular relations on hypergraphic polytopes are the image of modular relations on graphs by a suitable embedding map Z, introduced below.

The simple relations allow us to reduce the kernel problem to the subspace of **HGP** spanned by *fundamental hypergraphic polytopes*, that is polytopes of the form $\mathcal{F}^{-1}(A)$.

Remark 4.1.3. Note that the span of the fundamental hypergraphic polytopes does not form a Hopf algebra, as it is not stable for the coproduct.

Theorem 4.1.4 (Kernel and image of of $\Upsilon_{\mathbf{HGP}} : \mathbf{HGP} \to \mathbf{WQSym}$)**.** The space $\ker \Upsilon_{\mathbf{HGP}}$ is generated by the simple relations and the modular relations on hypergraphic polytopes. The image of $\Upsilon_{\mathbf{HGP}}$ is **SC**, a proper subspace of **WQSym** introduced in Definition 4.4.4 below.

Let us denote by $\mathbf{WQSym}_n$ the linear space of homogeneous word quasi-symmetric functions of degree n, and let $\mathbf{SC}_n = \mathbf{SC} \cap \mathbf{WQSym}_n$. A monomial basis for **SC** is presented in Definition 4.4.4. An asymptotic for the dimension of $\mathbf{SC}_n$ is computed in Proposition 4.4.14, where in particular it is shown that it is exponentially smaller than the dimension of $\mathbf{WQSym}_n$.

Two generalized permutahedra $\mathfrak{q}_1, \mathfrak{q}_2$ are isomorphic if $\mathfrak{q}_1$ can be obtained from $\mathfrak{q}_2$ by rearranging the coordinates of the points in $\mathfrak{q}_1$. If $\mathfrak{q}_1, \mathfrak{q}_2$ are isomorphic, the chromatic quasi-symmetric functions $\Psi_{\mathbf{GP}}(\mathfrak{q}_1)$ and $\Psi_{\mathbf{GP}}(\mathfrak{q}_2)$ are the same. We say that $\mathfrak{q}_1 - \mathfrak{q}_2$ is an *isomorphism relation on hypergraphic polytopes.*

Theorem 4.1.5 (Kernel and image of $\Psi_{\mathbf{HGP}} : \mathbf{HGP} \to QSym$)**.** The linear space $\ker \Psi_{\mathbf{HGP}}$ is generated by the simple relations, the modular relations and the isomorphism relations. The image of $\Psi_{\mathbf{HGP}}$ is $QSym$.

In [AA17], Aguiar and Ardila define the graph zonotope, a Hopf algebra embedding $Z : \mathbf{G} \to \mathbf{GP}$ discussed above. Remarkably, we have that $\Psi_{\mathbf{G}} \circ Z = \Psi_{\mathbf{GP}}$. They also define other polytopal embeddings from other combinatorial Hopf algebras $\mathbf{H}$, like matroids, to **GP**. One associates a universal morphism $\Psi_{\mathbf{H}}$ to these Hopf algebras that also satisfy $\Psi_{\mathbf{GP}} \circ Z = \Psi_{\mathbf{H}}$. These universal morphisms are discussed below.

In particular, we can see that $Z(\ker \Psi_{\mathbf{H}}) = \ker \Psi_{\mathbf{GP}} \cap Z(\mathbf{H})$. This relation between $\ker \Psi_{\mathbf{H}}$ and $\ker \Psi_{\mathbf{GP}}$ is the main motivation to describe $\ker \Psi_{\mathbf{GP}}$, and indicates that

$\ker \Psi_{\mathbf{GP}}$ is the kernel problem that deserves most attention. In this chapter, we leave the description of $\ker \Psi_{\mathbf{GP}}$ as an open problem.

Most of the combinatorial objects embedded in **GP** are also embedded in **HGP**, such as graphs and matroids, so a description of $\ker \Psi_{\mathbf{GP}}$ is already interesting.

We remark that a description of the generators of $\Psi_{\mathbf{GP}}$ or $\ker \Psi_{\mathbf{HGP}}$ does not entail a description of the generators of a generic $\ker \Psi_{\mathbf{H}}$. For that reason, the kernel problem on matroids and on simplicial complexes is still open, despite these Hopf algebras being realized as Hopf subalgebras of **HGP**.

Universal morphisms

For a Hopf algebra $\mathbf{H}$, a *character* η of $\mathbf{H}$ is a linear map $\eta : \mathbf{H} \to \mathbb{K}$ that preserves the multiplicative structure and the unit of $\mathbf{H}$. We define a *combinatorial Hopf algebra* as a pair $(\mathbf{H}, \eta)$ where $\mathbf{H}$ is a Hopf algebra and $\eta : \mathbf{H} \to \mathbb{K}$ a character of $\mathbf{H}$. For instance, consider the ring of quasi-symmetric functions $QSym$ introduced in [Ges84] with its monomial basis $\{M_\alpha\}$, indexed by compositions. Then, $QSym$ has a combinatorial Hopf algebra structure $(QSym, \eta_0)$, by setting $\eta_0(M_\alpha) = 1$ whenever α has one or zero parts.

In [ABS06], Aguiar, Bergeron, and Sottile showed that any combinatorial Hopf algebra $(\mathbf{H}, \eta)$ has a unique combinatorial Hopf algebra morphism $\Psi_{\mathbf{H}} : \mathbf{H} \to QSym$, *i.e.*, a Hopf algebra morphism that satisfies $\eta_0 \circ \Psi_{\mathbf{H}} = \eta$. In other words, $(QSym, \eta_0)$ is a terminal object in the category of combinatorial Hopf algebras. The construction of $\Psi_{\mathbf{H}}$ is given in [ABS06] and also presented below in Section 4.2. We will refer to these maps as the universal maps to $QSym$.

The commutative invariants previously shown on graphs $\Psi_{\mathbf{G}}$, on posets $\Psi_{\mathbf{Pos}}$ and on generalized permutahedra $\Psi_{\mathbf{GP}}$ can be obtained as universal maps to $QSym$. If we take the character $\eta(G) = \mathbb{1}[G \text{ has no edges}]$ on the graphs Hopf algebra, the unique combinatorial Hopf algebra morphism $\mathbf{G} \to QSym$ is exactly the map $\Psi_{\mathbf{G}}$. With the Hopf algebra structure imposed on **GP** in [AA17], if we consider the character $\eta(\mathfrak{q}) = \mathbb{1}[\mathfrak{q} \text{ is a point}]$, then $\Psi_{\mathbf{GP}}$ is the universal map from **GP** to $QSym$. On posets, the Hopf algebra structure considered is the one presented in [GR14] and the character that is considered is $\eta(P) = \mathbb{1}[P \text{ is an antichain}]$.

To see the maps $\Upsilon_{\mathbf{G}} : \mathbf{G} \to \mathbf{WSym}$ and $\Upsilon_{\mathbf{GP}} : \mathbf{GP} \to \mathbf{WQSym}$ as universal maps, we need a parallel of the universal property of $QSym$ in the non-commutative world. The fitting property is better described in the context of Hopf monoids in vector species. Consider the Hopf monoid $\mathtt{WQSym}$, which is presented in [AM10] as the Hopf monoid of faces. It is seen that for any connected Hopf monoid $\mathtt{h}$ there is a unique Hopf monoid morphism $\boldsymbol{\Upsilon}_{\mathtt{h}}$ between $\mathtt{h}$ and $\mathtt{WQSym}$. In the last chapter we establish another proof of this fact, using resources from character theory, and expand on that showing that instead of a connected Hopf monoid we can take any combinatorial Hopf monoid, for a suitable notion of combinatoric Hopf monoid.

The relationship between Hopf algebras and Hopf monoids is very well captured with the so called Fock functors, mapping Hopf monoids to Hopf algebras, and Hopf monoid morphisms to Hopf algebra morphism. In particular, the full Fock functor $\mathcal{K}$ satisfies $\mathcal{K}(\mathtt{WQSym}) = \mathbf{WQSym}$. Then, the universal property of $\mathtt{WQSym}$ gives us a Hopf algebra morphism $\mathcal{K}(\boldsymbol{\Upsilon}_{\mathtt{h}})$ from $\mathcal{K}(\mathtt{h})$ to $\mathbf{WQSym}$. The maps $\boldsymbol{\Upsilon}_{\mathbf{G}}, \boldsymbol{\Upsilon}_{\mathbf{GP}}$ arise precisely in this way, when applying $\mathcal{K}$ to the unique combinatorial Hopf monoid morphism from the Hopf monoid on graphs $\mathtt{Gr}$ and of generalized permutahedra $\mathtt{GP}$ to $\mathtt{WQSym}$. In particular, we observe that $\mathcal{K}(\mathtt{Gr}) = \mathbf{G}$ and $\mathcal{K}(\mathtt{GP}) = \mathbf{GP}$. If we consider the poset Hopf monoid $\mathtt{Pos}$, the universal property of the combinatorial Hopf monoid $\mathtt{WQSym}$ gives us a non-commutative analogue $\boldsymbol{\Upsilon}_{\mathbf{Pos}}$ of the Gessel invariant, which coincides with the one presented in [Fér15]. In particular, $\mathcal{K}(\mathtt{Pos}) = \mathbf{Pos}$. We will refer to these Hopf algebra morphisms as the universal maps to $\mathbf{WQSym}$.

Finally, our previous results have an interesting consequence. We show that, because $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$ is not surjective, there is no combinatorial Hopf monoid morphism from the Hopf monoid on posets to the Hopf monoid on hypergraphic polytopes. However, in [AA17] a Hopf monoid morphism from posets to extended generalized permutahedra is constructed. With this result we obtain that this map cannot be restricted from extended generalized permutahedra to generalized permutahedra.

This chapter and the following one are organized as follows: In Section 4.2 we address the preliminaries, where the reader can find the linear algebra tools that we use, the introduction to the main Hopf algebras of interest, and the proof that the several definitions of a generalized permutahedra are equivalent. In Section 4.3 we prove Theorems 4.1.1 and 4.1.2, and we study the augmented chromatic invariant. In Section 4.4 we prove Theorems 4.1.4 and 4.1.5, and we present asymptotics for the dimension of the graded Hopf algebra **SC**. In Chapter 5 we present the universal property of $\mathtt{WQSym}$. In Section 4.3.2 we find some relations between the coefficients of the augmented chromatic symmetric function and the coefficients of the original chromatic symmetric function on graphs.

4.2 Preliminaries

There are natural maps **WSym** $\to$ *Sym* and **WQSym** $\to$ *QSym* by allowing the variables to commute. We denote these maps by *comu*.

For an equivalence relation $\sim$ on a set A, we write $[x]_\sim$ for the equivalence class of x in $\sim$, and write $[x]$ when $\sim$ is clear from context. We write both $\mathcal{E}(\sim)$ and $A/\sim$ for the set of equivalence classes of $\sim$. All the vector spaces and algebras are over a generic field $\mathbb{K}$ of characteristic zero.

4.2.1 Linear algebra preliminaries

The following linear algebra lemmas will be useful to compute generators of the kernels and the images of Ψ and $\boldsymbol{\Upsilon}$. These lemmas describe a sufficient condition for a set $\mathcal{B}$ to span the kernel of a linear map $\phi : V \to W$.

Lemma 4.2.1. Let V be a finite dimensional vector space with basis $\{a_i | i \in [m]\}$, $\phi : V \to W$ be a linear map, and $\mathcal{B} = \{b_j | j \in J\} \subseteq \ker \phi$ be a family of relations.

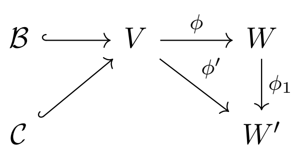
Assume that there exists $I \subseteq [m]$ such that:

- the family $\{\phi(a_i)\}_{i \in I}$ is linearly independent in W,
- for $i \in [m] \setminus I$ we have $a_i = b + \sum_{k=i+1}^{m} \lambda_{k,i} a_k$ for some $b \in \mathcal{B}$ and some scalars $\lambda_{k,i}$;

Then $\mathcal{B}$ spans $\ker \phi$. Additionally, we have that $\{\phi(a_i)\}_{i \in I}$ is a basis of the image of ϕ.

The following lemma will help us dealing with the composition $\Psi = \mathrm{comu} \circ \boldsymbol{\Upsilon}$: we give a sufficient condition for a natural enlargement of the set $\mathcal{B}$ to generate $\ker \Psi$, given that $\mathcal{B}$ already generates $\ker \boldsymbol{\Upsilon}$.

Lemma 4.2.2. We will use the same notation as in Lemma 4.2.1. Additionally, consider $\phi_1 : W \to W'$ linear map and write $\phi' = \phi_1 \circ \phi$. Take the equivalence relation $\sim$ in $\{a_i\}_{i \in [m]}$ that satisfies $a_i \sim a_j$ whenever $\phi'(a_i) = \phi'(a_j)$. Let $\mathcal{C} = \{a_i - a_j | \, a_i \sim a_j\}$ and write $\phi'([a_i]) = \phi'(a_i)$ with no ambiguity.



(4.5)

Assume the hypotheses in Lemma 4.2.1 and, additionally, suppose that the family $\{\phi'([a_i])\}_{[a_i]\in\mathcal{E}(\sim)}$ is linearly independent in W'.

Then, $\ker\phi'$ is generated by $\mathcal{B}\cup\mathcal{C}$. Furthermore, $\{\phi'([a_i])\}_{[a_i]\in\mathcal{E}(\sim)}$ is a basis of $\operatorname{im}\phi'$.

Proof of Lemma 4.2.1. Suppose, for sake of contradiction, that there is some element $c\in\ker\phi\setminus\operatorname{span}\mathcal{B}$. In particular $c\neq 0$. Write

$$c=\sum_{k=1}^{m}\tau_k a_k\,, \tag{4.6}$$

and note that if $\tau_i=0$ for every $i\notin I$, then

$$0=\phi(c)=\phi\left(\sum_{k\in I}\tau_k a_k\right)=\sum_{k\in I}\tau_k\phi(a_k),$$

which, by linear independence of $\{\phi(a_k)\}_{k\in I}$, implies that $\tau_k=0$ for every $k\in I$, contradicting $c\neq 0$. Therefore, we have $\tau_i\neq 0$ for $i\notin I$ whenever $c\in\ker\phi\setminus\operatorname{span}\mathcal{B}$.

Consider the smallest index $i_c\in[m]\setminus I$ such that τ_i is non-zero. Consider $c\in\ker\phi\setminus\langle\mathcal{B}\rangle$ that maximizes i_c.

Thus, we can write

$$c=\sum_{j\in I}\tau_j a_j+\sum_{\substack{j\in[m]\setminus I\\ j\geq i_c}}\tau_j a_j\,. \tag{4.7}$$

By hypotheses, because $i_c\notin I$, there is some $b'\in\mathcal{B}$ such that:

$$a_{i_c}=b'+\sum_{j=i_c+1}^{m}\lambda_{j,i_c}a_j\,.$$

So applying this to (4.7) gives us:

$$\begin{aligned}c-\tau_{i_c}b'&=\sum_{j\in I}\tau_j a_j+\sum_{\substack{j\in[m]\setminus I\\ j\geq i_c}}\tau_j a_j-\tau_{i_c}a_{i_c}+\sum_{k=i_c+1}^{m}\tau_{i_c}\lambda_{k,i_c}a_k\\&=\sum_{j\in I}\tau_j a_j+\sum_{\substack{j\in[m]\setminus I\\ j> i_c}}\tau_j a_j+\sum_{j=i_c+1}^{m}\tau_{i_c}\lambda_{j,i_c}a_j\,.\end{aligned} \tag{4.8}$$

Note that $c-\tau_{i_c}b_j\in\ker\phi\setminus\operatorname{span}\mathcal{B}$ which contradicts the maximality of i_c. From this we conclude that there are no elements c in $\ker\phi\setminus\operatorname{span}\mathcal{B}$.

To show that the family $\{\phi(a_i)\}_{i\in I}$ is a basis of $\operatorname{im}\phi$, we just need to establish that this is a generating set. Naturally, $\{\phi(a_i)\}_{i\in I} \cup \{\phi(a_i)\}_{i\in[m]}$ is a generating set because it is the image of a basis of V. We show by induction that $\{\phi(a_i)\}_{i\in I} \cup \{\phi(a_i)\}_{i\in[m]\setminus[k]}$ is a generating set for any non-negative $k \leq m+1$. This concludes the proof, since the original claim is this for $k = m+1$.

Indeed, if $\{\phi(a_i)\}_{i\in I} \cup \{\phi(a_i)\}_{i\in[m]\setminus[k]}$ is a generating set of $\operatorname{im}\phi$, then we note $a_k = b + \sum_{j=k+1}^{m} \lambda_{j,k} a_j$ for some $b \in \mathcal{B}$, so

$$\phi(a_k) \in \operatorname{span}\{\phi(a_i)\}_{i\in I} \cup \{\phi(a_i)\}_{i\in[m]\setminus[k+1]} .$$

This concludes the induction step. □

Proof of Lemma 4.2.2. Define $I' = \{\max A \cap I | \, A \in \mathcal{E}(\sim)\}$. Note that for every $j \in I \setminus I'$ there is $c \in \mathcal{C}$ such that $a_j = c + \sum_{k=j+1}^{m} \lambda_{k,j} a_k$. Indeed it is enough to choose $i \sim j$ with $i \in I'$, to write $a_j = \underbrace{a_j - a_i}_{\in \mathcal{C}} + a_i$.

So, the set $I' \subseteq [m]$ satisfies both that:

- We have by hypothesis that $\{\phi'(a_i)\}_{i\in I'} = \{\phi'([a_i])\}_{i\in I'}$ is linearly independent in W';
- For $i \in [m] \setminus I'$ we can write $a_i = b + \sum_{k=j+1}^{m} \lambda_{k,i} a_k$ for some $b \in \mathcal{B} \cup \mathcal{C}$ and some scalars $\lambda_{k,i}$.

Now applying Lemma 4.2.1 to I' instead of I, to ϕ' instead of ϕ and to $\mathcal{B} \cup \mathcal{C}$ instead of $\mathcal{B}$ tells us that $\mathcal{B} \cup \mathcal{C}$ generates $\ker \phi'$, and that $\{\phi'([a_i]) | i \in I'\} = \{\phi'(a_i) | i \in I\}$ spans the image of ϕ', as desired. □

4.2.2 Hopf algebras and associated combinatorial objects

In the following, all the Hopf algebras $\mathbf{H}$ have a grading, denoted by $\mathbf{H} = \oplus_{n\geq 0} \mathbf{H}_n$.

An *integer composition*, or simply a composition, of n, is a list $\alpha = (\alpha_1, \cdots, \alpha_k)$ of positive integers whose sum is n. We write $\alpha \models n$. We denote the length of the list by $l(\alpha)$ and we denote the set of compositions of size n by $\mathcal{C}_n$.

An *integer partition*, or simply a partition, of n, is a non-increasing list of positive integers $\lambda = (\lambda_1, \cdots, \lambda_k)$ whose sum is n. We write $\lambda \vdash n$. We denote the length of the list by $l(\lambda)$ and we denote the set of partitions of size n by $\mathcal{P}_n$. By disregarding the order of the parts on a composition α we obtain a partition $\lambda(\alpha)$.

A *set partition* $\boldsymbol{\pi} = \{\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_k\}$ of a set I is a collection of non-empty disjoint subsets of I, called *blocks*, that cover I. We write $\boldsymbol{\pi} \vdash I$. We denote the number of parts of the set partition by $l(\boldsymbol{\pi})$, and call it its length. We denote the family of set partitions of I by $\mathbf{P}_I$, or simply by $\mathbf{P}_n$ if $I = [n]$. By counting the elements on each block of $\boldsymbol{\pi}$, we obtain an integer partition denoted by $\lambda(\boldsymbol{\pi}) \vdash |I|$. We identify a set partition $\boldsymbol{\pi} \in \mathbf{P}_I$ with an equivalence relation $\sim_{\boldsymbol{\pi}}$ on I, where $x \sim_{\boldsymbol{\pi}} y$ if $x, y \in I$ are on the same block of $\boldsymbol{\pi}$.

A *set composition* $\vec{\boldsymbol{\pi}} = S_1 | \cdots | S_l$ of I is a list of non-empty disjoint subsets of I that cover I, which we call *blocks*. We write $\vec{\boldsymbol{\pi}} \models I$. We denote the size of the set composition by $l(\vec{\boldsymbol{\pi}})$. We write $\mathbf{C}_I$ for the family of set compositions of I, or simply $\mathbf{C}_n$ if $I = [n]$. By disregarding the order of a set composition $\vec{\boldsymbol{\pi}}$, we obtain a set partition $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}) \vdash I$. By counting the elements on each block of $\vec{\boldsymbol{\pi}}$, we obtain a composition denoted by $\alpha(\vec{\boldsymbol{\pi}}) \models |I|$. A set composition is naturally identified with a total preorder $R_{\vec{\boldsymbol{\pi}}}$ on I, where $x R_{\vec{\boldsymbol{\pi}}} y$ if $x \in S_i, y \in S_j$ for $i \leq j$.

Permutations act on set compositions and set partitions: for a set composition $\vec{\boldsymbol{\pi}} = (S_1, \cdots, S_k)$, a set partition $\pi = \{\pi^{(1)}, \cdots, \pi^{(k)}\}$ on I, and a permutation $\phi : I \to I$, we define the set composition $\phi(\vec{\boldsymbol{\pi}}) = (\phi(S_1), \cdots, \phi(S_k))$ and the set partition $\phi(\boldsymbol{\pi}) = \{\phi(\boldsymbol{\pi}^{(1)}), \cdots, \phi(\boldsymbol{\pi}^{(k)})\}$.

A *coloring* of the set I is a function $f : I \to \mathbb{N}$. The set composition type $\vec{\boldsymbol{\pi}}(f)$ of a coloring $f : I \to \mathbb{N}$ is the set composition obtained after deleting the empty sets of $f^{-1}(1)|f^{-1}(2)|\cdots$. This notation is extended to function $f : I \to \mathbb{R}$.

Definition 4.2.3. In partitions and in set partitions, we use the classical *coarsening orders* $\leq$ with the same notation, where we say that $\pi \leq \tau$ (resp. $\boldsymbol{\pi} \leq \boldsymbol{\tau}$) if τ is obtained from π by adding some parts of the original parts together (resp. if $\boldsymbol{\tau}$ is obtained from $\boldsymbol{\pi}$ by merging some blocks).

These objects relate to the Hopf algebras *Sym*, *QSym*, **WSym** and **WQSym**. The homogeneous component Sym_n (resp. $QSym_n$, $\mathbf{WSym}_n$ and $\mathbf{WQSym}_n$) of the Hopf algebra *Sym* (resp. *QSym*, **WSym**, **WQSym**) has a monomial basis indexed by partitions (resp. compositions, set partitions, set compositions), which we denote by $\{m_\lambda\}_{\lambda \in \mathcal{P}_n}$ (resp. $\{M_\alpha\}_{\alpha \in \mathcal{C}_n}$, $\{\mathbf{m}_{\boldsymbol{\pi}}\}_{\boldsymbol{\pi} \in \mathbf{P}_n}$ and $\{\mathbf{M}_{\vec{\boldsymbol{\pi}}}\}_{\vec{\boldsymbol{\pi}} \in \mathbf{C}_n}$).

4.2.3 Hopf algebras on graphs

Recall that for graphs, Gebhard and Sagan defined in [GS01] the non-commutative chromatic morphism. The following expression is given:

Lemma 4.2.4 ([GS01, Proposition 3.2])**.** For a graph G we say that a set partition $\boldsymbol{\tau}$ of $V(G)$ is proper if no block of $\boldsymbol{\tau}$ contains an edge. Then have that

$$\boldsymbol{\Upsilon}_{\mathbf{G}}(G) = \sum_{\boldsymbol{\tau}} \mathbf{m}_{\boldsymbol{\tau}} \,,$$

where the sum runs over all proper set partitions of $V(G)$.

4.2.4 Faces and a Hopf algebra structure of generalized permutahedra

In the following we identify I with $[n]$. For a set composition $\vec{\boldsymbol{\pi}} = S_k|\dots|S_1$ on $[n]$, recall that $R_{\vec{\boldsymbol{\pi}}}$ is a partial order on $[n]$. For a non-empty set $J \subseteq [n]$, define the set $J_{\vec{\boldsymbol{\pi}}} = \{\text{minima of J in } R_{\vec{\boldsymbol{\pi}}}\} = J \cap S_i$, where i is the smallest index with $J \cap S_i \neq \emptyset$. A coloring on $[n]$ is a map $f : [n] \to \mathbb{N}$. A real coloring on $[n]$ is a map $f : [n] \to \mathbb{R}$, and we identify the real coloring f with the linear function $f : \mathbb{R}^{[n]} \to \mathbb{R}$.

$$x \mapsto \sum_{i=1}^{n} f(i) x_i \,.$$

In the space $\mathbb{R}^{[n]}$, we define the simplices $\mathfrak{s}_J = \operatorname{conv}\{e_v | \, v \in J\}$ for each $J \subseteq [n]$. Recall that a *generalized permutahedron* is a Minkowski sum and difference of the form

$$\mathfrak{q} = \left(\sum_{\substack{J \neq \emptyset \\ a_J > 0}} a_J \mathfrak{s}_J \right) - \left(\sum_{\substack{J \neq \emptyset \\ a_J < 0}} |a_J| \mathfrak{s}_J \right) \,,$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$ that can be either positive, negative or zero.

Recall as well that a *hypergraphic polytope* is a generalized permutahedron of the form

$$\mathfrak{q} = \sum_{J \neq \emptyset} a_J \mathfrak{s}_J \,,$$

for non-negative reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$.

For a polytope $\mathfrak{q}$ and a real coloring f on $[n]$, we denote by $\mathfrak{q}_f$ the subset of $\mathfrak{q}$ on which f is minimized, that is

$$\mathfrak{q}_f := \arg\min_{x \in \mathfrak{q}} \sum_{i \in I} f(i) x_i \,.$$

A face of $\mathfrak{q}$ is the solution to such a linear optimization problem on $\mathfrak{q}$. A real coloring is said to be $\mathfrak{q}$-generic if the corresponding face is a point.

Note that if $J_1 \subseteq J_2$, then $\mathfrak{s}_{J_1}$ is a face of $\mathfrak{s}_{J_2}$. Incidentally, whenever f is a coloring that is minimal exactly in J_1, we have that $\mathfrak{s}_{J_1} = (\mathfrak{s}_{J_2})_f$. In fact, for a real coloring $f : [n] \to \mathbb{N}$ the face corresponding to f of a simplex is another simplex, specifically it we can directly compute that

$$(\mathfrak{s}_J)_f = \mathfrak{s}_{J_{\vec{\pi}(f)}} . \tag{4.9}$$

The following fact describes faces of the Minkowski sums and differences:

Lemma 4.2.5. Let f be a real coloring and $\mathfrak{a}, \mathfrak{b}$ two polytopes. Then $(\mathfrak{a}+\mathfrak{b})_f = \mathfrak{a}_f + \mathfrak{b}_f$ and, if the difference $\mathfrak{a} - \mathfrak{b}$ is well defined, $(\mathfrak{a} - \mathfrak{b})_f = \mathfrak{a}_f - \mathfrak{b}_f$.

Proof. Suppose that $m_\mathfrak{a}, m_\mathfrak{b}$ are the minima of f in the polytopes $\mathfrak{a}, \mathfrak{b}$. Let $x \in \mathfrak{a} + \mathfrak{b}$. So $x = a + b$ for some $a \in \mathfrak{a}, b \in \mathfrak{b}$.

Then $f(x) = f(a) + f(b) \geq m_\mathfrak{a} + m_\mathfrak{b}$. We have equality if and only if we have $a \in \mathfrak{a}_f, b \in \mathfrak{b}_f$, that is when $x \in \mathfrak{a}_f + \mathfrak{b}_f$.

Now $(\mathfrak{a} - \mathfrak{b})_f = \mathfrak{a}_f - \mathfrak{b}_f$ follows because $(\mathfrak{a} - \mathfrak{b})_f + \mathfrak{b}_f = \mathfrak{a}_f$ by the above. ☐

Definition 4.2.6 (Normal fan of a polytope)**.** A cone is a subset of an $\mathbb{R}$-vector space that is closed for addition and multiplication by positive scalars. For a polytope $\mathfrak{q}$ and $F \subseteq \mathfrak{q}$ one of its faces, we define its normal cone

$$\mathcal{N}_\mathfrak{q}(F) := \{f : [n] \to \mathbb{R} | \ \mathfrak{q}_f = F\} .$$

This is a cone in the dual space of $\mathbb{R}^n$. Moreover, the normal cones of all the faces of $\mathfrak{q}$ partition $(\mathbb{R}^n)^*$ into cones $\mathcal{N}_\mathfrak{q} = \{\mathcal{N}_\mathfrak{q}(F) | \, F \text{ is a face of } \mathfrak{q}\}$. This is the *normal fan* of $\mathfrak{q}$.

Example 4.2.7 (The normal fan of the n-permutahedron - The braid fan)**.** The faces of the permutahedron are indexed by $\mathbf{C}_n$. In particular, the corresponding normal cone of the face $F_{\vec{\pi}}$, corresponding to $\vec{\pi} \in \mathbf{C}_n$, is

$$\mathcal{N}(F_{\vec{\pi}}) = \{f : [n] \to \mathbb{R} | \vec{\pi}(f) = \vec{\pi}\} .$$

In the introduction we referred two other definitions of generalized permutahedra that are present in the literature. We recover them here, and justify their equivalence:

Lemma 4.2.8 (Definition 1 of generalized permutahedra, see [AA17])**.** A polytope $\mathfrak{q}$ is a generalized permutahedron in the sense of (4.3) if and only if its normal fan coarsens the one of the permutahedron. Specifically, for any two real colorings f_1, f_2, if $\vec{\pi}(f_1) = \vec{\pi}(f_2)$ then $\mathfrak{q}_{f_1} = \mathfrak{q}_{f_2}$.

Define the polytope $\mathcal{P}_n^z(\{z_I\}_{\emptyset\neq I\subseteq[n]})$ in the plane $\sum_i x_i = z_{[n]}$ given by the inequalities

$$\sum_{i\in I} x_i \geq z_I \,,$$

for some real numbers $\{z_I\}_{\emptyset\neq I\subseteq[n]}$.

Lemma 4.2.9 (Definition 2 of generalized permutahedra, see [Pos09])**.** A polytope is a generalized permutahedron if it can be expressed as $\mathcal{P}_n^z(\{z_I\}_{\emptyset\neq I\subseteq[n]})$ for real numbers $\{z_I\}_{I\subseteq[n]}$ such that

$$z_I + z_J \leq z_{I\cup J} + z_{I\cap J} \,,$$

for all non-empty sets $I, J \subseteq [n]$ that are not disjoint.

In [AA17, Theorem 12.3], is it shown that these two last notions of generalized permutahedra are equivalent. That is, a polytope $\mathfrak{q}$ is of the form $\mathfrak{q} = \mathcal{P}_n^z(\{z_I\}_{\emptyset\neq I\subseteq[n]})$ for real numbers $\{z_I\}_{\emptyset\neq I\subseteq[n]}$ if and only if its normal fan coarsens the one from the permutahedron.

In [ABD10, Proposition 2.4], Ardila, Benedetti and Doker show that any generalized permutahedron has an expression of the from given by Eq. (4.3). The main feature in that proof is the following: for real numbers $\{z_I\}_{\emptyset\neq I\subseteq[n]}$ such that $z_I + z_J \geq z_{I\cup J} + z_{I\cap J}$, if we choose reals $\{a_J\}_{\emptyset\neq J\subseteq[n]}$ such that $z_I = \sum_{\emptyset\neq J\subseteq I} a_J$, then Eq. (4.3) gives us a well defined polytope and in fact defines the same polytope as $\mathcal{P}_n^z(\{z_I\}_{\emptyset\neq I\subseteq[n]})$.

In the following we establish that the normal fan of a polytope of the form Eq. (4.3) coarsens the one of the n-permutahedron, concluding with the above that the three definitions of generalized permutahedra presented are equivalent.

Proposition 4.2.10. Let $\mathfrak{q}$ be a polytope of the form

$$\mathfrak{q} = \left(\sum_{J\in A_+} a_J \mathfrak{s}_J\right) - \left(\sum_{J\in A_-} |a_J| \mathfrak{s}_J\right) \,,$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset\neq J\subseteq[n]}$ that can be either positive, negative or zero, and $A_+ = \{J | a_J > 0\}$ and $A_- = \{J | a_J < 0\}$. Then its normal fan coarsens the one of the permutahedron.

Proof. Let f be a real coloring of $[n]$. As a consequence of Example 4.2.7 and as discussed in the end of Lemma 4.2.8, it is enough to establish that if f_1, f_2 satisfy $\vec{\pi}(f_1) = \vec{\pi}(f_2)$ then $\mathfrak{q}_{f_1} = \mathfrak{q}_{f_2}$. In fact, from Lemma 4.2.5 and (4.9), we have

$$\mathfrak{q}_f = \left(\sum_{J\in A_+} a_J (\mathfrak{s}_J)_f\right) - \left(\sum_{J\in A_-} |a_J| (\mathfrak{s}_J)_f\right) = \left(\sum_{J\in A_+} a_J \mathfrak{s}_{J_{\vec{\pi}(f)}}\right) - \left(\sum_{J\in A_-} |a_J| \mathfrak{s}_{J_{\vec{\pi}(f)}}\right) \,,$$

which clearly only depend on the set composition type of the real coloring f. □

Denote by $\mathfrak{q}_{\vec{\pi}}$ the face on $\mathfrak{q}$ that is the solution to any linear optimization problem on $\mathfrak{q}$ for a real coloring f with composition type $\vec{\pi}(f) = \vec{\pi}$, so that

$$\mathfrak{q}_{\vec{\pi}} = \left(\sum_{J \in A_+} a_J \mathfrak{s}_{J_{\vec{\pi}}} \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_{J_{\vec{\pi}}} \right) . \tag{4.10}$$

The following is a consequence of Lemma 4.2.8:

Proposition 4.2.11. If $\mathfrak{q}$ is a generalized permutahedron, then

$$\mathbf{\Upsilon}_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f\ \mathfrak{q}\text{-generic}} \mathbf{a}_f = \sum_{\mathfrak{q}_{\vec{\pi}} = \mathrm{pt}} \mathbf{M}_{\vec{\pi}} \in \mathbf{WQSym}_n . \tag{4.11}$$

We now turn away from the face structure of generalized permutahedra and debate its Hopf algebra structure, introduced in [AA17]. As usual, consider a generalized permutahedron $\mathfrak{q}$ given by (4.3). If $\vec{\pi} = A|B$ is a set composition of $[n]$, then $\mathfrak{q}_{\vec{\pi}}$ can be written as a Minkowski sum of polytopes

$$\mathfrak{q}_{\vec{\pi}} =: \mathfrak{q}|_A + \mathfrak{q}\backslash_A ,$$

where $\mathfrak{q}|_A$ is a generalized permutahedron in $\mathbb{R}^A$ and $\mathfrak{q}\backslash_A$ is a generalized permutahedron on $\mathbb{R}^B$. Note that $B = A^c$ so the dependence of $\mathfrak{q}|_A$ and $\mathfrak{q}\backslash_A$ on B is implicit.

We can obtain explicit expressions for $\mathfrak{q}|_A$ and $\mathfrak{q}\backslash_A$:

$$\mathfrak{q}|_A = \left(\sum_{\substack{J \in A_+ \\ J \not\subseteq B}} a_J \mathfrak{s}_{J \cap A} \right) - \left(\sum_{\substack{J \in A_- \\ J \not\subseteq B}} |a_J| \mathfrak{s}_{J \cap A} \right) , \mathfrak{q}\backslash_A = \left(\sum_{\substack{J \in A_+ \\ J \subseteq B}} a_J \mathfrak{s}_J \right) - \left(\sum_{\substack{J \in A_- \\ J \subseteq B}} |a_J| \mathfrak{s}_J \right) .$$

4.3 Main theorems on graphs

In this section we prove Theorems 4.1.1 and 4.1.2, which follow from Lemmas 4.2.1 and 4.2.2. We also discuss an application of Theorem 4.1.2 on the tree conjecture, by constructing a new graph invariant $\tilde{\Psi}(G)$ that satisfies the modular relations.

For a set partition $\boldsymbol{\pi}$, we define the graph $K_{\boldsymbol{\pi}}$ where $\{i, j\} \in E(K_{\boldsymbol{\pi}})$ if $i \sim_{\boldsymbol{\pi}} j$. This graph is the disjoint union of the complete graphs on the blocks of $\boldsymbol{\pi}$. We denote the complement of a graph G as G^c. Note that a set partition $\boldsymbol{\tau}$ is proper in $K_{\boldsymbol{\pi}}^c$ if and only if $\boldsymbol{\tau} \leq \boldsymbol{\pi}$ in the coarsening order on set partitions. Hence, as a consequence of Lemma 4.2.4,

$$\boldsymbol{\Upsilon}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) = \sum_{\boldsymbol{\tau} \leq \boldsymbol{\pi}} \mathbf{m}_{\boldsymbol{\tau}}. \tag{4.12}$$

We now show that the kernel of $\boldsymbol{\Upsilon}_{\mathbf{G}}$ is spanned by the modular relations.

Proof of Theorem 4.1.1. Recall that $\mathbf{G}_n$ is spanned by graphs with vertex set $[n]$. We choose an order $\tilde{\geq}$ in this family of graphs in a way that the number of edges is non-decreasing.

From (4.12), we know that the transition matrix of $\{\boldsymbol{\Upsilon}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) | \boldsymbol{\pi} \in \mathbf{P}_n\}$ over the monomial basis of **WSym** is upper triangular, hence forms a basis set of **WSym**. In particular, $\operatorname{im} \boldsymbol{\Upsilon}_{\mathbf{G}} = \mathbf{WSym}$.

In order to apply Lemma 4.2.1 to the set of modular relations on graphs, it suffices to show the following: if a graph G is not of the form $K^c_{\boldsymbol{\pi}}$, then we can find a formal sum $G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ that is a modular relation. Indeed, G is the graph with least edges in that expression, so it is the smallest in the order $\tilde{\geq}$. It follows from Lemma 4.2.1 that the modular relations generate the space $\ker \boldsymbol{\Upsilon}_{\mathbf{G}}$.

To find the desired modular relation, it is enough to find a triangle $\{e_1, e_2, e_3\}$ such that $e_1, e_2 \notin E(G)$ and $e_3 \in E(G)$. Consider $\boldsymbol{\tau}$, the set partition given by the connected components of G^c, so that $G \supseteq K^c_{\boldsymbol{\tau}}$. By hypothesis, $G \neq K^c_{\boldsymbol{\tau}}$, so there are vertices u, w in the same block of $\boldsymbol{\tau}$ that are not neighbors in G^c. Without loss of generality we can take such u, w that are at distance 2 in G^c, so they have a common neighbor v in G^c (see example in Fig. 4.2).

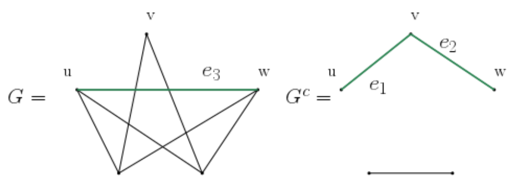


FIGURE 4.2: Choice of edges in proof of Theorem 4.1.1

The edges $e_1 = \{v, u\}$, $e_2 = \{v, w\}$ and $e_3 = \{u, w\}$ form the desired triangle, concluding the proof. □

Proof of Theorem 4.1.2. It is clear that $\Psi_{\mathbf{G}}$ is surjective, since $\boldsymbol{\Upsilon}_{\mathbf{G}}$ is surjective. Now, our goal is to apply Lemma 4.2.2 to the map $\Psi_{\mathbf{G}} = \text{comu} \circ \boldsymbol{\Upsilon}_{\mathbf{G}}$ and the equivalence relation corresponding to graph isomorphism. First, note that if $\lambda(\boldsymbol{\pi}) = \lambda(\boldsymbol{\tau})$ then $K^c_{\boldsymbol{\pi}}$ and $K^c_{\boldsymbol{\tau}}$ are isomorphic graphs. Define without ambiguity $r_{\lambda(\boldsymbol{\pi})} = \Psi_{\mathbf{G}}(K^c_{\boldsymbol{\pi}})$.

From the proof of Theorem 4.1.1, the hypotheses of Lemma 4.2.1 are satisfied. Therefore, to apply Lemma 4.2.2 it is enough to establish that the family $\{r_\lambda\}_{\lambda\in\mathcal{P}_n}$ is linearly independent. Indeed, it would follow that $\ker \Psi_{\mathbf{G}}$ is generated by the modular relations and the isomorphism relations, and $\{r_\lambda\}_{\lambda\in\mathcal{P}_n}$ is a basis of $\operatorname{im}\Psi_{\mathbf{G}}$ concluding the proof.

Recall that for set partitions $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ we have that $\boldsymbol{\pi}_1 \leq \boldsymbol{\pi}_2 \Rightarrow \lambda(\boldsymbol{\pi}_1) \leq \lambda(\boldsymbol{\pi}_2)$. The linear independence of $\{r_\lambda\}_{\lambda\in\mathcal{P}_n}$ follows from the fact that its transition matrix to the monomial basis is upper triangular under the coarsening order in integer partitions. Indeed, from (4.12), if we let $\boldsymbol{\tau}$ run over set partitions and σ run over integer partitions, we have

$$r_{\lambda(\boldsymbol{\pi})} = \Psi_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) = \sum_{\boldsymbol{\tau}\leq\boldsymbol{\pi}} m_{\lambda(\boldsymbol{\tau})} = \sum_{\sigma\leq\lambda(\boldsymbol{\pi})} a_{\boldsymbol{\pi},\sigma}\, m_\sigma\,,$$

where $a_{\boldsymbol{\pi},\sigma} = |\{\boldsymbol{\tau}\vdash[n] | \lambda(\boldsymbol{\tau}) = \sigma,\, \boldsymbol{\tau}\leq\boldsymbol{\pi}\}|$. Note that $a_{\boldsymbol{\pi},\lambda(\boldsymbol{\pi})} = 1$, so $\{r_\lambda\}_{\lambda\in\mathcal{P}_n}$ is linearly independent. □

Remark 4.3.1. We have obtained in the proof of Theorem 4.1.2 that $\{r_\lambda\}_{\lambda\vdash n}$ is a basis for Sym_n. This basis is different from other "chromatic bases" proposed in [CvW15]. The proof gives us a recursive way to compute the coefficients ζ_λ on the span $\Psi_{\mathbf{G}}(G) = \sum_\lambda \zeta_\lambda r_\lambda$. It is then natural to ask if combinatorial properties can be obtained for these coefficients, which are isomorphism invariants.

Similarly in the non-commutative case, we obtain that $\{\boldsymbol{\Upsilon}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}})\}_{\boldsymbol{\pi}\vdash[n]}$ is a basis of $\mathbf{WQSym}_n$, and so other coefficients arise. We can again ask for combinatorial properties of these coefficients.

4.3.1 The augmented chromatic invariant

Consider the ring of power series $\mathbb{K}[[x_1, x_2, \dots; q_1, q_2, \dots]]$ on two countably infinite collections of commuting variables, and let R be such ring modulo the relations $q_i(q_i-1)^2 = 0$.

Consider the graph invariant $\tilde{\Psi}(G) = \sum_f x_f \prod_i q_i^{c_G(f,i)}$ in R, where the sum runs over **all** colorings f of G, and $c_G(f,i)$ stands for the number of monochromatic edges of color i in the coloring f,*i.e.*, edges $\{v_1, v_2\}$ such that $f(v_1) = f(v_2) = i$.

For instance, if $G = K_2$, then $\tilde{\Psi}(G) = 2\sum_{1\leq i<j} x_i x_j + \sum_{1\leq i} x_i^2 q_i$. If we consider $G = K_3$ then we have

$$\tilde{\Psi}(G) = 6\sum_{1\leq i<j<k} x_i x_j x_k + 3\sum_{i\neq j} x_i x_j^2 q_j + \sum_{1\leq i} x_i^3 q_i^3\,.$$

Note that we can simplify further with the relation $q_i^3 = 2q_i^2 - q_i$.

A main property of this graph invariant is that it can be specialized to the chromatic symmetric function, by evaluating each variable q_i to zero. Another property of this graph invariant is the following:

Proposition 4.3.2. We have that $\ker \tilde{\Psi} = \ker \Psi_{\mathbf{G}}$. In particular, for graphs G_1, G_2 we have $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2)$ if and only if $\tilde{\Psi}(G_1) = \tilde{\Psi}(G_2)$.

Take, for instance, the celebrated tree conjecture introduced in [Sta95]:

Conjecture 4.3.3 (Tree conjecture on chromatic symmetric functions)**.** If two trees T_1, T_2 are not isomorphic, then $\Psi_{\mathbf{G}}(T_1) \neq \Psi_{\mathbf{G}}(T_2)$.

Consequently, from Proposition 4.3.2, the tree conjecture is equivalent to the following conjecture:

Conjecture 4.3.4. If two trees T_1, T_2 are not isomorphic, then $\tilde{\Psi}(T_1) \neq \tilde{\Psi}(T_2)$.

One strategy that has been employed to show that a family of non-isomorphic trees is distinguished by their chromatic symmetric function is to construct said trees using its coefficients over several bases, see for instance [OS14], [SST15] and [APZ14]. The graph invariant $\tilde{\Psi}$ provides more coefficients to reconstruct a tree, because Ψ results from $\tilde{\Psi}$ after the specialization $q_i = 0$. So, employing the same strategy to prove Conjecture 4.3.4 is *a priori* easier than to approach Conjecture 4.3.3 directly.

This shows us that the kernel method can also give us some light on other graph invariants: they may look stronger than Ψ, but are in fact as strong as Ψ if they satisfy the modular relations.

Proof of Proposition 4.3.2. Note that we have $\tilde{\Psi}(G)|_{q_i=0 \ i=1,2,\dots} = \Psi_{\mathbf{G}}(G)$. This readily yields $\ker \tilde{\Psi} \subseteq \ker \Psi_{\mathbf{G}}$. To show that $\ker \tilde{\Psi} \supseteq \ker \Psi_{\mathbf{G}}$, we need only to show that the modular relations and the isomorphism relations belong to $\ker \tilde{\Psi}$. For the isomorphism relations, this is trivial.

Let $l = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ be a generic modular relation on graphs, *i.e.*, $\{e_1, e_2, e_3\}$ are edges that form a triangle between the vertices $\{v_1, v_2, v_3\}$, with $e_3 \in G, e_1, e_2 \notin G$. Say that $e_1 = \{v_2, v_3\}$, $e_2 = \{v_3, v_1\}$ and $e_3 = \{v_1, v_2\}$. The proposition is proved if we show that $\tilde{\Psi}(l) = 0$.

For a coloring f of a graph H and a monochromatic edge e in H, define $c(e)$ the color of the vertices of e. Abbreviate $\mathbb{1}[e \text{ is monochromatic}] = m(e)$. With this, we use the abuse of notation $q_{c(e)}^{m(e)}$ even when e is not monochromatic, in which case we have $q_{c(e)}^{m(e)} = q_{c(e)}^{0} = 1$. Then

$$\prod_i q_i^{c_H(f,i)} = \prod_{e \text{ monochromatic}} q_{c(e)} = \prod_{e \in E(H)} q_{c(e)}^{m(e)} \,. \tag{4.13}$$

Set

$$\begin{aligned} s_f := &\prod_i q_i^{c_G(f,i)} - \prod_i q_i^{c_{G\cup\{e_1\}}(f,i)} - \prod_i q_i^{c_{G\cup\{e_2\}}(f,i)} + \prod_i q_i^{c_{G\cup\{e_1,e_2\}}(f,i)} \\ = &\left(1 - q_{c(e_1)}^{m(e_1)} - q_{c(e_2)}^{m(e_2)} + q_{c(e_1)}^{m(e_1)} q_{c(e_2)}^{m(e_2)}\right) \prod_{e \in E(G)} q_{c(e)}^{m(e)} \\ = &\left(1 - q_{c(e_1)}^{m(e_1)}\right)\left(1 - q_{c(e_2)}^{m(e_2)}\right) \prod_{e \in E(G)} q_{c(e)}^{m(e)} \,, \end{aligned} \tag{4.14}$$

and observe that $\tilde{\Psi}(l) = \sum_f x_f s_f$. Fix a coloring f. We now show that s_f is always zero.

It is easy to see that if either e_1 or e_2 are not monochromatic, then $q_{c(e_1)}^{m(e_1)} = 1$, respectively $q_{c(e_2)}^{m(e_2)} = 1$, which implies that $s_f = 0$.

It remains then to consider the case where $\{v_1, v_2, v_3\}$ is monochromatic. Without loss of generality, say it is of color a.

Then, $q_{c(e_1)}^{m(e_1)} = q_a = q_{c(e_2)}^{m(e_2)}$. Further, we have that $q_{c(e_3)}^{m(e_3)} = q_a$, so in R we have

$$s_f = (1 - q_a)^2 q_a \prod_{e \in E(G) \setminus \{e_3\}} q_{c(e)}^{m(e)} = 0 \,.$$

So $\tilde{\Psi}(l) = \sum_f x_f s_f = 0$.

In conclusion, any modular relation and any isomorphism relation is in $\ker \tilde{\Psi}$. From Theorem 4.1.2 we have that $\ker \Psi_{\mathbf{G}} \subseteq \ker \tilde{\Psi}$, so we conclude the proof. □

It is clear that Proposition 4.3.2 was established in an indirect way, by studying the kernel of the maps $\tilde{\Psi}$ and Ψ, instead of relating the coefficients of both invariants in some basis.

In Section 4.3.2 we relate the coefficients of both invariants in Corollary 4.3.7. Our original goal of establishing Proposition 4.3.2 without using Theorem 4.1.2 directly is not accomplished, which lends more strength to this kernel method.

4.3.2 Computing the augmented chromatic invariant on graphs

Recall that in Section 4.3, we define the ring R as the quotient ring of power series in $\mathbb{K}[[x_1, \dots; q_1, \dots]]$ by the relations $q_i(q_i - 1)^2 = 0$. We are then able to define a map $\tilde{\Psi} : \mathbf{G} \to R$, and we observed that $\ker \tilde{\Psi} = \ker \Psi$ in Proposition 4.3.2.

Here, we consider some specializations of $\tilde{\Psi}$ and obtain a linear combination of chromatic symmetric function of smaller graphs, in Theorem 4.3.5. The main motivation is to explore how to obtain Proposition 4.3.2 without using Theorem 4.1.1, and instead use a more direct way. This is not established in this chapter, which illustrates the strength and difficulty of the kernel approach. Let us first set up some necessary notation.

For an element $f \in R$, denote by $f\big|_{q_i=a}$ the specialization of the variable q_i to a in f, whenever defined (for $a = 0$ or $a = 1$). Additionally, denote by $f\big|_{q_i=1'}$ the specialization of the variable q_i to 1 in $\frac{\partial}{\partial q_i} f$. This is naturally an abuse of notation that allows us to denote the composition of several specializations in a more compact way. We also use this notation for the x_i variables. Further, we denote by $f\big|_{x_i=0''}$ the specialization $\frac{\partial^2}{\partial x_i^2} f\big|_{x_i=0}$.

We note that, in this ring, we can specialize infinitely many variables to zero. This however cannot be done with specializations to one, as the reader can readily check. Taking specializations of q_i to $a \notin \{0, 1, 1'\}$ is not well defined in the quotient ring.

For an edge $e \in E(G)$, denote by $G \setminus \mathcal{N}(e)$ the graph resulting after both endpoints of e are deleted from G, along with all its incident edges.

We say that a k-tuple of edges $m = (e_1, \dots, e_k)$ is an ordered matching if no two edges share a vertex, and write $\mathcal{M}_k(G)$ for the set of ordered matchings of size k on a graph G. We write $G \setminus \mathcal{N}(m)$ for the graph resulting after removing all vertices in the matching m from G, along with all its incident edges.

Finally, for a symmetric function f over the variables $x_1, x_2, \dots$, let $f \uparrow_k$ be the symmetric function over the variables $x_{k+1}, x_{k+2}, \dots$ with each index in f shifted up by k.

We obtain now a formula for $\tilde{\Psi}(G)$ that depends only on $\Psi_{\mathbf{G}}(H_i)$ for some graphs H_i that have less vertices than G.

Theorem 4.3.5. Let $k \geq 0$. We have the following relation between the graph invariant $\tilde{\Psi}$ and the chromatic symmetric function $\Psi_{\mathbf{G}}$:

$$\frac{1}{2^k} \tilde{\Psi}(G)\Big|_{\substack{q_i=1' \; i=1,\dots,k \\ q_i=0 \; i>k \\ x_i=0'' \; i=1,\dots,k}} = \sum_{m \in \mathcal{M}_k(G)} \Psi_{\mathbf{G}}(G \setminus \mathcal{N}(m)) \uparrow_k . \tag{4.15}$$

Proof. Recall that $c_G(f,i)$ counts the number of monochromatic edges in G with color i. With the expression given in Section 4.3 for the augmented chromatic symmetric function, we have

$$\tilde{\Psi}(G)\Big|_{q_i=1'\ i=1,\dots,k} = \sum_{f:V(G)\to\mathbb{N}} x_f \left(\prod_{i>k} q_i^{c_G(f,i)}\right) \prod_{i=1}^{k} c_G(f,i)\,.$$

We say that a coloring of G is *k-proper* if all monochromatic edges have color $j \leq k$. Observe that for a fixed coloring f, $\prod_{i=1}^{k} c_G(i,f)$ counts ordered matchings $(e_1,\dots,e_k)$ in G that satisfy $f(v)=\{i\}$ for any vertex v of e_i, $i=1,\dots,k$. Then, it is clear that

$$\begin{aligned}
\tilde{\Psi}(G)\Big|_{\substack{q_i=1'\ i=1,\dots,k\\ q_i=0\ i>k}} &= \sum_{f \text{ is k - proper}} x_f \prod_{i=1}^{k} c_G(i,f) = \sum_{f \text{ is k - proper}} \left[x_f \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)\\ f(e_i)=\{i\}}} 1 \right] \\
&= \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)}} \left[\sum_{\substack{f \text{ is k - proper}\\ f(e_i)=\{i\}}} x_f \right] \\
&= \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)}} \sum_{\substack{g \text{ coloring in } G\backslash\mathcal{N}(m)\\ g \text{ is k - proper}}} x_g (x_1\cdots x_k)^2\,.
\end{aligned} \tag{4.16}$$

So after the specialization $x_i = 0''$ for $i=1,\dots,k$, all colorings of $G\setminus\mathcal{N}(m)$ that use a color $j\leq k$ vanish, so

$$\begin{aligned}
\tilde{\Psi}(G)\Big|_{\substack{q_i=1'\ i=1,\dots,k\\ q_i=0\ i>k\\ x_i=0''\ i=1,\dots,k}} &= \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)}} \sum_{\substack{g \text{ proper in } G\backslash\mathcal{N}(m)\\ \operatorname{im} g\subseteq\mathbb{Z}_{>k}}} 2^k x_g \\
&= 2^k \sum_{m\in\mathcal{M}_k(G)} \Psi_{\mathbf{G}}(G\setminus\mathcal{N}(m))\uparrow_k
\end{aligned} \tag{4.17}$$

as desired. □

The right hand side of the expression of Theorem 4.3.5 can, in fact, be determined by the chromatic symmetric function of the graph G.

Proposition 4.3.6. If G_1, G_2 are two graphs such that $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2)$, and k a positive number, then

$$\sum_{m\in\mathcal{M}_k(G_1)} \Psi_{\mathbf{G}}(G_1\setminus\mathcal{N}(m)) = \sum_{m\in\mathcal{M}_k(G_2)} \Psi_{\mathbf{G}}(G_2\setminus\mathcal{N}(m))\,.$$

Proof. We use the power-sum basis $\{p_\lambda\}_{\lambda\vdash n}$ of Sym_n introduced in [Sta00].

We show that for a generic graph H, the coefficients of the symmetric function

$$\sum_{m\,\in\,\mathcal{M}_k(H)} \Psi_{\mathbf{G}}(H\setminus\,\mathcal{N}(m))\,,$$

in the power-sum basis are a function of the coefficients of $\Psi_{\mathbf{G}}(H)$ in the power-sum basis. Once this is established, the proposition follows.

For a graph G and a set of edges $S\subseteq E(S)$, write $\tau(S)$ for the integer partition recording the size of the connected components of the graph $(V(G),S)$. In [Sta95] the following expression for the coefficients in the power-sum basis is shown:

$$\Psi_{\mathbf{G}}(G)=\sum_{\lambda\vdash n}p_\lambda\sum_{\substack{S\subseteq E(G)\\ \tau(S)=\lambda}}(-1)^{|S|}\,.$$

Suppose that $\Psi_{\mathbf{G}}(H)=\sum_{\lambda\vdash n}c_\lambda p_\lambda$. For an integer partition λ write $m_2(\lambda)$ for the number of parts of size two, and write $\lambda\cup(2^k)$ for the integer partition resulting from λ by adding k extra parts of size two. Then we have that

$$\begin{aligned}\sum_{m\in\mathcal{M}_k(H)}\Psi_{\mathbf{G}}(H\setminus\mathcal{N}(m))&=\sum_{m\in\mathcal{M}_k(H)}\sum_{\lambda\vdash n}p_\lambda\sum_{\substack{S\subseteq E(H\setminus\mathcal{N}(m))\\ \tau(S)=\lambda}}(-1)^{|S|}\\ &=\sum_{\lambda\vdash n}p_\lambda\sum_{m\in\mathcal{M}_k(H)}\sum_{\substack{S\subseteq E(H\setminus\mathcal{N}(m))\\ \tau(S)=\lambda}}(-1)^{|S|}\,.\end{aligned}\tag{4.18}$$

Relabel the summands index by setting $R=S\cup m$, and note that for each $R\subseteq E(H)$ such that $\tau(R)=\lambda\cup(2^k)$, there are exactly $\binom{m_2(\lambda)+k}{k}k!$ pairs (S,m) of $m\in\mathcal{M}_k(H)$ and $S\subseteq E(H)$ such that $S\cup m=R$, $\tau(S)=\lambda$ and $S\subseteq E(H\setminus\mathcal{N}(m))$. Hence:

$$\begin{aligned}\sum_{m\in\mathcal{M}_k(H)}\Psi_{\mathbf{G}}(H\setminus\mathcal{N}(m))&=\sum_{\lambda\vdash n}p_\lambda\sum_{\substack{R\subseteq E(H)\\ \tau(R)=\lambda\cup 2^k}}\binom{m_2(\lambda)+k}{k}k!(-1)^{|R|-k}\\ &=\sum_{\lambda\vdash n}p_\lambda(-1)^k c_{\lambda\cup(2^k)}\binom{m_2(\lambda)+k}{k}k!\,.\end{aligned}\tag{4.19}$$

Therefore, the sum $\sum_{m\in\mathcal{M}_k(H)}\Psi_{\mathbf{G}}(H\setminus\mathcal{N}(m))$ is determined by $\Psi_{\mathbf{G}}(H)$. □

It follows from Theorem 4.3.5 and Proposition 4.3.6 that:

Corollary 4.3.7. If G_1, G_2 are graphs such that $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2)$, then for every integer $k \geq 0$ we have:

$$\tilde{\Psi}(G_1)\Big|_{\substack{q_i=1' \ i=1,\dots,k \\ q_i=0 \ i>k \\ x_i=0'' \ i=1,\dots,k}} = \tilde{\Psi}(G_2)\Big|_{\substack{q_i=1' \ i=1,\dots,k \\ q_i=0 \ i>k \\ x_i=0'' \ i=1,\dots,k}} .$$

The following fact is immediate from the definition of R:

Proposition 4.3.8. Suppose that $f_1, f_2 \in R$ are such that for every pair of finite disjoint sets $I, J \subseteq \mathbb{N}$ we have

$$f_1\Big|_{\substack{q_i=1' \ i\in I \\ q_i=0 \ i\in P \\ q_i=1 \ i\in J}} = f_2\Big|_{\substack{q_i=1' \ i\in I \\ q_i=0 \ i\in P \\ q_i=1 \ i\in J}}$$

where $P = \mathbb{N} \setminus (I \sqcup J)$. Then $f_1 = f_2$ in R.

In conclusion, to get an alternative proof of Proposition 4.3.2 we need to establish a generalization of Corollary 4.3.7 that introduces specializations of the type $q_i = 1$, in order to apply Proposition 4.3.8. Such a generalization has not been found by the author.

4.4 The CSF on hypergraphic polytopes

4.4.1 Poset structures on compositions

In this chapter we consider generalized permutahedra and hypergraphic polytopes in $\mathbb{R}^n$. Recall that with a set composition $\vec{\pi} = S_1|\dots|S_k \in \mathbf{C}_n$ we have the associated total preorder $R_{\vec{\pi}}$, and for a non-empty set $A \subseteq [n]$, we define the set $A_{\vec{\pi}} = A \cap S_i$ where i is as small as possible so that $A_{\vec{\pi}} \neq \emptyset$. We refer to $A_{\vec{\pi}}$ as the minima of A in $R_{\vec{\pi}}$.

Finally, recall as well that, for a hypergraphic polytope $\mathfrak{q}$, $\mathcal{F}(\mathfrak{q}) \subseteq 2^I \setminus \{\emptyset\}$ denotes the family of sets $J \subseteq I$ such that the coefficients in (4.3) satisfy $a_J > 0$. For a set $A \subseteq 2^I \setminus \{\emptyset\}$, we write $\mathcal{F}^{-1}(A)$ for the hypergraphic polytope $\mathfrak{q} = \sum_{J\in A} \mathfrak{s}_J$. We write $a = \text{pt}$ whenever a is a point polytope.

For a generalized permutahedron $\mathfrak{q}$ and a real coloring f with set composition type $\vec{\pi}$, recall that we write $\mathfrak{q}_f = \mathfrak{q}_{\vec{\pi}}$ for the corresponding face, without ambiguity, see Lemma 4.2.8.

Definition 4.4.1 (Basic hypergraphic polytopes and a preorder in set compositions)**.** For $\vec{\pi} \in \mathbf{C}_n$, the corresponding *basic hypergraphic polytope* is the fundamental hypergraphic polytope $\mathfrak{p}^{\vec{\pi}} = \mathcal{F}^{-1}\{A \text{ s.t. } |A_{\vec{\pi}}| = 1\}$.

Consider two set composition $\vec{\pi}_1, \vec{\pi}_2 \in \mathbf{C}_n$. If for any non-empty $A \subseteq [n]$ we have $|A_{\vec{\pi}_1}| = 1 \Rightarrow |A_{\vec{\pi}_2}| = 1$, we write $\vec{\pi}_1 \preceq \vec{\pi}_2$. Equivalently, $\vec{\pi}_1 \preceq \vec{\pi}_2$ if $\mathcal{F}(\mathfrak{p}^{\vec{\pi}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\pi}_2})$. With this, $\preceq$ is a preorder, called *singleton commuting preorder* or *SC preorder*. This nomenclature is motivated by Proposition 4.4.3.

Additionally, we define the equivalence relation $\sim$ in $\mathbf{C}_n$ as $\vec{\pi} \sim \vec{\tau}$ whenever $|A_{\vec{\pi}}| = 1 \Leftrightarrow |A_{\vec{\tau}}| = 1$ for all non-empty sets $A \subseteq [n]$. Note that $\vec{\pi} \sim \vec{\tau}$ if and only if $\mathfrak{p}^{\vec{\pi}} = \mathfrak{p}^{\vec{\tau}}$. We write $[\vec{\pi}]$ for the equivalence class of $\vec{\pi}$ under $\sim$, and write $\mathfrak{p}^{[\vec{\pi}]} = \mathfrak{p}^{\vec{\pi}}$, without ambiguity.

The preorder $\preceq$ projects to an order in $\mathbf{C}_n/_\sim$. In Proposition 4.4.14, we find an asymptotic formula for the number of equivalence classes of $\sim$.

Example 4.4.2. We see here the preorder $\preceq$ for $n = 3$, and the corresponding order in the equivalence classes of $\sim$. The set compositions $\vec{\pi}$ such that $\lambda(\vec{\pi}) = (1, 1, 1)$, are in bijection with permutations on $\{1, 2, 3\}$, we call these the *permutations in* $\mathbf{C}_3$. For a permutation $\vec{\pi}$ in $\mathbf{C}_3$ and a non-empty subset $A \subseteq \{1, 2, 3\}$, we have $|A_{\vec{\pi}}| = 1$. Hence, permutations are maximal elements in the singleton commuting preorder, and form an equivalence class of $\sim$. This is called the trivial equivalence class.

We also observe that if A is such that $|A_{23|1}| = 1$, then $\{2, 3\} \not\subseteq A$ and so we have that $|A_{1|23}| = 1$. It follows that $1|23 \succ 23|1$. The remaining structure of the preorder in $\mathbf{C}_3$ is in Fig. 4.3, where we collapse equivalence classes into vertices and draw the corresponding poset in its Hasse diagram.

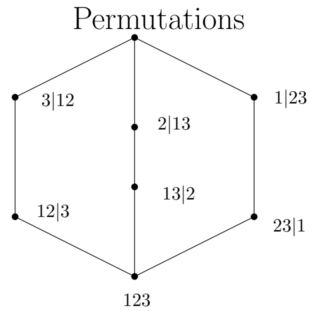


FIGURE 4.3: The SC order in $\mathbf{C}_3/_\sim$.

For $n = 4$ things are more interesting, as we have non-trivial equivalence classes. For instance, we have $[12|3|4] = \{12|3|4, 12|4|3\}$.

Proposition 4.4.3. Let $\vec{\pi}, \vec{\tau} \in \mathbf{C}_n$. Then we have that $\vec{\pi} \preceq \vec{\tau} \Rightarrow \boldsymbol{\lambda}(\vec{\pi}) \geq \boldsymbol{\lambda}(\vec{\tau})$.

Additionally, $\vec{\pi} \sim \vec{\tau}$ if and only if all the following happens:

1. We have $\boldsymbol{\lambda}(\vec{\pi}) = \boldsymbol{\lambda}(\vec{\tau})$, and;
2. For each pair (a, b) with $a, b \in [n]$ that satisfies both $a\, R_{\vec{\pi}}\, b$ and $b\, R_{\vec{\tau}}\, a$, either $\{a\}, \{b\} \in \boldsymbol{\lambda}(\vec{\pi})$ or $a \sim_{\boldsymbol{\lambda}(\vec{\pi})} b$.

In particular, $\alpha(\vec{\boldsymbol{\pi}}) = \alpha(\vec{\boldsymbol{\tau}})$.

Property 2. will be called the SC property. The equivalence classes of $\sim$ have a clear combinatorial description via Proposition 4.4.3. In particular, we see that $\vec{\boldsymbol{\pi}}_1 \sim \vec{\boldsymbol{\pi}}_2$ if all blocks are the same and in the same order, with possible exceptions between singletons. For instance, we have that $12|3|4|5|67 \sim 12|3|5|4|67$ but $12|3|4|5|67 \not\sim 3|12|4|5|67$.

We are also told in Proposition 4.4.3 that the map $\boldsymbol{\lambda} : \mathbf{C}_n \to \mathbf{P}_n$ flips the SC preorder with respect to the coarsening order $\leq$ in $\mathbf{P}_n$.

Proof of Proposition 4.4.3. Write $\boldsymbol{\pi}, \boldsymbol{\tau}$ for the underlying set partitions $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}), \boldsymbol{\lambda}(\vec{\boldsymbol{\tau}})$, respectively. Suppose that $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}$ and take i, j elements of $[n]$ such that $i \sim_{\boldsymbol{\tau}} j$. Then $\{i, j\} \not\in \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}}) \supseteq \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}})$. This implies that $|\{i, j\}_{\vec{\boldsymbol{\pi}}}| \neq 1$, hence $i \sim_{\boldsymbol{\pi}} j$. Since i, j are generic, we have that $\boldsymbol{\pi} \geq \boldsymbol{\tau}$. This concludes the first part.

For the second part, we will first show the direct implication. Suppose that $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathfrak{p}^{\vec{\boldsymbol{\tau}}}$. It follows from above that $\boldsymbol{\pi} = \boldsymbol{\tau}$. Our goal is to establish the SC property.

Take a, b that are in distinct blocks in $\boldsymbol{\pi}$, such that both $a \, R_{\vec{\boldsymbol{\pi}}} \, b$ and $b \, R_{\vec{\boldsymbol{\tau}}} \, a$. For sake of contradiction let $c \neq a$ be such that $c \sim_{\boldsymbol{\pi}} a$. Then $\{a, b, c\}_{\vec{\boldsymbol{\pi}}} = \{a, c\}$, which is not a singleton. However, we have that $\{a, b, c\}_{\vec{\boldsymbol{\tau}}} = \{b\}$ is a singleton, which is a contradiction with $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$. This contradicts the assumption that $a \sim_{\boldsymbol{\pi}} c$, so we conclude that $\{a\} \in \boldsymbol{\pi}$. Similarly we obtain that $\{b\} \in \boldsymbol{\pi}$. This shows the SC property.

For the reverse implication, suppose that $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}}$ are such that $\boldsymbol{\pi} = \boldsymbol{\tau}$ and satisfy the SC property. Our goal is to show that $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$. For sake of contradiction, take some nonempty set $A \subseteq [n]$ such that $A_{\vec{\boldsymbol{\pi}}} = \{a\}$ is a singleton, but $|A_{\vec{\boldsymbol{\tau}}}| \neq 1$. Finally, take an element $b \in A_{\vec{\boldsymbol{\tau}}}$, so that $a, b \in A$. We immediately have $a \, R_{\vec{\boldsymbol{\pi}}} \, b$, $b \, R_{\vec{\boldsymbol{\tau}}} \, a$.

Since $\boldsymbol{\pi} = \boldsymbol{\tau}$, either $A_{\vec{\boldsymbol{\pi}}} = A_{\vec{\boldsymbol{\tau}}}$ or $A_{\vec{\boldsymbol{\pi}}} \cap A_{\vec{\boldsymbol{\tau}}} = \emptyset$. Since $A_{\vec{\boldsymbol{\pi}}} \neq A_{\vec{\boldsymbol{\tau}}}$, they are disjoint and in particular $a \not\sim_{\boldsymbol{\pi}} b$. By the SC property we conclude that both $\{a\}, \{b\} \in \boldsymbol{\pi}$, contradicting that $|A_{\vec{\boldsymbol{\tau}}}| \neq 1$. That $|A_{\vec{\boldsymbol{\tau}}}| = 1 \Rightarrow |A_{\vec{\boldsymbol{\pi}}}| = 1$ follows similarly, concluding the proof.

Finally, whenever $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$, the SC property gives us $\alpha(\vec{\boldsymbol{\pi}}) = \alpha(\vec{\boldsymbol{\tau}})$. □

The following definition focus on the algebraic counterpart of $\sim$.

Definition 4.4.4. Consider the quasi-symmetric functions $\mathbf{N}_{[\vec{\boldsymbol{\pi}}]} = \sum_{\vec{\boldsymbol{\tau}} \in [\vec{\boldsymbol{\pi}}]} \mathbf{M}_{\vec{\boldsymbol{\tau}}}$, which are linearly independent. The *singleton commuting space*, or **SC** for short, is the graded vector subspace of **WQSym** spanned by $\biguplus_{n \geq 0} \{\mathbf{N}_{[\vec{\boldsymbol{\pi}}]} : [\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n / \sim\}$.

In Lemma 4.4.7, we show that $\mathbf{SC}$ is the image of $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$. As a consequence, $\mathbf{SC}$ is a Hopf algebra.

We turn to some properties of hypergraphic polytopes in the next lemma:

Lemma 4.4.5 (Vertices of a hypergraphic polytope)**.** Let $\mathfrak{q}$ be a hypergraphic polytope and let f be a real coloring.

Then, we have that $\mathfrak{q}_f = \mathrm{pt}$ if and only if $|A_{\vec{\pi}(f)}| = 1$ for each $A \in \mathcal{F}(q)$. In particular, if $\vec{\pi}(f_1) \sim \vec{\pi}(f_2)$ then $\mathfrak{q}_{f_1} = \mathrm{pt} \Leftrightarrow \mathfrak{q}_{f_2} = \mathrm{pt}$.

Proof. Write $\mathfrak{q} = \sum_{A \in \mathcal{F}(\mathfrak{q})} a_A \mathfrak{s}_A$, for coefficients $a_A \geq 0$. Computing the face corresponding to f on both sides, we obtain that $\mathfrak{q}_f = \mathrm{pt}$ if and only if

$$\sum_{A \in \mathcal{F}(\mathfrak{q})} a_A (\mathfrak{s}_A)_f = \mathrm{pt}\,,$$

or equivalently, if $(\mathfrak{s}_A)_f = \mathrm{pt}$ for each $A \in \mathcal{F}(\mathfrak{q})$.

We observed in Eq. (4.9) that $(\mathfrak{s}_A)_f = \mathfrak{s}_{A_{\vec{\pi}(f)}}$. Hence, we conclude that $\mathfrak{q}_f = \mathrm{pt}$ if and only if $|A_{\vec{\pi}(f)}| = 1$ for each $A \in \mathcal{F}(\mathfrak{q})$, as desired.

To show the last part of the lemma, just observe that $|A_{\vec{\pi}}| = 1$ only depends on the equivalence class of $\vec{\pi}$. □

The following corollary is immediate from (4.11) and Lemma 4.4.5.

Corollary 4.4.6. The image of $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$ is contained in the $\mathbf{SC}$ space, *i.e.*, for any hypergraphic polytope $\mathfrak{q}$ we have that

$$\boldsymbol{\Upsilon}_{\mathbf{HGP}}(\mathfrak{q}) \in \mathbf{SC}\,.$$

Another consequence of Lemma 4.4.5 is that we have $\mathfrak{p}^{\vec{\pi}}_{\vec{\tau}} = \mathrm{pt}$ precisely when $|A_{\vec{\pi}}| = 1 \Rightarrow |A_{\vec{\tau}}| = 1$, *i.e.*, when $\vec{\pi} \preceq \vec{\tau}$. It follow from (4.11) that:

$$\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}}) = \sum_{\vec{\pi} \preceq \vec{\tau}} \mathbf{M}_{\vec{\tau}}\,. \tag{4.20}$$

As presented, (4.20) seems to shows that the transition matrix of $\{\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}})\}_{\vec{\pi} \in \mathbf{C}_n}$ over the monomial basis is upper triangular. Since $\preceq$ is not an order but a preorder, that is not the case. The related result that we can establish is the following:

Lemma 4.4.7. The family $\{\mathbf{\Upsilon_{HGP}}(\mathfrak{p}^{[\vec{\pi}]})\}_{[\vec{\pi}]\in \mathbf{C}_n/\sim}$ forms a basis of $\mathbf{SC}$. In particular, we have $\operatorname{im}\mathbf{\Upsilon_{HGP}} = \mathbf{SC}$.

Proof. From (4.20) we have the following triangularity relation:

$$\mathbf{\Psi}(\mathfrak{p}^{[\vec{\pi}]}) = \sum_{\vec{\pi}\sim\vec{\tau}} M_{\vec{\tau}} + \sum_{\substack{\vec{\pi}\prec\vec{\tau}\\ \vec{\pi}\not\sim\vec{\tau}}} M_{\vec{\tau}} = N_{[\vec{\pi}]} + \sum_{[\vec{\pi}]\prec[\vec{\tau}]} N_{[\vec{\tau}]}\,, \tag{4.21}$$

where we take the projection of the preorder $\preceq$ into the corresponding order in $\mathbf{C}_n/\sim$.

Thus, $\{\mathbf{\Upsilon_{HGP}}(\mathfrak{p}^{[\vec{\pi}]})\}_{[\vec{\pi}]\in \mathbf{C}_n/\sim}$ is another basis of $\mathbf{SC}$. From Corollary 4.4.6, we conclude that $\operatorname{im}\mathbf{\Upsilon_{HGP}} = \mathbf{SC}$. □

In the commutative case, we wish to carry the triangularity of the monomial transition matrix in (4.21) into a new smaller basis in $QSym$.

For that, we project the order $\preceq$ into an order $\leq'$ in $\mathcal{C}_n$ as follows: we say that $\alpha \leq' \beta$ if we can find set compositions $\vec{\pi}, \vec{\tau}$ that satisfy $\vec{\pi} \preceq \vec{\tau}$, $\alpha(\vec{\pi}) = \alpha$ and $\alpha(\vec{\tau}) = \beta$. We will see that this projection is akin to the projection of the coarsening order of set partitions to the coarsening order on partitions. In particular, it preserves the desired upper triangularity.

Lemma 4.4.8. The relation $\leq'$ on $\mathcal{C}_n$ is an order and satisfies $\vec{\pi} \preceq \vec{\tau} \Rightarrow \alpha(\vec{\pi}) \leq' \alpha(\vec{\tau})$.

Recall that permutations of $[n]$ act on set compositions: if $\vec{\pi} = A_1|\dots|A_k \in \mathbf{C}_n$, and $\phi \in S_n$, then $\phi(\vec{\pi}) \in \mathbf{C}_n$ is the set composition $\phi(S_1)|\dots|\phi(S_k)$.

Proof of Lemma 4.4.8. We only need to check that $\leq'$ as defined is indeed an order, as it is straightforward that $\vec{\pi} \preceq \vec{\tau} \Rightarrow \alpha(\vec{\pi}) \leq' \alpha(\vec{\tau})$.

Reflexivity of $\leq'$ trivially follows from the definition of $\preceq$. To show antisymmetry of $\leq'$, it is enough to establish that if $\vec{\pi}_1 \preceq \vec{\tau}_2$, $\vec{\tau}_1 \preceq \vec{\pi}_2$ are set compositions such that $\alpha := \alpha(\vec{\pi}_1) = \alpha(\vec{\pi}_2)$ and $\beta := \alpha(\vec{\tau}_1) = \alpha(\vec{\tau}_2)$, then $\alpha = \beta$.

Indeed, if $\alpha(\vec{\pi}_1) = \alpha(\vec{\pi}_2)$ then there is a permutation ϕ in $[n]$ that satisfies $\phi(\vec{\pi}_1) = \vec{\pi}_2$. Then, ϕ lifts to a bijection between $\mathcal{F}(\mathfrak{p}^{\vec{\pi}_1})$ and $\mathcal{F}(\mathfrak{p}^{\vec{\pi}_2})$; in particular, they have the same cardinality. Similarly, $\mathcal{F}(\mathfrak{p}^{\vec{\tau}_1})$ and $\mathcal{F}(\mathfrak{p}^{\vec{\tau}_2})$ have the same cardinality.

But since $\vec{\pi}_1 \preceq \vec{\tau}_2$, $\vec{\tau}_1 \preceq \vec{\pi}_2$, *i.e.*, $\mathcal{F}(\mathfrak{p}^{\vec{\pi}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\tau}_2})$ and $\mathcal{F}(\mathfrak{p}^{\vec{\tau}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\pi}_2})$, it follows that $\mathcal{F}(\mathfrak{p}^{\vec{\pi}_1}) = \mathcal{F}(\mathfrak{p}^{\vec{\tau}_1})$, and so $\vec{\pi}_1 \sim \vec{\tau}_1$. From Proposition 4.4.3, we have that $\alpha = \alpha(\vec{\pi}_1) = \alpha(\vec{\tau}_1) = \beta$, and the antisymmetry follows.

To show transitivity, take compositions such that $\alpha \leq' \beta$ and $\beta \leq' \sigma$, *i.e.*, there are set compositions $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}_1$ and $\vec{\boldsymbol{\tau}}_2 \preceq \vec{\boldsymbol{\gamma}}$ such that $\alpha(\vec{\boldsymbol{\pi}}) = \alpha$, $\alpha(\vec{\boldsymbol{\tau}}_1) = \alpha(\vec{\boldsymbol{\tau}}_2) = \beta$ and $\alpha(\vec{\boldsymbol{\gamma}}) = \sigma$. Take a permutation ϕ in $[n]$ such that $\phi(\vec{\boldsymbol{\tau}}_1) = \vec{\boldsymbol{\tau}}_2$ and call $\vec{\boldsymbol{\delta}} = \phi(\vec{\boldsymbol{\pi}})$, note that $\alpha(\vec{\boldsymbol{\delta}}) = \alpha(\vec{\boldsymbol{\pi}}) = \alpha$.

We claim that $\vec{\boldsymbol{\delta}} \preceq \vec{\boldsymbol{\tau}}_2$. It follows that $\vec{\boldsymbol{\delta}} \preceq \vec{\boldsymbol{\gamma}}$ and $\alpha \leq' \sigma$, so the transitivity of $\leq'$ also follows. Take $A \subseteq [n]$ nonempty such that $|A_{\vec{\boldsymbol{\delta}}}| = 1$. Then $|\phi^{-1}(A)_{\vec{\boldsymbol{\pi}}}| = 1$ and from $\vec{\boldsymbol{\pi}} \leq \vec{\boldsymbol{\tau}}_1$ it follows that $|\phi^{-1}(A)_{\vec{\boldsymbol{\tau}}_1}| = 1$. From $\vec{\boldsymbol{\tau}}_2 = \phi(\vec{\boldsymbol{\tau}}_1)$ we have that $|A_{\vec{\boldsymbol{\tau}}_2}| = 1$. Since A is generic such that $|A_{\vec{\boldsymbol{\delta}}}| = 1$, we conclude that $\vec{\boldsymbol{\delta}} \preceq \vec{\boldsymbol{\tau}}_2$, as envisaged. □

4.4.2 The kernel and image problem on hypergraphic polytopes

Recall that a fundamental hypergraphic polytope in $\mathbb{R}^{[n]}$ is a polytope of the form $\sum_{\emptyset \neq J \subseteq [n]} a_J \mathfrak{s}_J$ where each $a_J \in \{0, 1\}$. In particular, a fundamental hypergraphic polytope can be written as $\mathfrak{q} = \mathcal{F}^{-1}(\mathcal{A})$ for some family of non-empty subsets of $[n]$.

In the following proposition, we reduce the problem of describing the kernel of $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$ to the subspace of **HGP** spanned by the fundamental hypergraphic polytopes.

Proposition 4.4.9 (Simple relations for $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$)**.** If $\mathfrak{q}_1, \mathfrak{q}_2$ are two hypergraphic polytopes such that $\mathcal{F}(\mathfrak{q}_1) = \mathcal{F}(\mathfrak{q}_2)$, then

$$\boldsymbol{\Upsilon}_{\mathbf{HGP}}(\mathfrak{q}_1) = \boldsymbol{\Upsilon}_{\mathbf{HGP}}(\mathfrak{q}_2)\,.$$

It remains to discuss the kernel of the map $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$ in the space of fundamental hypergraphic polytopes $\{\mathcal{F}^{-1}(\mathcal{A}) |\ \mathcal{A} \subseteq 2^{[n]} \setminus \{\emptyset\}\}$. For non-empty sets $A \subseteq [n]$, define $\operatorname{Orth} A = \{\vec{\boldsymbol{\pi}} \in \mathbf{C}_n \text{ s.t. } |A_{\vec{\boldsymbol{\pi}}}| = 1\}$. We now exhibit some linear relations of the chromatic function on fundamental hypergraphic polytopes.

Theorem 4.4.10 (Modular relations for $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$)**.** Let $\mathcal{A}, \mathcal{B}$ be two disjoint families of non-empty subsets of $[n]$. Consider the hypergraphic polytope $\mathfrak{q} = \mathcal{F}^{-1}(\mathcal{A})$, and take $\mathcal{K} = \cup_{A \in \mathcal{A}} (\operatorname{Orth} A)^c$, and $\mathcal{J} = \cup_{B \in \mathcal{B}} \operatorname{Orth} B$, families of set compositions.

Suppose that $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$. Then,

$$\sum\nolimits_{\mathcal{T} \subseteq \mathcal{B}} (-1)^{|\mathcal{T}|}\, \boldsymbol{\Upsilon}_{\mathbf{HGP}} \left[\mathfrak{q} + \mathcal{F}^{-1}(\mathcal{T})\right] = 0\,,$$

where the sum $\mathfrak{q} + \mathcal{F}^{-1}(\mathcal{T})$ is taken as the Minkowski sum.

The sum $\sum_{\mathcal{T} \subseteq \mathcal{B}} (-1)^{|\mathcal{T}|} \left[\mathfrak{q} + \mathcal{F}^{-1}(\mathcal{T})\right]$ is called a *modular relation on hypergraphic polytopes*. An example can be observed in Fig. 4.4 for $n = 4$, where we take the families $\mathcal{A} = \{\{1, 4\}, \{1, 2, 4\}\}$, $\mathcal{B} = \{\{1, 2\}, \{2, 3\}, \{2, 3, 4\}\}$.

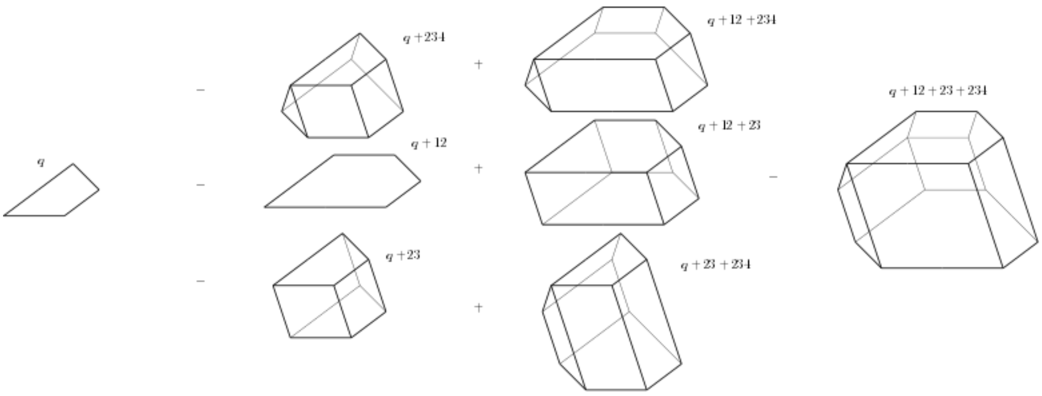


FIGURE 4.4: A modular relation on hypergraphic polytopes with eight terms, with the polytope $\mathfrak{q} = \mathcal{F}^{-1}(\{1,4\},\{1,2,4\})$.

Proof of Theorem 4.4.10. Write $\eta_f(\mathfrak{q}) = \mathbb{1}[f \text{ is } \mathfrak{q} - \text{generic }]$. The expansion of $\mathbf{\Upsilon}_{\mathbf{HGP}}$ for general hypergraphic polytopes $\mathfrak{q}$ is given by

$$\mathbf{\Psi}_{\mathbf{HGP}}(\mathfrak{q}) = \sum_f x_f \eta_f(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} x_f \,.$$

For short, write MRN for the modular relation for hypergraphic polytopes at hand. Hence:

$$\begin{aligned} \mathbf{\Psi}_{\mathbf{HGP}}(\text{MRN}) &= \sum_{\mathcal{T} \subseteq \mathcal{B}} (-1)^{|\mathcal{T}|} \mathbf{\Psi}_{\mathbf{HGP}} \left[\mathfrak{q} + \sum_{T \in \mathcal{T}} \mathfrak{s}_T \right] \\ &= \sum_{\mathcal{T} \subseteq \mathcal{B}} (-1)^{|\mathcal{T}|} \sum_f x_f \eta_f \left(\mathfrak{q} + \sum_{T \in \mathcal{T}} \mathfrak{s}_T \right) \\ &= \sum_f x_f \left[\sum_{\mathcal{T} \subseteq \mathcal{B}} (-1)^{|\mathcal{T}|} \eta_f \left(\mathfrak{q} + \sum_{T \in \mathcal{T}} \mathfrak{s}_T \right) \right] . \end{aligned} \tag{4.22}$$

We note from Lemmas 4.2.5 and 4.4.5 that for hypergraphic polytopes $\mathfrak{q}, \mathfrak{p}$, any coloring f that is not $\mathfrak{q}$-generic is not $(\mathfrak{q} + \mathfrak{p})$-generic. Hence, if $\eta_f(\mathfrak{q}) = 0$ then it follows that $\eta_f \left(\mathfrak{q} + \sum_{T \in \mathcal{T}} \mathfrak{s}_T \right) = 0$ for any $\mathcal{T} \subseteq \mathcal{B}$. We restrict the sum to $\mathfrak{q}$-generic colorings.

Further, define $\mathcal{B}(f) = \{B \in \mathcal{B} | f \text{ is } \mathfrak{s}_B - \text{ generic }\}$. Again according to Lemmas 4.2.5 and 4.4.5, we have that $\eta_f \left(\mathfrak{q} + \sum_{T \in \mathcal{T}} \mathfrak{s}_T \right) = 1$ exactly when $\mathcal{T} \subseteq \mathcal{B}(f)$, so the equation

(4.22) becomes

$$\begin{aligned}\mathbf{\Psi_{HGP}}(MRN) &= \sum_f x_f \left[\sum_{\mathcal{T}\subseteq\mathcal{B}} (-1)^{|\mathcal{T}|} \eta_f \left(\mathfrak{q} + \sum_{T\in\mathcal{T}} \mathfrak{s}_T \right) \right] \\ &= \sum_f x_f \left[\sum_{\mathcal{T}\subseteq\mathcal{B}(f)} (-1)^{|\mathcal{T}|} \right] = \sum_{\substack{f \text{ } \mathfrak{q}\text{- generic} \\ \mathcal{B}(f)=\emptyset}} x_f \,. \end{aligned} \tag{4.23}$$

It suffices to show that no coloring f is both $\mathfrak{q}$-generic and satisfies $B(f) = \emptyset$. Suppose otherwise, and take such f. If f is $\mathfrak{q}$-generic, we have $\vec{\boldsymbol{\pi}}(f) \notin \mathcal{K}$. If $B(f) = \emptyset$ we have that $\vec{\boldsymbol{\pi}}(f) \notin \operatorname{Orth} B_j$ for any j, hence $\vec{\boldsymbol{\pi}}(f) \notin \mathcal{J}$. Given that $f \in \mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$, we have a contradiction. It follows that Eq. (4.23) becomes $\mathbf{\Psi_{HGP}}(\mathrm{MRN}) = 0$, which concludes the proof. □

Remark 4.4.11. Recall that $Z : \mathbf{G} \to \mathbf{GP}$ is the graph zonotope map. It can be noted that, if $l = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ is a modular relation on graphs, then $Z(l)$ is the modular relation on hypergraphic polytopes corresponding to $\mathfrak{q} = Z(G)$, *i.e.*, $\mathcal{A} = E(G)$, and $\mathcal{B} = \{e_1, e_2\}$) . In this case, the condition $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$ follows from the fact that no proper coloring of G is monochromatic in both e_1 and e_2.

Recall that we set $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathcal{F}^{-1}(\{A \subseteq [n] \text{ s.t. } |A_{\vec{\boldsymbol{\pi}}}| = 1\})$. This only depends on the equivalence class $[\vec{\boldsymbol{\pi}}]$ under $\sim$, and we may write the same polytope as $\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]}$. These are called basic hypergraphic polytopes and are a particular case of fundamental hypergraphic polytopes.

To prove Theorem 4.1.4, we follow roughly the same idea as in the proof of Theorem 4.1.1: We use the family of hypergraphic polytopes $\{\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]}\}_{[\vec{\boldsymbol{\pi}}]\in\mathbf{C}_n/\sim}$ to apply Lemma 4.2.1, whose image by $\mathbf{\Upsilon_{GP}}$ is linearly independent. Recall that in Lemma 4.4.7, we established that it spans the image of $\mathbf{\Upsilon_{HGP}}$.

Proof of Theorem 4.1.4. First recall that $\mathbf{HGP}_n$ is a linear space generated by the hypergraphic polytopes in $\mathbb{R}^n$. According to Proposition 4.4.9, to compute the kernel of $\mathbf{\Upsilon_{HGP}}$, it suffices us study the span of the fundamental hypergraphic polytopes. Fix a total order $\tilde{\geq}$ on fundamental hypergraphic polytopes $\mathfrak{q}$ so that $|\mathcal{F}(\mathfrak{q})|$ is non decreasing.

We apply Lemma 4.2.1 with Theorem 4.4.10 to this finite dimensional subspace of $\mathbf{HGP}_n$.

Lemma 4.4.7 guarantees that $\{\mathbf{\Upsilon_{GP}}(\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]})\}_{[\vec{\boldsymbol{\pi}}]\in\mathbf{C}_n/\sim}$ is linearly independent. Therefore, it suffices to show that for any fundamental hypergraphic polytopes $\mathfrak{q}$ that is not a basic hypergraphic polytope, we can write some modular relation b as $b = \mathfrak{q} + \sum_i \lambda_i \mathfrak{q}_i$, where $|\mathcal{F}(\mathfrak{q})| < |\mathcal{F}(\mathfrak{q}_i)| \, \forall i$. Indeed, it would follow from Lemma 4.2.1 that the simple relations and the modular relations on hypergraphic polytopes span $\ker \boldsymbol{\Upsilon}_{\mathbf{HGP}}$.

The desired modular relation is constructed by taking $\mathcal{A} = \mathcal{F}(\mathfrak{q})$ and $\mathcal{B} = \mathcal{F}(\mathfrak{q})^c$ in Theorem 4.4.10. Let us write $\mathcal{K} = \cup_{A \in \mathcal{F}(\mathfrak{q})} (\operatorname{Orth} A)^c$ and $\mathcal{J} = \cup_{B \in \mathcal{F}(\mathfrak{q})^c} \operatorname{Orth} B$. We claim that $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$.

Take, for sake of contradiction, some $\vec{\pi} \notin \mathcal{K} \cup \mathcal{J}$. Note that from $\vec{\pi} \notin \mathcal{K}$ we have $|A_{\vec{\pi}}| = 1$ for every $A \in \mathcal{F}(\mathfrak{q})$. Note as well that from $\vec{\pi} \notin \mathcal{J}$ we have that $|B_{\vec{\pi}}| \neq 1$ for every $B \notin \mathcal{F}(\mathfrak{q})$. Therefore, if $\vec{\pi} \notin \mathcal{K} \cup \mathcal{J}$, then $\mathfrak{q} = \mathfrak{p}^{\vec{\pi}}$, contradicting the assumption that $\mathfrak{q}$ is not a basic hypergraphic polytope. We obtain that $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$. Finally, note that

$$\mathfrak{q} + \sum\nolimits_{\mathcal{T} \subseteq \mathcal{F}(\mathfrak{q})^c} (-1)^{|\mathcal{T}|} \left[\mathfrak{q} + \mathcal{F}^{-1}(\mathcal{T}) \right] ,$$

is a modular relation that respects the order $\tilde{\geq}$. This shows that the hypotheses of Lemma 4.2.1 are satisfied. □

For the commutative case we use Lemma 4.2.2. Note that we already have a generator set of $\ker \boldsymbol{\Upsilon}_{\mathbf{HGP}}$, so similarly to the proof of Theorem 4.1.2, we just need to establish some linear independence.

Recall that two hypergraphic polytopes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are isomorphic if there is a permutation matrix P such that $x \in \mathfrak{q}_2 \Leftrightarrow xP \in \mathfrak{q}_1$. If $\vec{\pi}_1$ and $\vec{\pi}_2$ share the same composition type, then $\mathfrak{p}^{\vec{\pi}_1}$ and $\mathfrak{p}^{\vec{\pi}_2}$ are isomorphic, and so we have $\Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}_1}) = \Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}_2})$. Set $R_{\alpha(\vec{\pi})} := \Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}})$ without ambiguity.

Proof of Theorem 4.1.5. We use Lemma 4.2.2 with the map $\Psi_{\mathbf{HGP}} = \operatorname{comu} \circ \boldsymbol{\Upsilon}_{\mathbf{HGP}}$.

From the proof of Theorem 4.1.4, to apply Lemma 4.2.2 it is enough to establish that the family $\{R_\alpha\}_{\alpha \in \mathcal{C}_n}$ is linearly independent. It would follow that $\ker \Psi_{\mathbf{HGP}}$ is generated by the modular relations, the simple relations and the isomorphism relations, and that $\{R_\alpha\}_{\alpha \in \mathcal{C}_n}$ is a basis of $\operatorname{im} \Psi_{\mathbf{G}}$, concluding the proof.

To show that $\{R_\alpha\}_{\alpha \in \mathcal{C}_n}$ is linear independent, write R_α on the monomial basis of $QSym$, and use the order $\leq'$ mentioned in Lemma 4.4.8.

As a consequence of (4.20), if we write $A_{\vec{\pi},\beta} = |\{\vec{\tau} \in \mathbf{C}_n | \vec{\pi} \preceq \vec{\tau}, \alpha(\vec{\tau}) = \beta\}|$, from Lemma 4.4.8 we have:

$$R_{\alpha(\vec{\pi})} = \Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}}) = \sum_{\vec{\pi} \preceq \vec{\tau}} M_{\alpha(\vec{\tau})} = A_{\vec{\pi},\alpha(\vec{\pi})} M_{\alpha(\vec{\pi})} + \sum_{\alpha(\vec{\pi}) <' \beta} A_{\vec{\pi},\beta} M_\beta \, . \tag{4.24}$$

It is clear that $A_{\vec{\pi},\alpha(\vec{\pi})} > 0$, so independence follows, which completes the proof. □

Remark 4.4.12. We have obtained in the proof of Theorem 4.1.5 that $\{R_\alpha\}_{\alpha\in\mathcal{C}_n}$ is a basis for Qym_n. The proof gives us a recursive way to compute the coefficients ζ_α on the expression $\Psi_{\mathbf{HGP}}(\mathfrak{q}) = \sum_{\alpha\in\mathcal{C}_n}\zeta_\alpha R_\alpha$. It is then natural to ask if combinatorial properties can be obtained for these coefficients, which are isomorphism invariants.

Similarly, in the non-commutative case, we can write the chromatic quasi-symmetric function of a hypergraphic polytope as

$$\Upsilon_{\mathbf{HGP}}(\mathfrak{q}) = \sum_{[\vec{\pi}]\in\mathbf{C}_n/\sim} \zeta_{[\vec{\pi}]}(\mathfrak{q})\Upsilon_{\mathbf{HGP}}(\mathfrak{p}^{[\vec{\pi}]}),$$

and ask for the combinatorial meaning of the coefficients $(\zeta_{[\vec{\pi}]})_{[\vec{\pi}]\in\mathbf{C}_n/\sim}$. These questions are not answered in this chapter.

4.4.3 The dimension of SC space

Let $sc_n := \dim \mathbf{SC}_n$. Recall that, from Definition 4.4.4, the elements of the Hopf algebra **SC** are of the form $\sum_{\vec{\pi}\models[n]}\mathbf{M}_{\vec{\pi}}a_{\vec{\pi}}$, where $a_{\vec{\pi}_1} = a_{\vec{\pi}_2}$ whenever $\vec{\pi}_1 \sim \vec{\pi}_2$ in the SC equivalence relation. Hence, sc_n counts the equivalence classes of $\sim$.

The goal of this section is to compute the asymptotics of sc_n, by using the combinatorial description in Proposition 4.4.3.

Proposition 4.4.13. Let $F(x) = \sum_{n\geq 0} sc_n\frac{x^n}{n!}$ be the exponential power series enumerating the dimensions of $\mathbf{SC}_n$. Then

$$F(x) = \frac{e^x}{1+(1+x)e^x - e^{2x}},$$

where e is the Napier constant.

Proposition 4.4.14. The dimension of **SC** has an asymptotic growth of

$$sc_n = n!\gamma^{-n}\left(\tau + o(\delta^{-n})\right),$$

where $\delta < 1$ is some real number, $\gamma \approxeq 0.814097 \approxeq 1.1745\log(2)$ is the unique positive root of the equation

$$e^{2x} = 1 + (1+x)e^x,$$

and $\tau = \mathrm{Res}_\gamma(F) \approxeq 0.588175$ is the residue of the function F at γ.

In particular, $\dim \mathbf{SC}_n$ is exponentially smaller than $\dim \mathbf{WQSym}_n = |\mathbf{C}_n|$, which is asymptotically

$$n! \log(2)^{-n} \left(\frac{1}{2\log(2)} + o(1) \right) ,$$

according to [Bar80]. Before we prove Propositions 4.4.13 and 4.4.14, we introduce a useful combinatorial family.

A barred set composition of $[n]$ is a set composition of $[n]$ where some of the blocks may be barred. For instance, $13|\overline{45}|2$ and $12|4|\overline{35}$ are bared set compositions of $\{1, 2, 3, 4, 5\}$. A barred set composition is *integral* if

- No two barred blocks occur consecutively, and;
- Every block of size one is barred;

An integral barred set composition is also called an IBSC for short. In Table 4.1 we have all the integral barred set compositions of small size:

n	IBSC	Equivalent classes under $\sim$
0	$\emptyset$	$\{\vec{\emptyset}\}$
1	$\overline{1}$	$\{1\}$
2	$\overline{12}, 12$	$\{1\|2, 2\|1\}, \{12\}$
3	$\overline{123}, \overline{1}\|23, \overline{2}\|13, \overline{3}\|12,$ $12\|\overline{3}, 13\|\overline{2}, 23\|\overline{1}, 123$	$[1\|2\|3)], \{1\|23\}, \{2\|13\}, \{3\|12\},$ $\{12\|3\}, \{13\|2\}, \{23\|1\}, \{123\}$

TABLE 4.1: Small IBSCs and equivalence classes of $\sim$

According to Proposition 4.4.3, we can construct a map from equivalence classes of $\sim$ and integral barred set compositions: from a set composition, we squeeze all consecutive singletons into one bared block. This map is a bijection, as is inverted by splitting all bared blocks into singletons, and the equivalence classed obtained is independent on the order that this splitting is done. So, for instance, $\overline{13}|24 \leftrightarrow \{1|3|24, 3|1|24\}$ and $13|24 \leftrightarrow \{13|24\}$. See Table 4.1 for more examples.

Proof of Proposition 4.4.13. We use the framework developed in [FS09] of labeled combinatorial classes. In the following, a calligraphic style letter denotes a combinatorial class, and the corresponding upper case letter denotes its exponential generating function. Let $\mathcal{B}$ and $\mathcal{U}$ be the collections $\{\overline{1}, \overline{12}, \cdots\}$ and $\{12, 123, 1234, \cdots\}$, respectively, with exponential generating functions $B(x) = e^x - 1$ and $U(x) = e^x - 1 - x$. Additionally, let $\mathcal{O} = \{\emptyset\}$ with $O(x) = 1$.

Let $\mathcal{F}$ be the class of IBSCs, and we denote by $\mathcal{F}^o$ the class of IBSCs that start with an unbarred set. Denote by $\overline{\mathcal{F}}$ the class of IBSCs that start with a barred set.

n	0	1	2	3	4	5	6	7	8	9
sc_n	1	1	2	8	40	242	1784	15374	151008	1669010
$\boldsymbol{\pi}_n$	1	1	3	13	75	541	4683	47293	545835	7087261

TABLE 4.2: First elements of the sequences sc_n and $\boldsymbol{\pi}_n = \dim \boldsymbol{WQSym}_n$.

Our goal is to show that $F(x) = \frac{e^x}{1+(1+x)e^x - e^{2x}}$. By definition we have that $\mathcal{F} = \overline{\mathcal{F}} \sqcup \mathcal{F}^o \sqcup \mathcal{O}$. Further, we can recursively describe $\overline{\mathcal{F}}$ and $\mathcal{F}^o$ as $\overline{\mathcal{F}} = \mathcal{B} \times (\mathcal{F}^o \sqcup \mathcal{O})$ and $\mathcal{F}^o = \mathcal{U} \times \mathcal{F}$.

According to the dictionary rules in [FS09], this implies that $F = \overline{F} + F^o + O$ and that

$$\overline{F}(x) = (e^x - 1)(F^o(x) + 1)\,,$$

$$F^o(x) = (e^x - 1 - x)(\overline{F}(x) + F^o(x) + 1)\,,$$

The unique solution of the system has $F^o(x) = \frac{e^{2x} - (1+x)e^x}{1 - 1e^{2x} + (x+1)e^x}$ so it follows

$$F(x) = \frac{e^x}{1 + (1+x)e^x - e^{2x}}$$

as desired. □

With this we can easily compute the dimension of $\mathbf{SC}_n$ for small n, and compare it with $\dim \boldsymbol{WQSym}_n$, as done in Table 4.2.

Proof of Proposition 4.4.14. Let $l(x) := e^{2x} - (1+x)e^x - 1$, then $F(x) = -\frac{e^x}{l(x)}$ is the quotient of two entire functions with non-vanishing numerator, so the poles are the zeros of $l(x)$. Note that $F(x)$ is a counting exponential power series around zero, so it has positive coefficients. By Pringsheim's Theorem as in [FS09], one of the dominant singularities of $F(x)$ is a positive real number, call it γ.

We show now that any other singularity $z \neq \gamma$ of F, that is, a zero of l, has to satisfy $|z| > |\gamma|$. Thus, showing that γ is the unique dominant singularity and allowing us to compute a simple asymptotic formula. Suppose, that z is a singularity of F distinct from γ, such that $|z| = |\gamma|$. So, we have that $l(z) = 0$ and that $z \notin \mathbb{R}^+$. The equation $l(z) = 0$ can easily be rewritten as

$$1 = l(z) + 1 = \sum_{n \geq 1} z^n \frac{2^n - 1 - n}{n!}\,.$$

Note that $2^n \geq n+1$ for $n \geq 1$. Now we apply the strict triangular inequality on the right hand side to obtain

$$1 < \sum_{n\geq 1} |z|^n \frac{2^n - n - 1}{n!} = \sum_{n\geq 1} \gamma^n \frac{2^n - n - 1}{n!} = l(\gamma) + 1 = 1\,,$$

where we note that the inequality is strict for $z \notin \mathbb{R}^+$ because some of the terms $|z|^n$ do not lie in the same ray through the origin. This is a contradiction with the assumption that there exists such a pole, as desired.

We additionally prove that γ is the unique positive real root, so we can easily approximate it by some numerical method, for instance the bisection method. The function l in the positive real line satisfies $\lim_{x\to+\infty} l(x) = +\infty$ and $l(0) = -1$, so it has at least one zero. Note that such zero is unique, as $l'(x) > 0$ for x positive. Also, since $l'(\gamma) > 0$, the zero γ is simple.

Since the function $F(x)$ is meromorphic in $\mathbb{C}$, and γ is the dominant singularity, we conclude that

$$\frac{sc_n}{n!} = \gamma^{-n} \left(\mathrm{Res}_\gamma(F) + o(\delta^{-n}) \right) \,,$$

for any δ such that $1 > \delta > |\gamma/\gamma_2|$, where γ_2 is a second smallest singularity of F, if it exists, and arbitrarily large otherwise.

We can approximate $\gamma \approxeq 0.814097$, and also estimate the residue of $F(x)$ at γ as $\tau = \mathrm{Res}_\gamma(F) = \frac{e^\gamma}{l'(\gamma)} \approxeq 0.588175$. This proves the desired asymptotic formula. □

4.5 Faces of generalized permutahedra and the singleton commuting equivalence relation

This section is an original work of this thesis. The main result of this section is the following:

Theorem 4.5.1 (Image of $\mathbf{\Upsilon_{GP}}$)**.** The image of $\mathbf{\Upsilon_{GP}}$ is precisely **SC**.

Recall that **SC** is the Hopf algebra spanned by $\bigcup_{n\geq 0}\{\mathbf{N}_{[\vec{\pi}]}\}_{\vec{\pi}\in \mathbf{C}_n/\sim}$, a basis indexed by equivalence classes of set compositions on the *singleton commuting equivalence relation.* This Hopf algebra is precisely the image of $\mathbf{\Upsilon_{HGP}}$. Because we have the following inclusion of combinatorial Hopf algebras, $\mathbf{HGP} \subseteq \mathbf{GP}$, it follows that $\mathbf{SC} = \mathrm{im}\, \mathbf{\Upsilon_{HGP}} \subseteq \mathrm{im}\, \mathbf{\Upsilon_{HGP}}$.

In the remaining of this section we present the proof of the other inclusion. We remark that an immediate consequence of this result is Corollary 5.5.2, where we compare the image of the chromatic map on generalized permutahedra with the image of the chromatic map on posets.

We first establish a lemma about generalized permutahedra and some other relevant propositions:

Lemma 4.5.2. Let $\{a_J\}_{\substack{J\subseteq[n]\\J\neq\emptyset}}$ be a family of real numbers such that $\mathfrak{q}=\sum_{\substack{J\subseteq[n]\\J\neq\emptyset}} a_J\mathfrak{s}_J$ is a well defined generalized permutahedron that is a point. Then, for any set J such that $|J|\geq 2$, we have that $a_J=0$.

We remark that this lemma is trivial for hypergraphic polytopes (it is a simple application of Lemma 4.2.5), and it follows that $\operatorname{im}\mathbf{\Upsilon}_{\mathbf{HGP}}=\mathbf{SC}$. Before we prove this lemma let us establish first some general claims regarding the coefficients $\{a_J\}_{\substack{J\subseteq[n]\\J\neq\emptyset}}$.

Proposition 4.5.3. Consider $n\geq 0$, and fix a set of real numbers $\{a_J\}_{\substack{J\subseteq[n]\\J\neq\emptyset}}$. Define for each non-empty $J\subseteq[n]$ the following

$$\mathcal{U}_J=\sum_{\substack{K\cap J\neq\emptyset\\K\subseteq[n]}} a_K\,,\qquad \mathcal{W}_J=\sum_{\substack{K\supseteq J\\K\subseteq[n]}} a_K\,.$$

Then, we have the following relation between $\{\mathcal{U}_J\}_{\substack{J\subseteq[n]\\J\neq\emptyset}}$ and $\{\mathcal{W}_J\}_{\substack{J\subseteq[n]\\J\neq\emptyset}}$:

$$\mathcal{U}_J=\sum_{\substack{K\subseteq J\\K\neq\emptyset}}(-1)^{|K|+1}\mathcal{W}_K\,.$$

Furthermore, for J a singleton, we have that

$$\mathcal{U}_J=\mathcal{W}_J\,. \tag{4.25}$$

Proof. First, (4.25) is immediate because we observe that the formulas for $\mathcal{U}_J$ and $\mathcal{W}_J$ are the same.

Then, observe that for any finite set J, we have that

$$\sum_{K\subseteq X}(-1)^{|K|}=\mathbb{1}[X=\emptyset]\,.$$

Thus, we have that

$$\begin{aligned}\sum_{\substack{K\subseteq J\\K\neq\emptyset}}(-1)^{|K|+1}\mathcal{W}_K &= \sum_{\substack{K\subseteq J\\K\neq\emptyset}}\sum_{S\supseteq K}(-1)^{|K|+1}a_S = \sum_{S\subseteq[n]}\sum_{\substack{K\subseteq J\\K\subseteq S\\K\neq\emptyset}}(-1)^{|K|+1}a_S\\ &= \sum_{S\subseteq[n]}a_S\left((-1)^{|\emptyset|}-\sum_{K\subseteq J\cap S}(-1)^{|K|}\right)\\ &= \sum_{\substack{S\subseteq[n]\\S\cap J\neq\emptyset}}a_J=\mathcal{U}_J\,,\end{aligned}$$

as desired. □

Proposition 4.5.4. Consider $n\geq 0$, and fix a set of real numbers $\{a_J\}_{\substack{J\subseteq[n]\\J\neq\emptyset}}$ such that the generalized permutahedron $\mathfrak{q}=\sum_{\substack{J\subseteq[n]\\J\neq\emptyset}}a_J\mathfrak{s}_J$ is well defined. Consider $\mathcal{U}_J$ as in Proposition 4.5.3, and for a set $K\subseteq[n]$, let $\vec{e}_K$ be the characteristic vector of K, that is if $i\in K$, then $(\vec{e}_K)_i=1$, and $(\vec{e}_K)_i=0$ otherwise.

Then $\mathcal{U}_K=\max_{x\in\mathfrak{q}}\{\vec{e}_K^T\cdot x\}$.

Proof. If $\mathfrak{s}_J$ is a simplex, then $\max_{x\in\mathfrak{s}_J}\{\vec{e}_K^T\cdot x\}=\mathbb{1}[J\cap K\neq\emptyset]$. Thus, as we have seen in Lemma 4.2.5, optimization problems commute with the Minkowski operations, so we have the following:

$$\begin{aligned}\max_{x\in\mathfrak{q}}\{\vec{e}_K^T\cdot x\} &= \sum_{\substack{J\subseteq[n]\\J\neq\emptyset}}a_J\max_{x\in\mathfrak{s}_J}\{\vec{e}_K^T\cdot x\}\\ &= \sum_{\substack{J\subseteq[n]\\K\cap J\neq\emptyset}}a_J=\mathcal{U}_K\,,\end{aligned}\tag{4.26}$$

as desired. □

We are now ready to present the proof of Lemma 4.5.2.

Proof of Lemma 4.5.2. Assume that $\sum_{\substack{J\subseteq[n]\\J\neq\emptyset}}a_J\mathfrak{s}_J$ is a well defined generalized permutahedron that is a point, say $\mathfrak{q}=\{\vec{x}\}$. Define $\mathcal{U}_J,\mathcal{W}_J$ as in Proposition 4.5.3. We will show that there is no set J such that $\mathcal{W}_J\neq 0$ and $|J|\geq 2$. This readily implies that there is no set J such that $a_J\neq 0$ and $|J|\geq 2$, concluding the lemma.

Assume otherwise, by contradiction, that there is some set J such that $\mathcal{W}_J\neq 0$ and $|J|\geq 2$. Let J_0 be the smallest such set. In particular, observe that $\mathcal{W}_{\mathrm{J}_0}\neq 0$ but $\mathcal{W}_J=0$ for any $J\subsetneq\mathrm{J}_0$ such that $|J|\geq 2$.

Then, from Proposition 4.5.4, for any set J we have that

$$\begin{aligned}\mathcal{U}_J = \max_{x\in\mathfrak{q}}\{\vec{e}_J^T \cdot x\} &= \left(\sum_{j\in J} \vec{e}_{\{j\}}^T\right) \cdot \vec{x} \\ &= \sum_{j\in J} \vec{e}_{\{j\}}^T \cdot \vec{x} = \sum_{j\in J} \mathcal{U}_{\{j\}} = \sum_{j\in J} \mathcal{W}_{\{j\}} \,.\end{aligned} \tag{4.27}$$

Comparing with Proposition 4.5.3, we have that

$$\sum_{\substack{K\subseteq J\\ |K|\geq 2}} (-1)^{1+|K|}\mathcal{W}_K = \mathcal{U}_J - \sum_{j\in J} \mathcal{W}_{\{j\}} = 0\,,$$

however, if we let $J = \mathtt{J}_0$, we get

$$\sum_{\substack{K\subseteq \mathtt{J}_0\\ |K|\geq 2}} (-1)^{1+|K|}\mathcal{W}_K = (-1)^{1+|\mathtt{J}_0|}\mathcal{W}_{\mathtt{J}_0} \neq 0\,.$$

This is a contradiction with the fact that such set $\mathtt{J}_0$ exists, as desired. □

Proof of Theorem 4.5.1. We know that

$$\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{q}) = \sum_{\vec{\boldsymbol{\pi}} \text{ is } \mathfrak{q}-\text{generic}} \mathbf{M}_{\vec{\boldsymbol{\pi}}}\,.$$

Suppose that $\vec{\boldsymbol{\pi}}_1 = A_1|\dots|A_k$, and $\vec{\boldsymbol{\pi}}_2 = B_1|\dots|B_k$ are set compositions such that $\mathfrak{q}_{\vec{\boldsymbol{\pi}}_1}$ is a point and $\vec{\boldsymbol{\pi}}_1 \sim \vec{\boldsymbol{\pi}}_2$. Define, for $i = 1,\dots,k$, the set $F_i = \bigcup_{j=i+1}^k A_j$, and $G_i = \bigcup_{j=i+1}^k B_j$. Observe that $F_k = G_k = \emptyset$. From the assumption that $\vec{\boldsymbol{\pi}}_1 \sim \vec{\boldsymbol{\pi}}_2$ we get that for a given $i = 1,\dots,k$ and $K \subseteq B_i$ with $|K| \geq 2$, we have that $A_i = B_i$ and $F_i = G_i$.

We wish to show that $\mathfrak{q}_{\vec{\boldsymbol{\pi}}_2}$ is also a point. This concludes the proof, because in this way we can group the sum above as

$$\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{q}) = \sum_{\text{all } \vec{\tau}\in[\vec{\boldsymbol{\pi}}] \text{ are } \mathfrak{q}-\text{generic}} \mathbf{N}_{[\vec{\boldsymbol{\pi}}]}\,,$$

where the sum runs over equivalence classes $[\vec{\boldsymbol{\pi}}]$, and this is trivially an element of **SC**.

We can rearrange the sum obtained in Eq. (4.10) as follows: for each $i = 1,\dots k$ and non-empty $K \subseteq A_i$, we group together all the sets I such that $I_{\vec{\boldsymbol{\pi}}} = K$. Those are precisely all the sets $I = J \cup K$ for some $J \subseteq F_i$. Thus, we have that

$$\mathfrak{q}_{\vec{\pi}_1} = \sum_{i=1}^{k} \sum_{\substack{K \subseteq A_i \\ K \neq \emptyset}} \mathfrak{s}_K \left(\sum_{J \subseteq F_i} a_{J \cup K} \right) .$$

From Lemma 4.5.2, we have that

$$\sum_{J \subseteq F_i} a_{J \cup K} = 0, \text{ for each } i = 1, \ldots, k \text{ and each } K \subseteq A_i \text{ with } |K| \geq 2. \tag{4.28}$$

Similarly, we have that

$$\mathfrak{q}_{\vec{\pi}_2} = \sum_{i=1}^{k} \sum_{\substack{K \subseteq B_i \\ K \neq \emptyset}} \mathfrak{s}_K \left(\sum_{J \subseteq G_i} a_{J \cup K} \right) . \tag{4.29}$$

The proof is concluded when we establish that $\sum_{J \subseteq G_i} a_{J \cup K} = 0$ for each $i = 1, \ldots, k$ and each $K \subseteq B_i$ with $|K| \geq 2$. This is precisely (4.28) because in this case we have that$A_i = B_i$ and $F_i = G_i$. Therefore $\mathfrak{q}_{\vec{\pi}_2}$ is a point, as desired. $\square$

Chapter 5

Hopf species and the non-commutative universal property

This chapter is a work based on the article [Pen18], to be published in *Journal of Combinatorial Theory A*. A short version was published in the proceedings of *Formal Power Series and Algebraic Combinatorics* (talk presentation). The article [Pen18] is split into this chapter and Chapter 4.

In [ABS06], a character in a Hopf algebra is defined as a multiplicative linear map that preserves unit, and a combinatorial Hopf algebra (or CHA, for short) is a Hopf algebra endowed with a character. For instance, a character η_0 in $QSym$ is $\eta_0(M_\alpha) = \mathbb{1}[l(\alpha) = 1]$. In fact, the CHA of quasi-symmetric functions $(QSym, \eta_0)$ is a terminal object in the category of CHAs, *i.e.*, for each CHA $(\mathbf{H}, \eta)$ there is a unique combinatorial Hopf algebra morphism $\Psi_{\mathbf{H}} : \mathbf{H} \to QSym$.

Our goal here is to draw a parallel for Hopf monoids in vector species. We see that the Hopf species $\mathtt{WQSym}$ plays the role of $QSym$. Specifically, we construct a unique Hopf monoid morphism from any combinatorial Hopf monoid $\mathtt{h}$ to $\mathtt{WQSym}$, in line with what was done in [ABS06] and [Whi16].

In Section 5.5, we investigate this universal property applied to the Hopf structure of generalized permutahedra and posets. We use Theorem 4.5.1 and Proposition 4.4.14 to obtain that no combinatorial Hopf monoid morphism from $\mathtt{GP}$ to $\mathtt{Pos}$ exists.

Remark 5.0.1. The category of combinatorial Hopf monoids was introduced in two distinct ways, by [AA17] in vector species, and by [Whi16] in pointed set species, which we call here a *comonoidal combinatorial Hopf monoid*. Here we consider the notion of [AA17].

In [Whi16], White shows that a comonoidal combinatorial Hopf monoid in coloring problems is a terminal object on the category of CCHM. Nevertheless, it is already advanced there that, if we consider a weaker notion of combinatorial Hopf monoid, the terminal object in such category is indexed by set compositions. No counterpart of $\mathtt{WQSym}$ in pointed set species is discussed here.

5.1 Hopf monoids in vector species

In this section, we recall the basic notions on Hopf monoids in vector species introduced in [AM10, Chapter 8]. We write $\mathtt{Set}^{\times}$ for the category of finite sets with bijections as only morphisms, and write $\mathtt{Vec}_{\mathbb{K}}$ for the category of vector spaces over $\mathbb{K}$ with linear maps as morphisms. A *vector species*, or simply a species, is a functor $\overline{a} : \mathtt{Set}^{\times} \to \mathtt{Vec}_{\mathbb{K}}$. Species form a category $\mathtt{Sp}_{\mathbb{K}}$, where morphisms are natural transformations between species. For a species $\mathtt{a}$, we denote by $\mathtt{a}[I]$ the vector space mapped from I through $\mathtt{a}$. For a natural transformation $\eta : \mathtt{a} \Rightarrow \mathtt{b}$, we may write either $\eta[I]$ or η_I to the corresponding map $\mathtt{a}[I] \to \mathtt{b}[I]$.

The Cauchy product is defined on species $\mathtt{a}, \mathtt{b}$ as follows:

$$\mathtt{a} \odot \mathtt{b}[I] = \bigoplus_{I = S \sqcup T} \mathtt{a}[S] \otimes \mathtt{b}[T] \, .$$

Two fundamental species are of interest. The first one $\mathcal{E}$ acts as the identity for the Cauchy product, and is defined as $\mathcal{E}[\emptyset] = \mathbb{K}$ for the empty set, and $\mathcal{E}[A] = 0$ otherwise. The functor $\mathcal{E}$ maps morphisms to the identity. The exponential species $\mathtt{Exp}$ is defined as $\mathtt{Exp}[A] = \mathbb{K}$ for any set A, and maps morphisms to the identity as well.

A vector species $\mathtt{a}$ is called a bimonoid if there are natural transformations $\mu : \mathtt{a} \odot \mathtt{a} \Rightarrow \mathtt{a}$, $\iota : \mathcal{E} \Rightarrow \mathtt{a}$, $\Delta : \mathtt{a} \Rightarrow \mathtt{a} \odot \mathtt{a}$ and $\varepsilon : \mathtt{a} \Rightarrow \mathcal{E}$ that satisfy some properties which we recover here only informally. We address the reader to [AM10, Section 8.2 - 8.3] for a detailed introduction of bimonoids in species.

- The natural transformation μ is associative.
- The natural transformation ι acts as unit on both sides.
- The natural transformation Δ is coassociative.

- The natural transformation ε acts as a counit on both sides.
- Both μ, Δ are determined by maps

$$\mu_{A,B} : \mathtt{a}[A] \otimes \mathtt{a}[B] \to \mathtt{a}[A \sqcup B]\,,$$

$$\Delta_{A,B} : \mathtt{a}[A \sqcup B] \to \mathtt{a}[A] \otimes \mathtt{a}[B]\,.$$

- The natural transformations satisfy some coherence relations typical for Hopf algebras. In particular it satisfies diagram 5.1 below, which enforces that the multiplicative and comultiplicative structure agree.

$$\begin{array}{ccc} \mathtt{a}[I] \otimes \mathtt{a}[J] & \xrightarrow{\mu_{I,J}} & \mathtt{a}[S] \\ \downarrow{\scriptstyle \Delta_{R,T} \otimes \Delta_{U,V}} & & \downarrow{\scriptstyle \Delta_{M,N}} \\ \mathtt{a}[R] \otimes \mathtt{a}[T] \otimes \mathtt{a}[U] \otimes \mathtt{a}[V] & \xrightarrow{(\mu_{R,U} \otimes \mu_{T,V}) \circ \mathbf{twist}} & \mathtt{a}[M] \otimes \mathtt{a}[N] \end{array} \tag{5.1}$$

We consider the canonical isomorphism $\beta : V \otimes W \to W \otimes V$, and also refer to any composition of tensors of identity maps and β as a *twist*. Whenever needed, we consider a suitable twist function without defining it explicitly, by letting the source and the target of the map clarify its precise definition. For instance, that is done above in Diagram (5.1).

We use $\mu_{A,B}$ and $\cdot_{A,B}$ interchangeably for the monoidal product. Namely, $\cdot_{A,B}$ will be employed for in line notation

We say that a bimonoid $\mathtt{h}$ is *connected* if the dimension of $\mathtt{h}[\emptyset]$ is one. A bimonoid is called a *Hopf monoid* if there is a natural transformation $s : \mathtt{h} \Rightarrow \mathtt{h}$, called the *antipode*, that satisfies

$$\mu \circ (\mathrm{id}_{\mathtt{h}} \odot s) \circ \Delta = \iota \circ \varepsilon = \mu \circ (s \odot \mathrm{id}_{\mathtt{h}}) \circ \Delta\,.$$

Proposition 5.1.1 (Proposition 8.10 in [AM10])**.** If $\mathtt{h}$ is a connected bimonoid, then there is an antipode on $\mathtt{h}$ that makes it a Hopf monoid.

Example 5.1.2 (Hopf monoids)**.**

- The exponential vector species can be endowed with a trivial product and coproduct. This is a connected bimonoid, hence it is a Hopf monoid.
- The vector space $\mathtt{Gr}[I]$ (resp. $\mathtt{Pos}$, $\mathtt{GP}$ and $\mathtt{HGP}$ has a basis given by all graphs on the vertex set I (resp. partial orders on the set I, generalized permutahedra in $\mathbb{R}^I$, hypergraphic polytopes in $\mathbb{R}^I$) defines a Hopf monoid with the operations introduced above.

5.2 Combinatorial Hopf monoids

The notion of characters in Hopf monoids was already brought to light in [AA17], where it is used to settle, for instance, a conjecture of Humpert and Martin [HM12] on graphs.

Definition 5.2.1. Let $\mathtt{h}$ be a Hopf monoid. A *Hopf monoid character* $\eta : \mathtt{h} \Rightarrow \mathtt{Exp}$, or simply a *character*, is a monoid morphism such that $\eta_\emptyset = \varepsilon_\emptyset$ and the following diagram commutes:

$$\begin{array}{ccc} \mathtt{h}[I] \otimes \mathtt{h}[J] & \xrightarrow{\mu_{I,J}} & \mathtt{h}[A] \\ \downarrow{\scriptstyle \eta_I \otimes \eta_J} & & \downarrow{\scriptstyle \eta_A} \\ \mathbb{K} \otimes \mathbb{K} & \xrightarrow{\cong} & \mathbb{K} \end{array} \tag{5.2}$$

A *combinatorial Hopf monoid* is a pair $(\mathtt{h}, \eta)$ where $\mathtt{h}$ is a Hopf monoid, and η a character of $\mathtt{h}$.

The condition that η and ε coincide in the $\emptyset$ level is commonly verified in Hopf monoids of combinatorial objects. In particular, this condition is always verified in connected Hopf monoids.

Example 5.2.2 (Combinatorial Hopf monoids)**.** From the examples on Hopf monoids above and the characters defined in Section 4.2, we can construct combinatorial Hopf monoids: in $\mathtt{Gr}$ with the character $\eta(G) = \mathbb{1}[G \text{ has no edges}]$, in $\mathtt{Pos}$ with the character $\eta(P) = \mathbb{1}[P \text{ is antichain}]$, and in $\mathtt{GP}$ with the character given by $\eta(\mathtt{q}) = \mathbb{1}[\mathtt{q} \text{ is a point }]$.

A combinatorial Hopf monoid morphism $\alpha : (\mathtt{h}_1, \eta_1) \Rightarrow (\mathtt{h}_2, \eta_2)$ is a Hopf monoid morphism $\alpha : \mathtt{h}_1 \Rightarrow \mathtt{h}_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathtt{h}_1 & \xrightarrow{\alpha} & \mathtt{h}_2 \\ & {\scriptstyle \eta_1} \searrow \quad \swarrow {\scriptstyle \eta_2} & \\ & \mathtt{Exp} & \end{array} \tag{5.3}$$

We introduce the Fock functors, that give us a construction of several graded Hopf algebras from a Hopf monoid and, more generally, construct graded vector spaces from vector species. The topic is carefully developed in [AM10, Section 3.1, Section 15.1].

Definition 5.2.3 (Fock functors)**.** Denote by $\mathtt{gVec}_\mathbb{K}$ the category of graded vector spaces over $\mathbb{K}$. We focus on the following Fock functors $\mathcal{K}, \overline{\mathcal{K}} : \mathtt{Sp}_\mathbb{K} \to \mathtt{gVec}_\mathbb{K}$, called full Fock functor and bosonic Fock functor, respectively, defined as:

$$\mathcal{K}(q) := \bigoplus_{n\geq 0} q[\{1,\dots,n\}] \text{ and } \overline{\mathcal{K}}(q) := \bigoplus_{n\geq 0} q[\{1,\dots,n\}]_{S_n} ,$$

where V_{S_n} stands for the vector space of coinvariants on V over the action of S_n, *i.e.*, the quotient of V under all relations of the form $x - \sigma(x)$, for $\sigma \in S_n$.

If $\mathtt{h}$ is a combinatorial Hopf monoid with structure morphisms $\mu, \iota, \Delta, \varepsilon$, then $\mathcal{K}(\mathtt{h})$ and $\overline{\mathcal{K}}(\mathtt{h})$ are Hopf algebras with related structure maps. If η is a character of $\mathtt{h}$, then $\mathcal{K}(\mathtt{h})$ and $\overline{\mathcal{K}}(\mathtt{h})$ also have a character.

Example 5.2.4 (Fock functors of some Hopf monoids)**.**

- The Hopf algebra $\mathcal{K}(\mathcal{E})$ is the linear Hopf algebra $\mathbb{K}$. The Hopf algebra $\mathcal{K}(\mathtt{Exp})$ is the polynomial Hopf algebra $\mathbb{K}[x]$.
- The Hopf algebras $\mathcal{K}(\mathtt{Gr})$, $\mathcal{K}(\mathtt{Pos})$ and $\mathcal{K}(\mathtt{GP})$ are the Hopf algebras of graphs **G**, of posets **Pos** and of generalized permutahedra **GP** introduced above.
- The Hopf algebra $\mathcal{K}(\mathtt{HGP})$ is the Hopf algebra **HGP**, the Hopf subalgebra of **GP** introduced above.

5.3 The word quasi-symmetric function Hopf monoid

Recall that a coloring of a set I is a map $f : I \to \mathbb{N}$, and that $\mathfrak{C}_I$ be the set of colorings of I. Recall as well that a set composition $\vec{\boldsymbol{\pi}} = S_1|\cdots|S_l$ can be identified with a total preorder $R_{\vec{\boldsymbol{\pi}}}$, where we say $a\, R_{\vec{\boldsymbol{\pi}}}\, b$ if $a \in S_i$ and $b \in S_j$ satisfy $i \leq j$. For a set composition $\vec{\boldsymbol{\pi}}$ of A and a non-empty subset $I \subseteq A$, we define $\vec{\boldsymbol{\pi}}|_I$ as the set composition of I obtained by restricting the preorder $R_{\vec{\boldsymbol{\pi}}}$ to I.

If I, J are disjoint sets, and $f \in \mathfrak{C}_I$ and $g \in \mathfrak{C}_J$, then we set $f * g \in \mathfrak{C}_{I\sqcup J}$ as the unique coloring in $I \sqcup J$ that satisfies both $f * g|_I = f$ and $f * g|_J = g$.

For a set composition $\vec{\boldsymbol{\pi}} \in \mathbf{C}_I$, let $\mathbb{M}_{\vec{\boldsymbol{\pi}}} = \sum_{\substack{f\in\mathfrak{C}_I \\ \vec{\boldsymbol{\pi}}(f)=\vec{\boldsymbol{\pi}}}} [f]$ be a formal sum of colorings, and define $\mathtt{WQSym}[I]$ as the span of $\{\mathbb{M}_{\vec{\boldsymbol{\pi}}}\}_{\vec{\boldsymbol{\pi}}\in\mathbf{C}_I}$. This gives us a $\mathbb{K}$-linear space with basis enumerated by $\mathbf{C}_I$, so that $\mathtt{WQSym}$ is a species.

Further, define the monoidal product operation with

$$\mathbb{M}_{\vec{\boldsymbol{\pi}}} \cdot_{A,B} \mathbb{M}_{\vec{\boldsymbol{\tau}}} = \sum_{\substack{f\in\mathfrak{C}_A \\ \vec{\boldsymbol{\pi}}(f)=\vec{\boldsymbol{\pi}}}} \sum_{\substack{g\in\mathfrak{C}_B \\ \vec{\boldsymbol{\pi}}(g)=\vec{\boldsymbol{\tau}}}} [f * g] = \sum_{\substack{\vec{\boldsymbol{\lambda}}|_A=\vec{\boldsymbol{\pi}} \\ \vec{\boldsymbol{\lambda}}|_B=\vec{\boldsymbol{\tau}}}} \mathbb{M}_{\vec{\boldsymbol{\lambda}}} \,. \tag{5.4}$$

We write $I <_{\vec{\pi}} J$ whenever there is no $i \in I$ and $j \in J$ such that $j \, R_{\vec{\pi}} \, i$. The coproduct $\Delta_{I,J} M_{\vec{\pi}}$ is defined as

$$\mathbb{M}_{\vec{\pi}|_I} \otimes \mathbb{M}_{\vec{\pi}|_J} ,$$

whenever $I <_{\vec{\pi}} J$, and is zero otherwise.

If we set the unit as $\iota_\emptyset(1) = \mathbb{M}_\emptyset$ and the counit acting on the basis as $\varepsilon(\mathbb{M}_{\vec{\pi}}) = \mathbb{1}[\vec{\pi} = \emptyset]$ we get a Hopf monoid. In fact, this is the dual Hopf monoid of faces $\Sigma^* = \Sigma_1^*$ in [AM10].

Proposition 5.3.1 ([AM10, Definition 12.19])**.** With these operations, the species `WQSym` becomes a Hopf monoid.

Proposition 5.3.2 ([AM10, Section 17.3.1])**.** The Hopf algebra $\mathcal{K}(\texttt{WQSym})$ is the Hopf algebra on word quasi-symmetric functions **WQSym**, and $\overline{\mathcal{K}}(\texttt{WQSym})$ is the Hopf algebra on quasi-symmetric functions $QSym$.

The identification is as follows: $\mathcal{K}(\texttt{WQSym})$ and **WQSym** by identifying a coloring $f : [n] \to \mathbb{N}$ with the non-commutative monomial $\prod_{i=1}^n a_{f(i)} =: a_f$, and extend this to identify $\mathbb{M}_{\vec{\pi}}$ with $\mathbf{M}_{\vec{\pi}}$.

Proposition 5.3.3 (Combinatorial Hopf monoid on `WQSym`)**.** Take the character $\eta : \texttt{WQSym} \Rightarrow \texttt{Exp}$ defined in the basis elements as

$$\eta_0[I](\mathbb{M}_{\vec{\pi}}) = \mathbb{1}[l(\vec{\pi}) \leq 1] \, . \tag{5.5}$$

This turns $(\texttt{WQSym}, \eta_0)$ into a combinatorial Hopf monoid.

Proof. We write $\eta_{0,I} = \eta_0[I]$ for short. That η_0 is a natural transformation is trivial, and also $\eta_{0,\emptyset}(\mathbb{M}_\emptyset) = 1$, so it preserves the unit.

To show that η_0 is multiplicative, we just need to check that the diagram (5.2) commutes for the basis elements, *i.e.*, if $A = I \sqcup J$, then

$$\eta_{0,I}(\mathbb{M}_{\vec{\pi}})\eta_{0,J}(\mathbb{M}_{\vec{\tau}}) = \eta_{0,A}(\mathbb{M}_{\vec{\pi}}\mathbb{M}_{\vec{\tau}}) = \sum_{\substack{\vec{\gamma} \in \mathbf{C}_A \\ \vec{\gamma}|_I = \vec{\pi} \\ \vec{\gamma}|_J = \vec{\tau}}} \eta_{0,A}(\mathbb{M}_{\vec{\gamma}}) \, . \tag{5.6}$$

Note that if $\vec{\gamma}$ is a set composition of A such that $\vec{\gamma}|_I = \vec{\pi}$, then trivially we have that $l(\vec{\pi}) \leq l(\vec{\gamma})$, so from (5.5), $\eta_{0,I}(\mathbb{M}_{\vec{\pi}}) = 0 \Rightarrow \eta_{0,A}(\mathbb{M}_{\vec{\gamma}}) = 0$. Similarly, if $\vec{\gamma}|_J = \vec{\tau}$, we have $\eta_{0,J}(\mathbb{M}_{\vec{\tau}}) = 0 \Rightarrow \eta_{0,A}(\mathbb{M}_{\vec{\gamma}}) = 0$.

So it is enough to consider the case where $\eta_{0,I}(\mathbb{M}_{\vec{\pi}}) = \eta_{0,J}(\mathbb{M}_{\vec{\tau}}) = 1$, *i.e.*, $l(\vec{\pi}), l(\vec{\tau}) \leq 1$. Now, if γ has only one part, it does indeed hold that $\gamma|_I = \vec{\pi}$ and $\gamma|_J = \vec{\tau}$, so there is a unique $\vec{\gamma}$ on the right hand side of (5.6) that satisfies $l(\vec{\gamma}) \leq 1$, and this concludes the proof. □

5.4 Universality of WQSym

The following theorem is the main theorem of this section. For connected Hopf monoids, this is a corollary of [AM10, Theorem 11.23].

Theorem 5.4.1 (Terminal object in combinatorial Hopf monoids)**.** Given $\mathtt{h}$ a Hopf monoid with a character $\eta : \mathtt{h} \Rightarrow \mathtt{Exp}$, there is a unique combinatorial Hopf monoid morphism $\Upsilon_{\mathtt{h}} : \mathtt{h} \Rightarrow \mathtt{WQSym}$, *i.e.*, a unique Hopf monoid morphism $\Upsilon_{\mathtt{h}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathtt{h} & \xrightarrow{\Upsilon_{\mathtt{h}}} & \mathtt{WQSym} \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\eta_0} & \\ & \mathtt{Exp} & \end{array} \tag{5.7}$$

We remark that this is a claim motivated in [AM10, Theorem 11.23], which applies to any connected Hopf monoid. There, the notion of *positive monoid* was introduced, a monoid in species such that $\mathtt{h}[\emptyset] = 0$ and with no unit axioms. Any Hopf monoid $\mathtt{h}$ can become a positive monoid $\mathtt{h}^+$ by setting $\mathtt{h}^+[\emptyset] = 0$ and $\mathtt{h}^+[I] = \mathtt{h}[I]$ for any non-empty set I. A functor $\mathcal{T}^\vee$, mapping positive monoids to Hopf monoids, was constructed, so that any *positive monoid* q, connected Hopf monoid $\mathtt{h}$ and monoid morphism $\eta : \mathtt{h}^+ \Rightarrow q$, there exists a unique Hopf monoid morphism $\eta_\vee : h \Rightarrow \mathcal{T}^\vee(q)$ with the following commuting diagram on positive monoids:

$$\begin{array}{ccc} \mathtt{h}^+ & \xrightarrow{\eta_\vee^+} & \mathcal{T}^\vee(q)^+ \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\varepsilon(q)} & \\ & q & \end{array} \tag{5.8}$$

where $\varepsilon(q) : \mathcal{T}^\vee(q)^+ \Rightarrow q$ is a map that comes from the construction of $\mathcal{T}^\vee$. In the case where q is the positive exponential monoid, the resulting Hopf monoid $\mathcal{T}^\vee(q)$ is precisely $\mathtt{WQSym}$, thus obtaining Theorem 5.4.1 for connected Hopf monoids.

In fact, the result presented in Theorem 5.4.1 is a minor extension of [AM10, Theorem 11.23] to Hopf monoids that are not necessarily connected, but whose character agrees with the counit in $\mathtt{h}[\emptyset]$. First we will present a self contained proof by means of multi-characters. In Remark 5.4.6, we present a more direct proof, using [AM10, Theorem

11.23] and a suitably constructed connected Hopf monoid. This proof was kindly pointed out by a reviewer.

Definition 5.4.2 (Multi-character and other notations)**.** For a set composition on a non-empty set I, say $\vec{\pi} = S_1|\cdots|S_l$ with k 0, denote for short

$$\mathtt{h}[\vec{\pi}] = \bigotimes_{i=1}^{l} \mathtt{h}[S_i]\,,$$

and similarly define for a natural transformation $\zeta : \mathtt{h} \Rightarrow \mathtt{b}$ the linear transformation $\zeta[\vec{\pi}] : \mathtt{h}[\vec{\pi}] \to \mathtt{b}[\vec{\pi}]$ as $\zeta[\vec{\pi}] = \bigotimes_{i=1}^{l} \zeta[S_i]$. For a character, $\zeta[\vec{\pi}] : \mathtt{h}[\vec{\pi}] \to \mathbb{K}^{\otimes l} \cong \mathbb{K}$.

For a set composition $\vec{\pi}$ on I of length k, let us define $\Delta_{\vec{\pi}}$ as a map

$$\Delta_{\vec{\pi}} : \mathtt{h}[I] \to \mathtt{h}[\vec{\pi}]\,,$$

inductively as follows:

- If the length of $\vec{\pi}$ is 1, then $\Delta_{\vec{\pi}} = \mathrm{id}_{\mathtt{h}[I]}$.
- If $\vec{\pi} = S_1|\cdots|S_k$ for $k > 1$, let $\vec{\tau} = S_1|\cdots|S_{k-1}$ and define

$$\Delta_{\vec{\pi}} = (\Delta_{\vec{\tau}} \otimes \mathrm{id}_{S_k}) \circ \Delta_{I\setminus S_k, S_k}\,. \tag{5.9}$$

Note that, by coassociativity, this definition of $\Delta_{\vec{\pi}}$ is independent of the chosen order in the inductive definition in (5.9), *i.e.*, for any set $I = A \sqcup B$ and set composition $\vec{\pi} \in \mathbf{C}_I$ such that $A <_{\vec{\pi}} B$, we have

$$\Delta_{\vec{\pi}} = (\Delta_{\vec{\pi}|_A} \otimes \Delta_{\vec{\pi}|_B}) \circ \Delta_{A,B}\,. \tag{5.10}$$

We can define $f_{\vec{\pi},\eta} : \mathtt{h}[I] \to \mathbb{K}$ as

$$\mathtt{h}[I] \xrightarrow{\Delta_{\vec{\pi}}} \mathtt{h}[\vec{\pi}] \xrightarrow{\eta[\vec{\pi}]} \mathbb{K}^{\otimes l} \cong \mathbb{K}\,,$$

Finally, if $A = I \sqcup J$ with $I \neq \emptyset \neq J$ and $\vec{\pi} \in \mathbf{C}_I, \vec{\tau} \in \mathbf{C}_J$, then we write both $\vec{\pi}|\vec{\tau}$ and $(\vec{\pi}, \vec{\tau})$ for the unique set composition $\vec{\gamma} \in \mathbf{C}_A$ such that $\vec{\gamma}|_I = \vec{\pi}$, $\vec{\gamma}|_J = \vec{\tau}$, and $I <_{\vec{\gamma}} J$.

Example 5.4.3. In the graph combinatorial Hopf monoid $\mathtt{Gr}$, take the labeled cycle C_5 on $\{1, 2, 3, 4, 5\}$ given in Fig. 5.1. Denote by K_J the complete graph on the labels J and by 0_J the empty graph on the labels J.

Consider the set compositions $\vec{\pi}_1 = 13|2|45$ and $\vec{\pi}_2 = 24|13|5$. Then

$$\Delta_{\vec{\pi}_1}(C_5) = 0_{\{1,3\}} \otimes 0_{\{2\}} \otimes K_{\{4,5\}}\,,$$

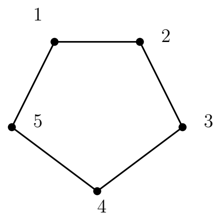


FIGURE 5.1: Cycle on the set [5]

$$\Delta_{\vec{\pi}_2}(C_5) = 0_{\{2,4\}} \otimes 0_{\{1,3\}} \otimes 0_{\{5\}} ,$$

in particular, $f_{\vec{\pi}_1,\eta}(C_5) = 0$ and $f_{\vec{\pi}_2,\eta}(C_5) = 1$. Generally,

$$f_{\vec{\pi},\eta}(G) = \mathbb{1}[\boldsymbol{\lambda}(\vec{\pi}) \text{ is a stable set partition on } G] . \tag{5.11}$$

From (5.11) and from Lemma 4.2.4 we have that $\mathcal{K}(\boldsymbol{\Upsilon}_{\texttt{Gr}}) = \boldsymbol{\Upsilon}_{\mathbf{G}}$ is the chromatic symmetric function in non-commutative variables, and that $\overline{\mathcal{K}}(\boldsymbol{\Upsilon}_{\texttt{Gr}}) = \Psi_{\mathbf{G}}$ is the chromatic symmetric function.

In a similar way we can establish that $\mathcal{K}(\boldsymbol{\Upsilon}_{\texttt{Pos}}) = \boldsymbol{\Upsilon}_{\mathbf{Pos}}$, that $\overline{\mathcal{K}}(\boldsymbol{\Upsilon}_{\texttt{Pos}}) = \Psi_{\mathbf{Pos}}$, that $\mathcal{K}(\boldsymbol{\Upsilon}_{\mathbf{GP}}) = \boldsymbol{\Upsilon}_{\texttt{GP}}$ and that $\overline{\mathcal{K}}(\boldsymbol{\Upsilon}_{\mathbf{GP}}) = \Psi_{\texttt{GP}}$, where GP is the combinatorial species on generalized permutahedra.

With this notation, we can rephrase diagram (5.1) in a different way:

Proposition 5.4.4. Consider a Hopf monoid $(\mathtt{h}, \mu, \iota, \Delta, \varepsilon)$. Let $\vec{\gamma} = C_1|\dots|C_k$ be a set composition on $S = I \sqcup J$, where I, J are non-empty sets. Write $A_i := C_i \cap I$ and $B_i := C_i \cap J$, and let $\vec{\pi} := (\vec{\gamma}|_I, \vec{\gamma}|_J) = A_1|\dots|A_k|B_1|\dots|B_k$, erasing the empty blocks.

Define $\mu_{(\vec{\gamma},I,J)} : \mathtt{h}[\vec{\pi}] \to \mathtt{h}[\vec{\gamma}]$ as the tensor product of the maps

$$\mathtt{h}[A_i] \otimes \mathtt{h}[B_i] \xrightarrow{\mu_{A_i,B_i}} \mathtt{h}[C_i] ,$$

composed with the necessary twist so that it maps $\mathtt{h}[\vec{\pi}] \to \mathtt{h}[\vec{\gamma}]$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathtt{h}[I] \otimes \mathtt{h}[J] & \xrightarrow{\mu_{I,J}} & \mathtt{h}[S] \\ \big\downarrow (\Delta_{\vec{\gamma}|_I} \otimes \Delta_{\vec{\gamma}|_J}) & & \big\downarrow \Delta_{\vec{\gamma}} \\ \mathtt{h}[\vec{\pi}] & \xrightarrow{\mu_{(\vec{\gamma},I,J)}} & \mathtt{h}[\vec{\gamma}] \end{array} \tag{5.12}$$

Note that diagram (5.1) corresponds to diagram (5.12) when $k = 2$. We prove now that Diagram (5.12) is obtained by gluing diagrams of the form of Diagram (5.1):

Proof. We act by induction on the length of $\vec{\gamma}$, $k := l(\vec{\gamma})$. The base case is for $k = 2$, where we recover diagram (5.1).

Suppose now that $k \geq 3$. Applying $I = A_1 \sqcup A_2, J = B_1 \sqcup B_2$ to (5.1) we have the following commuting diagram:

$$\begin{array}{ccc} \mathtt{h}[A_1 \sqcup A_2|B_1 \sqcup B_2] & \xrightarrow{\mu_{A_1\sqcup A_2, B_1 \sqcup B_2}} & \mathtt{h}[C_1 \sqcup C_2] \\ \downarrow{\scriptstyle \Delta_{A_1,A_2}\otimes\Delta_{B_1,B_2}} & & \downarrow{\scriptstyle \Delta_{C_1,C_2}} \\ \mathtt{h}[A_1|A_2|B_1|B_2] & \xrightarrow{(\mu_{A_1,B_1}\otimes\mu_{A_2,B_2})\circ\textbf{twist}} & \mathtt{h}[C_1|C_2] \end{array} \tag{5.13}$$

Write $I' = I \setminus (C_1 \sqcup C_2)$ and $J' = J \setminus (C_1 \sqcup C_2)$, let $\vec{\gamma}' = C_3|\dots|C_l$ and take $\vec{\gamma}^o = C_1 \sqcup C_2|C_3|\dots|C_l = \{C_1 \sqcup C_2\}|\gamma'$. Observe that γ' is a partition of a non-empty set. By tensoring diagram (5.13) with

$$\begin{array}{ccc} \mathtt{h}[A_3|B_3|A_4|\dots|B_l] & \xrightarrow{\mu_{(\vec{\gamma}', I', J')}} & \mathtt{h}[C_3|\dots|C_l] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathtt{h}[A_3|B_3|A_4|\dots|B_l] & \xrightarrow{\mu_{(\vec{\gamma}', I', J')}} & \mathtt{h}[C_3|\dots|C_l] \end{array} \tag{5.14}$$

we have:

$$\begin{array}{ccc} \mathtt{h}[A_1 \sqcup A_2|B_1 \sqcup B_2|A_3|B_3|\dots] & \xrightarrow{\mu_{A_1\sqcup A_2, B_1 \sqcup B_2}\otimes\mu_{(\vec{\gamma}', I', J')}} & \mathtt{h}[\vec{\gamma}^o] \\ \downarrow{\scriptstyle \Delta_{A_1,A_2}\otimes\Delta_{B_1,B_2}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Delta_{C_1,C_2}\otimes\mathrm{id}} \\ \mathtt{h}[A_1|A_2|B_1|B_2|A_3|B_3|A_4|\dots] & \xrightarrow{(\mu_{A_1,B_1}\otimes\mu_{A_2,B_2}\otimes\mu_{(\vec{\gamma}', I', J')})\circ\textbf{twist}} & \mathtt{h}[\vec{\gamma}] \end{array} \tag{5.15}$$

Note that

$$\mu_{A_1,B_1} \otimes \mu_{A_2,B_2} \otimes \mu_{(\vec{\gamma}', I', J')} = \mu_{(\vec{\gamma}, I, J)} \,,$$

$$\mu_{A_1 \sqcup A_2, B_1 \sqcup B_2} \otimes \mu_{(\vec{\gamma}', I', J')} = \mu_{(\vec{\gamma}^o, I, J)} \,.$$

So, by induction hypothesis, (5.12) commutes for the set composition $\vec{\gamma}^o = C_1 \sqcup C_2|C_3|\dots|C_l$. Apply the necessary twists so as to glue with with diagram (5.15) as follows:

$$\begin{array}{ccc} \mathtt{h}[I] \otimes \mathtt{h}[J] & \xrightarrow{\mu_{I,J}} & \mathtt{h}[S] \\ \downarrow{\scriptstyle \mathbf{twist} \circ (\Delta_{\vec{\gamma}^o|_I} \otimes \Delta_{\vec{\gamma}^o|_J})} & & \downarrow{\scriptstyle \Delta_{\vec{\gamma}^o}} \\ \mathtt{h}[A_1 \sqcup A_2 | B_1 \sqcup B_2 | A_3 | B_3 | \dots | B_l] & \xrightarrow{\mu_{(\vec{\gamma}^o, I, J)}} & \mathtt{h}[\vec{\gamma}'] \\ \downarrow{\scriptstyle \mathbf{twist} \circ (\Delta_{A_1,A_2} \otimes \Delta_{B_1,B_2} \otimes \mathrm{id})} & & \downarrow{\scriptstyle \Delta_{C_1,C_2} \otimes \mathrm{id}} \\ \mathtt{h}[A_1 | B_1 | A_2 | \dots | B_l] & \xrightarrow{\mu_{(\vec{\gamma}, I, J)} \circ \mathbf{twist}} & \mathtt{h}[\vec{\gamma}] \end{array} \tag{5.16}$$

We note that absorbing the twist in the bottom left vector space and erasing the middle line gives us the desired diagram. □

Proposition 5.4.5. Consider a combinatorial Hopf monoid $(\mathtt{h}, \eta)$. Let $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}}$ be set compositions of the disjoint non-empty sets I and J, respectively, and take $\vec{\boldsymbol{\lambda}}$ set composition of $\mathrm{S} = I \sqcup J$. Take $a \in \mathtt{h}[I]$, $b \in \mathtt{h}[J]$, $c \in \mathtt{h}[I \sqcup J]$. Then we have that

$$f_{\vec{\boldsymbol{\lambda}},\eta}(a \cdot_{I,J} b) = f_{\vec{\boldsymbol{\lambda}}|_I,\eta}(a) f_{\vec{\boldsymbol{\lambda}}|_J,\eta}(b)\,, \tag{5.17}$$

and that

$$f_{\vec{\boldsymbol{\pi}},\eta} \otimes f_{\vec{\boldsymbol{\tau}},\eta} \circ \Delta_{I,J}(a) = f_{(\vec{\boldsymbol{\pi}},\vec{\boldsymbol{\tau}}),\eta}(a)\,. \tag{5.18}$$

Proof. Note that (5.17) reduces to

$$f_{\vec{\gamma},\eta} \circ \mu_{I,J} = f_{\vec{\gamma}|_I,\eta} \otimes f_{\vec{\gamma}|_J,\eta}. \tag{5.19}$$

Now Proposition 5.4.4 tells us that

$$\Delta_{\vec{\gamma}} \circ \mu_{I,J} = \mu_{(\vec{\gamma},I,J)} \circ (\Delta_{\vec{\gamma}|_I} \otimes \Delta_{\vec{\gamma}|_J})\,. \tag{5.20}$$

Suppose that $\vec{\gamma} = C_1 | \dots | C_k$ for $k \geq 1$. Then by tensoring diagrams of the form (5.2) for each decomposition $(I \cup C_i) \sqcup (J \cup C_i) = C_i$, we obtain

$$\eta[\vec{\gamma}] \circ \mu_{(\vec{\gamma},I,J)} = \eta[(\vec{\gamma}|_I, \vec{\gamma}|_J)]\,. \tag{5.21}$$

From (5.20) and (5.21) we get that

$$\eta[\vec{\gamma}] \circ \Delta_{\vec{\gamma}} \circ \mu_{I,J} = \eta[\vec{\gamma}] \circ \mu_{(\vec{\gamma},I,J)} \circ (\Delta_{\vec{\gamma}|_I} \otimes \Delta_{\vec{\gamma}|_J}) = \eta[(\vec{\gamma}|_I, \vec{\gamma}|_J)] \circ (\Delta_{\vec{\gamma}|_I} \otimes \Delta_{\vec{\gamma}|_J})\,.$$

So

$$f_{\vec{\gamma},\eta} \circ \mu_{I,J} = (\eta[\vec{\gamma}|_I] \otimes \eta[\vec{\gamma}|_J]) \circ (\Delta_{\vec{\gamma}|_I} \otimes \Delta_{\vec{\gamma}|_J}) = \eta_{\vec{\gamma}|_I} \otimes \eta_{\vec{\gamma}|_J}\,.$$

This concludes the proof of (5.19). Remains to show (5.18), which follows from (5.10) via

$$(f_{\vec{\pi},\eta} \otimes f_{\vec{\tau},\eta}) \circ \Delta_{I,J} = (\eta[\vec{\pi}] \otimes \eta[\vec{\tau}]) \circ (\Delta_{\vec{\pi}} \otimes \Delta_{\vec{\tau}}) \circ \Delta_{I,J} = \eta[(\vec{\pi}, \vec{\tau})] \circ \Delta_{(\vec{\pi},\vec{\tau})} = f_{(\vec{\pi},\vec{\tau}),\eta} \,,$$

whenever both I and J are non empty, establishing the equality as desired. □

Proof of Theorem 5.4.1. Let $a \in \mathtt{h}[I]$. We define

$$\boldsymbol{\Upsilon}_{\mathtt{h}}(a) = \sum_{\vec{\pi} \in \mathbf{C}_I} \mathbb{M}_{\vec{\pi}} \eta_{\vec{\pi}}(a) \,,$$

The commutativity of Diagram (5.7) follows because $f_{\vec{\pi},\eta} = \eta$ whenever $\vec{\pi}$ has length 1 or 0. Remains to show that such map is a combinatorial Hopf monoid morphism, *i.e.*, that we have:

- $\boldsymbol{\Upsilon}_{\mathtt{h}}(a \cdot_{I,J} b) = \boldsymbol{\Upsilon}_{\mathtt{h}}(a)\boldsymbol{\Upsilon}_{\mathtt{h}}(b)$.
- $\boldsymbol{\Upsilon}_{\mathtt{h} \odot \mathtt{h}}(\Delta_{I,J} a) = \Delta_{I,J} \circ \boldsymbol{\Upsilon}_{\mathtt{h}}(a)$.
- $\boldsymbol{\Upsilon}_{\mathtt{h}} \circ \iota_{\mathtt{h}} = \iota_{\mathtt{WQSym}}$.
- $\varepsilon_{\mathtt{WQSym}} \circ \boldsymbol{\Upsilon}_{\mathtt{h}} = \varepsilon_{\mathtt{h}}$.

The last two equations follow from direct computation. Taking the coefficients on the monomial basis for the first two items, this reduces to Proposition 5.4.5 whenever I, J are non empty.

Further, when $I = \emptyset$, we can assume $a = \lambda \mathbb{M}_{\vec{\emptyset}}$ and the first equation follows immediately. Also, when $I = \emptyset$, $\Delta_{I,J}(a) = a \otimes \mathbb{M}_{\vec{\emptyset}}$, and the second equation follows. This concludes that $\boldsymbol{\Upsilon}_{\mathtt{h}}$ is a combinatorial Hopf monoid morphism.

It remains to establish the uniqueness. Suppose that $\phi : \mathtt{h} \Rightarrow \mathtt{WQSym}$ is a combinatorial Hopf monoid morphism. Note that $\eta_0[\emptyset]$ is an isomorphism, so from (5.7) applied to both ϕ and $\boldsymbol{\Upsilon}_{\mathtt{h}}$ we get $\eta_0[\emptyset]^{-1}\eta[\emptyset] = \phi[\emptyset] = \boldsymbol{\Upsilon}_{\mathtt{h}}[\emptyset]$.

Now take I non-empty. For each $a \in \mathtt{h}[I]$, write $\phi[I](a) = \sum_{\vec{\pi} \in \mathbf{C}_I} \mathbb{M}_{\vec{\pi}} \phi_{\vec{\pi}}(a)$ and apply $\Delta_{\vec{\pi}}$ on both sides. Since ϕ is a comonoid morphism, we have:

$$\phi[\vec{\pi}]\Delta_{\vec{\pi}}(a) = \Delta_{\vec{\pi}}\phi[I](a) = \sum_{\vec{\tau}} \phi_{\vec{\tau}}(a)\Delta_{\vec{\pi}}\mathbb{M}_{\vec{\tau}} = \phi_{\vec{\pi}}(a)\Delta_{\vec{\pi}}\mathbb{M}_{\vec{\pi}} \,, \tag{5.22}$$

because $\Delta_{\vec{\pi}}(\mathbb{M}_{\vec{\tau}}) = 0$ whenever $\vec{\pi} \neq \vec{\tau}$ and $|\vec{\pi}| = |\vec{\tau}|$.

However, since $\eta_0 \phi = \eta$, we have that $\eta_0[\vec{\pi}]\phi[\vec{\pi}] = \eta[\vec{\pi}]$, so

$$\eta_0[\vec{\pi}]\phi[\vec{\pi}]\Delta_{\vec{\pi}}(a) = \eta[\vec{\pi}]\Delta_{\vec{\pi}}(a) = f_{\vec{\pi},\eta}(a)\,. \tag{5.23}$$

Applying $\eta_0[\vec{\pi}]$ on (5.22) and using (5.23), gives us $\phi_{\vec{\pi}}(a) f_{\vec{\pi},\eta_0}(\mathbb{M}_{\vec{\pi}}) = \eta_{\vec{\pi}} a$. However, $f_{\vec{\pi},\eta_0}(\mathbb{M}_{\vec{\pi}}) = 1$, so we have that

$$\phi_{\vec{\pi}}(a) = f_{\vec{\pi},\eta}(a)\,.$$

which concludes the uniqueness. □

Remark 5.4.6. We would like to point out that this theorem also follows from [AM10, Theorem 11.23] directly. However, we present here a proof with multi-characters in the interest of self containment.

Specifically, let $(\eta, \mathtt{h})$ be a combinatorial Hopf monoid, and consider the following Hopf submonoid $\mathtt{h}'$, where $\mathtt{h}'[I] = \mathtt{h}[I]$ for $I \neq \emptyset$, and $\mathtt{h}'[\emptyset] = \mathbb{K}$. This is also a combinatorial Hopf monoid, as the maps ε and η, suitably redefined to be the relevant restriction to $\mathtt{h}'[\emptyset]$, satisfy the Hopf monoid axioms. The unit and bialgebra axioms guarantee that $\mathtt{h}'[\emptyset]$ is stable for the product and the coproduct.

Therefore, because this a connected Hopf monoid, from [AM10, Theorem 11.23] there is a unique Hopf monoid morphism $\Psi : \mathtt{h}' \Rightarrow \mathtt{WQSym}$. This map can be extended to a Hopf monoid morphism $\Psi : \mathtt{h} \Rightarrow \mathtt{WQSym}$ by setting $\Psi_{\emptyset} = \eta_{\emptyset}$. It must be checked that this map satisfies the Hopf monoid morphism axioms. It is a direct observation that (5.7) commutes, making this a Hopf monoid morphism.

Conversely, if there are two distinct combinatorial Hopf monoids $\Psi_1, \Psi_2 : \mathtt{h} \Rightarrow \mathtt{WQSym}$, then they must agree on $\mathtt{h}[\emptyset]$ in accordance with (5.7). On the other hand, if we consider the compositions $\mathrm{inc} \circ \Psi_1$, $\mathrm{inc} \circ \Psi_2$, these correspond to two combinatorial Hopf monoid morphisms $\mathtt{h}' \Rightarrow \mathtt{WQSym}$, so they must coincide in accordance with [AM10, Theorem 11.23]. Thus, Ψ_1, Ψ_2 must agree on $\mathtt{h}[I]$ for any non-empty set I.

5.5 Generalized permutahedra and posets

In the following, we see that the universal map that we constructed above in Theorem 5.4.1 is well behaved with respect to combinatorial Hopf monoid morphisms. This is in fact a classical property of terminal objects in any category.

Lemma 5.5.1. If $\phi : \mathtt{h}_1 \Rightarrow \mathtt{h}_2$ is a combinatorial Hopf monoid morphism between two Hopf monoids with characters η_1 and η_2 respectively, then the following diagram commutes

$$\begin{array}{ccc} \mathtt{h}_1 & \xrightarrow{\ \phi\ } & \mathtt{h}_2 \\ & {\scriptstyle \boldsymbol{\Psi}_{\mathtt{h}_1}} \searrow \quad \swarrow {\scriptstyle \boldsymbol{\Psi}_{\mathtt{h}_2}} & \\ & \mathtt{WQSym} & \end{array} \tag{5.24}$$

Proof. It is a direct observation that the composition of combinatorial Hopf monoids morphism is still a combinatorial Hopf monoid morphism. Hence, we have that $\boldsymbol{\Upsilon}_{\mathtt{h}_2} \circ \phi : \mathtt{h}_1 \to \mathtt{WQSym}$ is a combinatorial Hopf monoid morphism. By uniqueness it is $\boldsymbol{\Upsilon}_{\mathtt{h}_2} \circ \phi = \boldsymbol{\Upsilon}_{\mathtt{h}_1}$. □

Corollary 5.5.2. There are no Hopf monoid morphisms $\phi : \mathtt{Pos} \to \mathtt{GP}$ that preserve the corresponding characters.

Proof. For sake of contradiction, suppose that such ϕ exists, hence it satisfies $\mathcal{K}(\boldsymbol{\Upsilon}_{\mathtt{GP}}) \circ \mathcal{K}(\phi) = \mathcal{K}(\boldsymbol{\Upsilon}_{\mathtt{Pos}})$, according to Lemma 5.5.1, so

$$\boldsymbol{\Upsilon}_{\mathbf{Pos}} = \boldsymbol{\Upsilon}_{\mathbf{GP}} \circ \mathcal{K}(\phi) \, .$$

However, $\boldsymbol{\Upsilon}_{\mathbf{Pos}}$ is surjective, whereas we have seen in Theorem 4.5.1 that $\boldsymbol{\Upsilon}_{\mathbf{GP}}$ is not surjective. This is the desired contradiction. □

Chapter 6

The feasible region for consecutive patterns of permutations is a cycle polytope

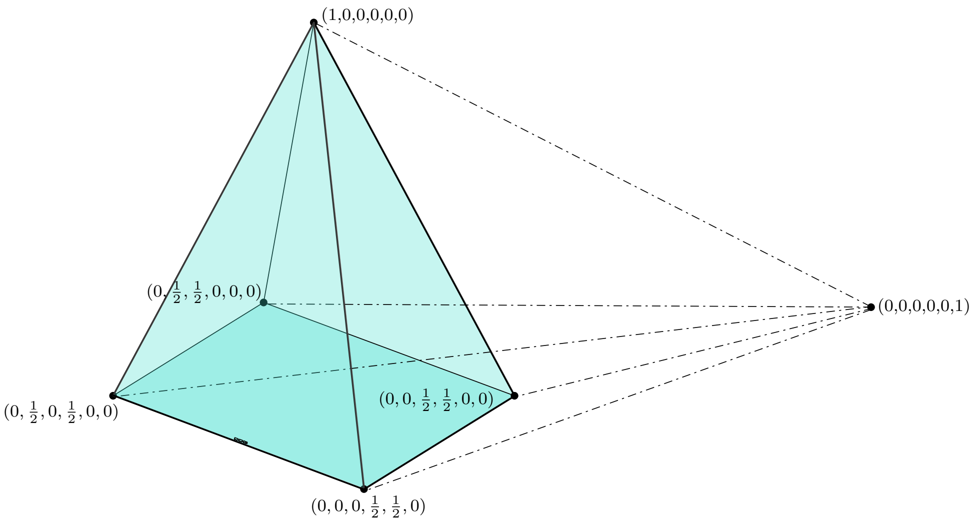


FIGURE 6.1: The four-dimensional polytope P_3 given by the six patterns of size three (see Eq. (6.2) for a precise definition). We highlight in light-blue one of the six three-dimensional facets of P_3. This facet is a pyramid with square base. The polytope itself is a four-dimensional pyramid, whose base is the highlighted facet.

This chapter is a joint work with Jacopo Borga, based on the article [BP19] submitted for publication. A short version of this chapter is accepted for publication in the proceedings of *Formal Power Series and Algebraic Combinatorics* (poster presentation).

6.1 Introduction

There are two central notions of patterns in permutations that we wish to study in this chapter: the notion of classical pattern and the notion of consecutive pattern. Classical patterns match the definition of pattern in the presheaf of permutations given in Definition 1.5.3, while for the consecutive patterns we require the occurrence to be an interval.

We will consider a *feasible regions* for each of these types of patterns. The feasible region for classical patterns has been already studied in the literature, whereas the feasible region for consecutive patterns is new. Both were introduced in Chapter 1 and we recall them now.

We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}$ the space of all permutations, and by $\widetilde{\text{occ}}(\pi, \sigma)$ (resp. $\widetilde{\text{c-occ}}(\pi, \sigma)$) the proportion of classical occurrences (resp. consecutive occurrences) of a permutation π in σ (see Section 6.1.7 for notation and basic definitions).

We consider the classical pattern limiting sets, sometimes called the *feasible region* for (classical) patterns, defined as

$$clP_k := \{\vec{v} \in [0,1]^{\mathcal{S}_k} | \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\}, \tag{6.1}$$

and we introduce the consecutive pattern limiting sets, called here the *feasible region* for consecutive patterns,

$$P_k := \{\vec{v} \in [0,1]^{\mathcal{S}_k} | \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\}. \tag{6.2}$$

We will present some known facts regarding clP_k in Section 6.1.1. The main results on this chapter relate to P_k, the feasible region for consecutive patterns. Specifically, we will describe P_k as a cycle polytope of an explicit graph in Section 6.3, and furthermore detail on its face structure in Theorem 6.2.12. Lastly, we will attempt to describe the feasible region when we mix classical and consecutive patterns.

6.1.1 The feasible region for classical patterns

The feasible region clP_k was first studied in [KKRW15] with a different technical definition, for a generic family of patterns instead of the whole $\mathcal{S}_k$. More precisely, given a list of finite sets of permutations $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$, the authors considered the *feasible region* for $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$, that is, the set

$$\left\{ \vec{v} \in [0,1]^\ell \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty, \sum_{\tau \in \mathcal{P}_i} \widetilde{\text{occ}}(\tau, \sigma^m) \to \vec{v}_i, \text{ for } i = 1, \ldots, \ell \right\}.$$

They first studied the simplest case when $\mathcal{P}_1 = \{12\}$ and $\mathcal{P}_2 = \{123, 213\}$ showing that the corresponding feasible region for $(\mathcal{P}_1, \mathcal{P}_2)$ is the region of the square $[0,1]^2$ bounded from below by the parameterized curve $(2t - t^2, 3t^2 - 2t^3)_{t\in[0,1]}$ and from above by the parameterized curve $(1 - t^2, 1 - t^3)_{t\in[0,1]}$ (see [KKRW15, Theorem 13]).

They also proved in [KKRW15, Theorem 14] that if each $\mathcal{P}_i = \{\tau_i\}$ is a singleton, and there is some value p such that, for all permutations τ_i, the final element $\tau_i(|\tau_i|)$ is equal to p , then the corresponding feasible region is convex. They remarked that one can construct examples where the feasible region is not strictly convex: e.g. in the case where $\mathcal{P}_1 = \{231, 321\}$ and $\mathcal{P}_2 = \{123, 213\}$.

They finally studied two additional examples: the feasible regions for $(\{12\}, \{123\})$ (see [KKRW15, Theorem 15]) and for the patterns $(\{123\}, \{321\})$ (see [KKRW15, Section 10]). In the first case, they showed that the feasible region is equal to the so-called "scalloped triangle" of Razborov [Raz08, Raz07] (this region also describes the space of limit densities for edges and triangles in graphs). For the second case, they showed that the feasible region is equal to the limit of densities of triangles versus the density of anti-triangles in graphs, see [HLN+14, HLN+16].

The set clP_k was also studied in [GHK+17], even though with a different goal. There, it was shown that clP_k contains an open ball B with dimension $|I_k|$, where I_k is the set of $\oplus$-indecomposable permutations of size at most k. Specifically, for a particular ball $B \subseteq \mathbb{R}^{I_k}$, the authors constructed permutons $P_{\vec{x}}$ such that $\Delta_\pi(P_{\vec{x}}) = \vec{x}_\pi$, for each point $\vec{x} \in B$.

This work opened the problem of finding the maximal dimension of an open ball contained in clP_k, and placed a lower bound on it. In [Var14] an upper bound for this maximal dimension was indirectly given as the number of so-called *Lyndon permutations* of size at most k, whose set we denote $\mathcal{L}_k$. In this article, the author showed that for any permutation π that is not a Lyndon permutation, $\widetilde{\text{occ}}(\pi, \sigma)$ can be expressed as a polynomial on the functions $\{\widetilde{\text{occ}}(\tau, \sigma) | \tau \in \mathcal{L}_k\}$ that does not depend on σ. It follows that clP_k sits inside an algebraic variety of dimension $|\mathcal{L}_k|$. We expect that this bound is sharp since, as referred in Conjecture 1.7.1.

6.1.2 First main result

Unlike with the case of classical patterns, we are able to obtain here a full description of the feasible region P_k as the cycle polytope of a specific graph, called the *overlap graph* $\mathcal{O}v(k)$.

The graph $\mathcal{O}v(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for every $\pi \in \mathcal{S}_k$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

The overlap graph $\mathcal{O}v(4)$ is displayed in Fig. 6.2.

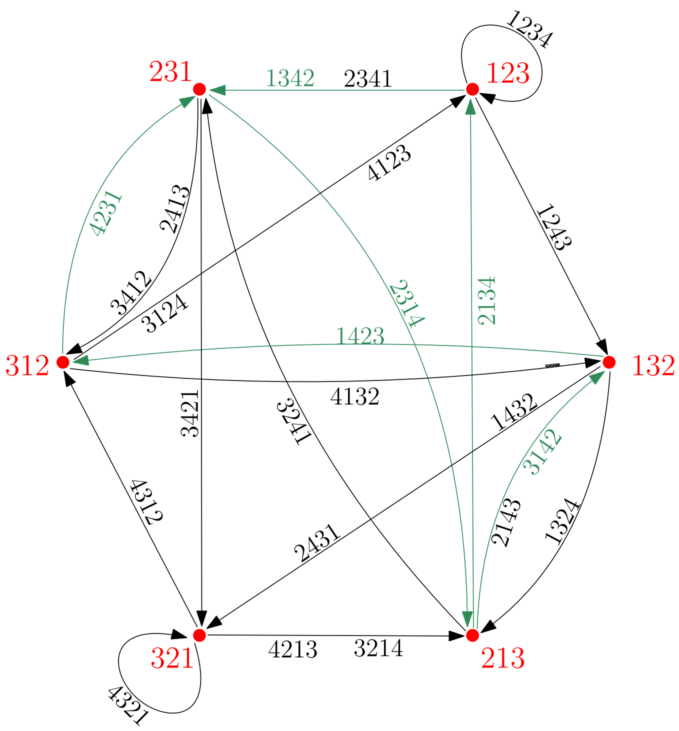


FIGURE 6.2: The overlap graph $\mathcal{O}v(4)$. The six vertices are painted in red and the edges are drawn as labeled arrows. Note that in order to obtain a clearer picture we did not draw multiple edges, but we use multiple labels (for example the edge $231 \to 312$ is labeled with the permutations 3412 and 2413 and should be thought of as two distinct edges labeled with 3412 and 2413 respectively). The role of the green arrows is clarified in Example 6.3.7.

Our first main result is the following.

Theorem 6.1.1. P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. Its dimension is $k! - (k-1)!$ and its vertices are given by the simple cycles of $\mathcal{O}v(k)$.

In addition, we also determine the equations that describe the polytope P_k (for a precise statement see Theorem 6.3.12).

In order to establish the dimension, the vertices and the equations describing P_k, we first prove general results for cycle polytopes of directed multigraphs (see Section 6.1.4) and then we transfer them to the specific case of our graph of interest.

6.1.3 The overlap graph

Overlap graphs were already studied in previous works. We give here a brief summary of the relevant literature. The overlap graph $\mathcal{O}v(k)$ is the line graph of the *de Bruijn graph for permutations* of size $k-1$. The latter was introduced in [CDG92], where the authors studied universal cycles (sometime also called *de Bruijn cycles*) of several combinatorial structures, including permutations. In this case, a universal cycle of order n is a cyclic word of size $n!$ on an alphabet of N letters that contains all the patterns of size n as consecutive patterns. In [CDG92] it was conjectured (and then proved in [Joh09]) that such universal cycles always exist when the alphabet is of size $N = n + 1$.

The *de Bruijn graph for permutations* was also studied under the name of *graph of overlapping permutations* in [KPV19] again in relation with universal cycles and universal words. Further, in [AF18] the authors enumerate some families of cycles of these graphs.

We mainly use overlap graphs as a tool to state and prove our results on P_k, rather than exploiting its properties. We remark that, applying the same ideas used to show the existence of Eulerian and Hamiltonian cycles in classical de Bruijn graphs (see [CDG92]), it is easy to prove the existence of both Eulerian and Hamiltonian cycles in $\mathcal{O}v(k)$. In particular, with an Eulerian path in $\mathcal{O}v(k)$, we can construct (although not uniquely) a permutation σ of size $k! + k - 1$ such that $\text{c-occ}(\pi, \sigma) = 1$ for any $\pi \in \mathcal{S}_k$.

6.1.4 Polytopes and cycle polytopes

As said before, we obtain general results for cycle polytopes of directed multigraphs:

Theorem 6.1.2. The cycle polytope of a strongly connected directed multigraph $G = (V, E)$ has dimension $|E| - |V|$.

We also determine the equations defining the polytope (see Theorem 6.2.12) and we show that all its faces can be identified with some subgraphs of G (see Theorem 6.2.13). This gives us a description of the face poset of the polytope. Further, the computation of the dimension is generalized for any cycle polytope, even those that do not come from strongly connected graphs (see Theorem 6.2.11).

Some weaker versions of our results already appeared in the literature. Polytopes similar to the cycle polytopes studied here, called *unrescaled cycle polytopes* (*U-cycle polytopes* for short), were introduced in [BO00] in the directed version and in [CP89] in the undirected version[1]. Balas & Oosten [BO00] and Balas & Stephan [BS09] computed the dimension of the U-cycle polytope of the *complete graph* (that is, the complete directed graph without loops) and described the facets of the corresponding polytope. Notice that we study cycle polytopes for general directed multigraphs and we do not restrict to the case of complete graphs as in [BO00, BS09].

The U-cycle polytopes for undirected graphs were initially considered to tackle the Simple Cycle Problem (SCP) [GP02], that also goes by the name of Weighted Girth Problem [Bau97]. This problem consists in finding a minimum weighted simple cycle in an undirected graph with costs associated with each edge. The related decision problem is known to be NP-hard, as it can be reduced to the "traveling salesman" problem (TSP), that asks the following question: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?". The Simple Cycle Problem was also considered later in [LdCS13].

Remark 6.1.3. In [BO00, Proposition 4], the dimension of the U-cycle polytope for the complete graph on n vertices without loops is computed as $(n-1)^2$. We point out that in Theorem 6.1.2 we compute the dimension of the cycle polytope of the complete graph as $(n^2-n)-n=(n-1)^2-1$. This is coherent with the previous result, because a cycle polytope has an extra equation given by $\sum_e x_e = 1$ when compared with its corresponding U-cycle polytope.

We point out that other instances of polytopes related to paths in graphs were also investigated. For instance, there is a path version of U-cycle polytopes, considered in [Ste09]. Specifically, the (s,t)-p-path polytope of a directed graph G is the convex hull of the incidence vectors of simple directed (s,t)-paths in G of size p. There, the authors gave some characterizations of the facets of the path polytopes. More concerning this polytope can be found, for instance, in [DG04, DR00].

6.1.5 Mixing classical patterns and consecutive patterns

We saw in Section 6.1.1 that the feasible region clP_k for classical pattern occurrences has been studied in several papers. In this chapter we study the feasible region P_k of limiting points for consecutive pattern occurrences. A natural question is the following: what is the feasible region if we mix classical and consecutive patterns?

[1]The cycle polytopes introduced in this chapter are intrinsically related to the U-cycle polytopes. The vertices of the U-cycle polytope of a directed multigraph G are defined as the incidence vectors of simple cycles of G. We *additionally rescale* each of the vertices so that the coordinates sum up to one. The U-cycle polytopes were considered in the literature simply under the name of *cycle polytopes*. We adapt the name of *cycle polytopes* to our family of polytopes for the sake of simplifying the terminology in this chapter.

We answer this question showing that:

Theorem 6.1.4. For any two points $\vec{v}_1 \in clP_k$ and $\vec{v}_2 \in P_k$, there exists a sequence of permutations $(\sigma^m)_{m\in\mathbb{N}}$ such that $|\sigma^m| \to \infty$, satisfying

$$(\widetilde{\text{occ}}(\pi, \sigma^m))_{\pi\in\mathcal{S}_k} \to \vec{v}_1 \quad \text{and} \quad (\widetilde{\text{c-occ}}(\pi, \sigma^m))_{\pi\in\mathcal{S}_k} \to \vec{v}_2.$$

This result shows a sort of independence between classical patterns and consecutive patterns, in the sense that knowing the proportion of classical patterns of a certain sequence of permutations gives no constraints for the proportion of consecutive patterns of the same sequence and *vice versa*.

We stress that we provide an explicit construction of the sequence $(\sigma^m)_{m\in\mathbb{N}}$ in the theorem above (for a more precise and general statement, see Theorem 6.4.1).

We conclude this section with the following observation on local and scaling limits of permutations.

Observation 6.1.5. In Theorems 1.7.2 and 1.7.3 we saw that the proportion of occurrences (resp. consecutive occurrences) in a sequence of permutations $(\sigma^m)_{m\in\mathbb{N}}$ characterizes the permuton limit (resp. Benjamini–Schramm limit) of the sequence. Theorem 6.1.4 proves that the permuton limit of a sequence of permutations induces no constraints for the Benjamini–Schramm limit and *vice versa*. For instance, we can construct a sequence of permutations where the permuton limit is the decreasing diagonal and the Benjamini–Schramm limit is the classical increasing total order on the integer numbers.

We remark that a particular instance of this "independence phenomenon" for local/scaling limits of permutations was recently also observed by Bevan, who pointed out in the abstract of [Bev19] that "the knowledge of the local structure of uniformly random permutations with a specific fixed proportion of inversions reveals nothing about their global form". Here, we prove that this is a *universal phenomenon* which is not specific to the framework studied by Bevan.

6.1.6 Outline of the chapter

This chapter is organized as follows:

- In Section 6.2 we analyze directed multigraphs and consider their *cycle polytopes*. There, we prove Theorem 6.1.2 and the results mentioned immediately below it.
- Our results regarding P_k come in Section 6.3, where we prove Theorem 6.1.1.

- Finally, we prove in Section 6.4 a more precise version of Theorem 6.1.4.

6.1.7 Notation

We summarize here the notation and some basic definitions used in this chapter.

Permutations and patterns

For every $n \in \mathbb{N}$, we view permutations of $[n] = \{1, 2, \dots, n\}$ as words of size n, and write them using the one-line notation $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$. We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}_{\geq n}$ the set of permutations of size at least n, and by $\mathcal{S}$ the set of permutations of finite size.

We often view a permutation $\sigma \in \mathcal{S}_n$ as a diagram, specifically as an $n \times n$ board with n points at positions $(i, \sigma(i))$ for all $i \leq n$.

If $x_1, \dots, x_n$ is a sequence of distinct numbers, let $\text{std}(x_1, \dots, x_n)$ be the unique permutation π in $\mathcal{S}_n$ whose elements are in the same relative order as $x_1, \dots, x_n$, *i.e.*, $\pi(i) < \pi(j)$ if and only if $x_i < x_j$. Given a permutation $\sigma \in \mathcal{S}_n$ and a subset of indices $I \subseteq [n]$, let $\sigma|_I$ be the permutation induced by $(\sigma(i))_{i \in I}$, namely, $\sigma|_I := \text{std}\left((\sigma(i))_{i \in I}\right)$. For example, if $\sigma = 87532461$ and $I = \{2, 4, 7\}$, then $87532461|_{\{2,4,7\}} = \text{std}(736) = 312$.

Given two permutations, $\sigma \in \mathcal{S}_n$, $\pi \in \mathcal{S}_k$ for some positive integers $n \geq k$, we say that σ contains π as a *pattern* if there exists a *subset* $I \subseteq [n]$ such that $\sigma|_I = \pi$, that is, if σ has a subsequence of entries order-isomorphic to π. Denoting by $i_1, i_2, \dots, i_k$ the elements of I in increasing order, the subsequence $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ is called an *occurrence* of π in σ. In addition, we say that σ contains π as a *consecutive pattern* if there exists an *interval* $I \subseteq [n]$ such that $\sigma|_I = \pi$, that is, if σ has a subsequence of adjacent entries order-isomorphic to π. Using the same notation as above, $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ is then called a *consecutive occurrence* of π in σ.

Recall that we denote by $\mathbf{p}_\pi(\sigma)$ the number of occurrences of a pattern π in σ, more precisely

$$\mathbf{p}_\pi(\sigma) := \left|\left\{I \subseteq [n] \text{ s.t. } \sigma|_I = \pi\right\}\right|.$$

We denote by $\text{c-occ}(\pi, \sigma)$ the number of consecutive occurrences of a pattern π in σ, more precisely

$$\text{c-occ}(\pi, \sigma) := \left|\left\{I \subseteq [n] | \; I \text{ is an interval, } \sigma|_I = \pi\right\}\right|.$$

Moreover, we denote by $\widetilde{\text{occ}}(\pi,\sigma)$ (resp. by $\widetilde{\text{c-occ}}(\pi,\sigma)$) the proportion of occurrences (resp. consecutive occurrences) of a pattern π in σ, that is,

$$\widetilde{\text{occ}}(\pi,\sigma) \coloneqq \frac{\mathbf{p}_\pi(\sigma)}{\binom{n}{k}} \in [0,1], \qquad \widetilde{\text{c-occ}}(\pi,\sigma) \coloneqq \frac{\text{c-occ}(\pi,\sigma)}{n} \in [0,1]\,.$$

For a fixed $k \in \mathbb{N}$ and a permutation $\sigma \in \mathcal{S}_{\geq k}$, we let $\widetilde{\text{occ}}_k(\sigma), \widetilde{\text{c-occ}}_k(\sigma) \in [0,1]^{\mathcal{S}_k}$ be the vectors

$$\widetilde{\text{occ}}_k(\sigma) \coloneqq \left(\widetilde{\text{occ}}(\pi,\sigma)\right)_{\pi\in\mathcal{S}_k}, \qquad \widetilde{\text{c-occ}}_k(\sigma) \coloneqq \left(\widetilde{\text{c-occ}}(\pi,\sigma)\right)_{\pi\in\mathcal{S}_k}.$$

Finally, we denote with $\oplus$ the direct sum of two permutations, *i.e.*, for $\tau \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$,

$$\tau \oplus \sigma = \tau(1)\dots\tau(k)(\sigma(1)+m)\dots(\sigma(n)+m)\,,$$

and we denote with $\sigma^{\oplus \ell}$ the direct sum of ℓ copies of σ.

Polytopes

Given a set $S \subseteq \mathbb{R}^n$, we define the *convex hull* (resp. *affine span*, *linear span*) of S as the set of all *convex combinations* (resp. *affine combinations*, *linear combinations*) of points in S, that is,

$$\text{conv}(S) = \left\{ \sum_{i=1}^k \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in [0,1] \text{ for } i = 1,\dots k\,, \sum_{i=1}^k \alpha_i = 1 \right\},$$

$$\text{Aff}(S) = \left\{ \sum_{i=1}^k \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \text{ for } i = 1,\dots k\,, \sum_{i=1}^k \alpha_i = 1 \right\},$$

$$\text{span}(S) = \left\{ \sum_{i=1}^k \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \text{ for } i = 1,\dots k \right\}.$$

Definition 6.1.6 (Polytope)**.** A polytope $\mathfrak{p}$ in $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ described by m linear inequalities. That is, there is some $m \times n$ real matrix A and a vector $b \in \mathbb{R}^m$ such that

$$\mathfrak{p} = \{\vec{x} \in \mathbb{R}^n | A\vec{x} \geq b\}\,.$$

The dimension of a polytope $\mathfrak{p}$ in $\mathbb{R}^n$ is the dimension of $\text{Aff}\,\mathfrak{p}$ as an affine space, and we denote it as $\dim \mathfrak{p}$.

For any polytope, there is a unique minimal finite set of points $\mathcal{P} \subset \mathbb{R}^n$ such that $\mathfrak{p} = \text{conv}\,\mathcal{P}$, see [Zie12]. This family $\mathcal{P}$ is called the set of *vertices* of $\mathfrak{p}$.

Definition 6.1.7 (Faces of a polytope)**.** Let $\mathfrak{p}$ be a polytope in $\mathbb{R}^n$. A linear form in $\mathbb{R}^n$ is a linear map $f : \mathbb{R}^n \to \mathbb{R}$. Its minimizing set on $\mathfrak{p}$ is the subset $\mathfrak{p}_f \subseteq \mathfrak{p}$ where f takes minimal values. This set always exists because $\mathfrak{p}$ is compact.

A face of $\mathfrak{p}$ is a subset $\mathfrak{f} \subseteq \mathfrak{p}$ for which there exists a linear form f that satisfies $\mathfrak{p}_f = \mathfrak{f}$. A face is also a polytope, and any vertex of $\mathfrak{p}$ is a face of $\mathfrak{p}$. The faces of a polytope $\mathfrak{p}$ form a poset when ordered by inclusion, called the *face poset*.

We observe that the vertices of a polytope are exactly the singletons that are faces.

Remark 6.1.8 (Computing faces and vertices of a polytope)**.** If f is a linear form in $\mathbb{R}^n$ and $\mathfrak{p} = \operatorname{conv} A \subseteq \mathbb{R}^n$, then

$$\mathfrak{p}_f = \operatorname{conv}\{\arg\min_{a\in A} f(a)\} = \arg\min_{a\in\mathfrak{p}} f(a)\,. \tag{6.3}$$

In particular, the vertices V of $\operatorname{conv} A$ satisfy $V \subseteq A$. Also, when computing $\mathfrak{p}_f$, it suffices to evaluate f on A.

Directed graphs

All graphs, their subgraphs and their subtrees are considered to be directed multigraphs in this chapter (and we often refer to them as directed graphs or simply as graphs). In a directed multigraph $G = (V(G), E(G))$, the set of edges $E(G)$ is a multiset, allowing for loops and parallel edges. An edge $e \in E(G)$ is an oriented pair of vertices, (v, u), often denoted by $e = v \to u$. We write $\mathrm{s}(e)$ for the starting vertex v and $\mathrm{a}(e)$ for the arrival vertex u. We often consider directed graphs G with labeled edges, and write $\mathrm{lb}(e)$ for the label of the edge $e \in E(G)$. In a graph with labeled edges we refer to edges by using their labels. Given an edge $e = v \to u \in E(G)$, we denote by $N_G(e)$ the neighboring edges, that is set of edges $e' \in E(G)$ such that $e' = u \to w$ for some $w \in V(G)$, *i.e.*, $N_G(e) = \{e' \in E(G) |\, \mathrm{s}(e') = \mathrm{a}(e)\}$.

A *walk* of size k on a directed graph G is a sequence of k edges $(e_1, \dots, e_k) \in E(G)^k$ such that for all $i \in [k-1]$, $\mathrm{a}(e_i) = \mathrm{s}(e_{i+1})$. A walk is a *cycle* if $\mathrm{s}(e_1) = \mathrm{a}(e_k)$. A walk is a *path* if all the edges are distinct, as well as its vertices, with a possible exception that $\mathrm{s}(e_1) = \mathrm{a}(e_k)$ may happen. A cycle that is a path is called a *simple cycle*. Given two walks $w = (e_1, \dots, e_k)$ and $w' = (e'_1, \dots, e'_{k'})$ such that $\mathrm{a}(e_k) = \mathrm{s}(e'_1)$, we write $w \bullet w'$ for the concatenation of the two walks, *i.e.*, $w \bullet w' = (e_1, \dots, e_k, e'_1, \dots, e'_{k'})$. For a walk w, we denote by $|w|$ the number of edges in w.

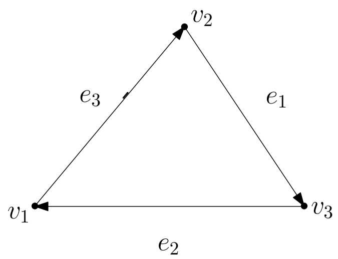
Given a walk $w = (e_1, \dots, e_k)$ and an edge e, we denote by $n_e(w)$ the number of times the edge e is traversed in w, *i.e.*, $n_e(w) := |\{i \leq k | e_i = e\}|$.

For a vertex v in a directed graph G, we define $\deg_G^i(v)$ to be the number of incoming edges to v, *i.e.*, edges $e \in E(G)$ such that $\mathrm{a}(e) = v$, and $\deg_G^o(v)$ to be the number of outgoing edges in v, *i.e.*, edges $e \in E(G)$ such that $\mathrm{s}(e) = v$. Whenever it is clear from the context, we drop the subscript G. A vertex v that satisfies $\deg^i(v) = 0$ is called a *source*.

The incidence matrix of a directed graph G is the matrix $L(G)$ with rows indexed by $V(G)$, and columns indexed by $E(G)$, such that for any edge $e = v \to u$, the corresponding column in $L(G)$ has $(L(G))_{v,e} = -1$, $(L(G)))_{u,e} = 1$ and is zero everywhere else.

For instance, we show in Fig. 6.3 a graph G with its incidence matrix $L(G)$.

$G =$



$$, \qquad L(G) = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \ .$$

FIGURE 6.3: A graph G with its incidence matrix $L(G)$.

A directed graph G is said to be *strongly connected* if for any two vertices $v_1, v_2 \in V(G)$, there is a path starting in v_1 and arriving in v_2. For instance, the graph in Fig. 6.3 is strongly connected.

For a graph G with a distinguished vertex r, we say that T is a *rooted spanning tree* with root r if T is tree with $T \subseteq G$ such that $V(T) = V(G)$ and any edge of T is directed away from the root.

6.2 The cycle polytope of a graph

In this section we establish general results about the cycle polytope of a graph. Here, all graphs are considered to be directed multigraphs that may have loops and parallel edges, unless stated otherwise. We recall the definition of cycle polytope.

We set, for each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^{E(G)}$ so that

$$(\vec{e}_{\mathcal{C}})_e := \frac{n_e(\mathcal{C})}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E(G),$$

and the *cycle polytope* of G as $P(G) := \text{conv}\{\vec{e}_{\mathcal{C}} | \mathcal{C} \text{ is a simple cycle of } G\}$.

6.2.1 Vertices of the cycle polytope

We start by giving a full description of the vertices of this polytope.

Proposition 6.2.1. The vertices of $P(G)$ are precisely $\{\vec{e}_{\mathcal{C}} | \mathcal{C} \text{ is a simple cycle of } G\}$.

Proof. We only need to show that any point of the form $\vec{e}_{\mathcal{C}}$ is indeed a vertex. Consider now a simple cycle $\mathcal{C}$, and recall that vertices of a polytope are characterized by being the only singletons that are faces. Define:

$$f_{\mathcal{C}}(\vec{x}) := -\sum_{e\in\mathcal{C}} x_e, \quad \text{for all} \quad \vec{x} = (x_e)_{e\in E(G)} \in \mathbb{R}^{E(G)},$$

where we identify $\mathcal{C}$ with the set of edges in $\mathcal{C}$. We will show that $P(G)_{f_{\mathcal{C}}} = \{\vec{e}_{\mathcal{C}}\}$. That is, that $\vec{e}_{\mathcal{C}}$ is the unique minimizer of $f_{\mathcal{C}}$ in $P(G)$, concluding the proof.

It is easy to check that $f_{\mathcal{C}}(\vec{e}_{\mathcal{C}}) = -1$. From Eq. (6.3), we only need to establish that any simple cycle $\tilde{\mathcal{C}}$ that satisfies $f_{\mathcal{C}}(\vec{e}_{\tilde{\mathcal{C}}}) \leq -1$ is equal to $\mathcal{C}$. Take a generic simple cycle $\tilde{\mathcal{C}}$ in G such that $f_{\mathcal{C}}(\vec{e}_{\tilde{\mathcal{C}}}) \leq -1$. Then, $\sum_{e\in\mathcal{C}}(\vec{e}_{\tilde{\mathcal{C}}})_e \geq 1$. Since $\vec{e}_{\tilde{\mathcal{C}}}$ satisfies the equation $\sum_{e\in E(G)} (\vec{e}_{\tilde{\mathcal{C}}})_e = 1$ and has non-negative coordinates, we must have that $(\vec{e}_{\tilde{\mathcal{C}}})_e = 0$ for all $e \notin \mathcal{C}$. Thus $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ as sets of edges. However, because both $\tilde{\mathcal{C}}, \mathcal{C}$ are simple cycles, we conclude that $\mathcal{C} = \tilde{\mathcal{C}}$, as desired. □

6.2.2 Dimension of the cycle polytope

The goal of this section is to prove the following result.

Theorem 6.2.2 (Dimension of the cycle polytope)**.** If G is a strongly connected graph, then the cycle polytope of G has dimension $|E(G)| - |V(G)|$.

To compute the dimension of the polytope $P(G)$ we start by finding some linear relations that are satisfied in $P(G)$ and that define an affine space of dimension $|E(G)| - |V(G)|$ (see Lemma 6.2.4). This gives us an upper bound on the dimension of $P(G)$. For the lower bound, we first assume that the graph G has a loop lp. We find a rooted spanning tree T of G, and construct $|E(G)| - |V(G)|$ many distinct points $\vec{v}^{(e)}$ (indexed by $E(G) \setminus (E(T) \cup \{lp\})$) in a suitable translation of $P(G)$. Finally we observe that the set

$$\{\vec{v}^{(e)} | e \in E(G) \setminus (E(T) \cup \{lp\})\},$$

is linearly independent. Finally, we reduce the problem on loopless graphs G to the remaining ones.

The construction of the rooted spanning tree is done in Lemma 6.2.3, the construction of $\vec{v}^{(e)}$ is done with the help of Lemma 6.2.5, and the desired linear independence is proved in Lemma 6.2.9. In Theorem 6.2.11, we establish a generalization of this theorem, where we compute the dimension of $P(G)$ for any graph G, not only the ones that are strongly connected.

Lemma 6.2.3. Let G be a directed graph that is strongly connected, and r be a vertex of G. Then, there exists a rooted spanning tree T of G with root r.

Proof. We construct the desired tree T algorithmically. Start with a tree T with only one vertex r and no edges. Order the vertices in $V(G) \setminus \{r\}$ and successively for each v, consider the shortest path P from a vertex of the tree T to v in G. This exists because G is strongly connected and it has at most one vertex in common with T.

After going through all vertices, we obtain a spanning tree of G. Further, it is easy to see that all the edges are oriented away from r at every step of the algorithm. So this is a rooted spanning tree, as desired. □

In what follows we assume that we have a strongly connected graph G with at least one loop. We fix a particular loop lp in G, and a spanning rooted tree T with a root r, which exists by Lemma 6.2.3 above.

Lemma 6.2.4. The points $\vec{x} \in P(G)$ satisfy the following relations:

$$\sum_{e \in E(G)} x_e = 1 \,, \tag{6.4}$$

$$\sum_{\mathrm{s}(e)=v} x_e = \sum_{\mathrm{a}(e)=v} x_e \,, \forall v \in V(G) \,. \tag{6.5}$$

Moreover, these equations define an affine space with dimension $|E(G)| - |V(G)|$.

Proof. Because these equations are linear, to observe that any $\vec{x} \in P(G)$ satisfies Eqs. (6.4) and (6.5) we only need to show that the vertices $\{\vec{e}_{\mathcal{C}} | \mathcal{C}$ is a simple cycle of $G\}$ satisfy them, which is trivial.

Thus, the claim is proven once we establish the dimension of the affine space. Let A be the matrix with rows indexed by $\{\triangleleft\} \cup V(G)$, where $\triangleleft$ is a formal symbol, and columns indexed by $E(G)$ defined as

$$A_{\lhd,e} = 1 \qquad A_{v,e} = \begin{cases} -1, & \text{if } \mathrm{s}(e) = v,\ \mathrm{a}(e) \neq v, \\ 1, & \text{if } \mathrm{s}(e) \neq v,\ \mathrm{a}(e) = v, \\ 0, & \text{otherwise}, \end{cases}$$

for any vertex v in $V(G)$, and any edge e in $E(G)$.

Then, Eqs. (6.4) and (6.5) are equivalent to

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We want to show that this equation defines an affine space with dimension $|E(G)| - |V(G)|$. First, we observe that this system has a non-empty set of solutions. For instance $\vec{e}_{lp}$ satisfies Eqs. (6.4) and (6.5). Hence, it suffices to show that $\operatorname{rank}(A) = |V(G)|$. We claim that $\operatorname{rank}(A) \leq |V(G)|$. Indeed

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \end{bmatrix} A = \vec{0},$$

and by the rank nullity theorem on A^T, $\operatorname{rank}(A) + 1 \leq \operatorname{rank}(A^T) + \dim\ker(A^T) = |V(G)| + 1$. Then, the result is established if we find a non-singular $|V(G)| \times |V(G)|$ minor of A.

Let $V' = V(G) \setminus \{r\}$, where r is the root of the tree T in G, and consider the minor M given by the rows indexed with $\{\lhd\} \cup V'$ and the columns indexed with $\{lp\} \cup E(T)$. We denote by $L'(H)$ the incidence matrix of a subgraph H of G, with one row (corresponding to r) removed. We define

$$M := \begin{array}{c} \begin{array}{cc} lp & \qquad E(T) \end{array} \\ \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & L'(T) & \\ 0 & & & \end{bmatrix} \end{array} \begin{array}{c} \lhd \\ \\ V' \\ \\ \end{array}.$$

Note that because T is a spanning tree, $|E(T) \cup \{lp\}| = |V(G)| = |\{\lhd\} \cup V'|$, and so M is a square matrix. Observe that M is non-singular whenever $L'(T)$ is non-singular.

Then, it suffices to establish that $L'(T)$ is non-singular. We proceed by induction on the size of T.

The base case is when the tree T has one vertex. Then, $L'(T)$ is the empty matrix, which is by convention non-singular. For the induction step, consider a leaf w of T. We reorder the rows and columns of $L'(T)$, so that the column corresponding to the edge e incident to w is the leftmost one, and the row corresponding to the leaf w is the uppermost one. Then we have

$$L'(T) = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{cc} e & E(T') \\ \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & L'(T') & \\ * & & & \end{array}\right] & \begin{array}{c} w \\ V(T') \end{array} \end{array},$$

where T' is the tree corresponding to T after deleting the vertex w, along with its incident edge e. By induction hypothesis, this is a non-singular matrix, and that completes the proof. □

Lemma 6.2.5. Let $e \in E(G) \setminus (E(T) \cup \{lp\})$ be an edge in G. Then there are two non-empty cycles $\mathcal{C}_1^{(e)}, \mathcal{C}_2^{(e)}$ such that $e \in \mathcal{C}_1^{(e)}$, $e \notin \mathcal{C}_2^{(e)}$ and $\{f \in E(G) | n_f(\mathcal{C}_1^{(e)}) \neq n_f(\mathcal{C}_2^{(e)})\} \subseteq E(T) \cup \{e, lp\}$.

Recall that we denote the concatenation of walks by $\bullet$.

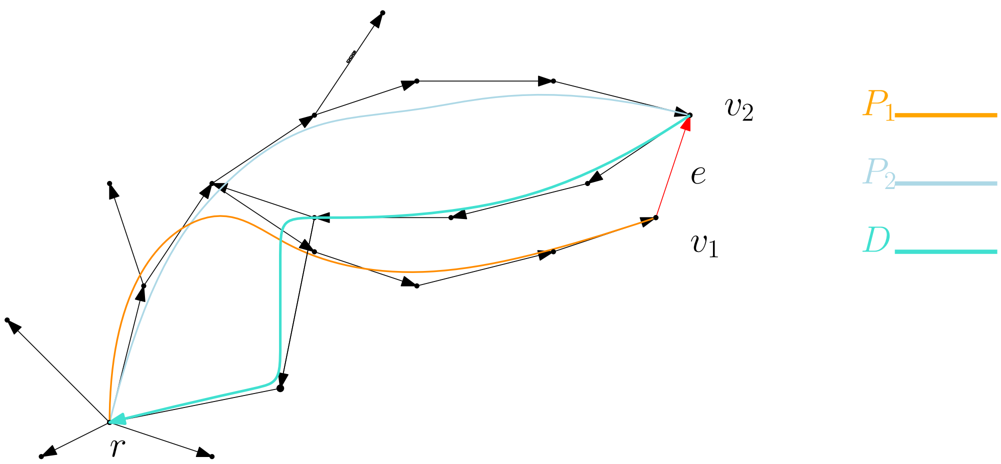


FIGURE 6.4: The construction of the cycles $\mathcal{C}_1^{(e)}, \mathcal{C}_2^{(e)}$.

Proof. Let $v_1 = \mathrm{s}(e)$, $v_2 = \mathrm{a}(e)$ and recall that r is the root of the rooted spanning tree T. We suggest to compare what follows with Fig. 6.4. We can find a path P_1 (resp. P_2)

from r to v_1 (resp. v_2) in T. Let D be a walk from v_2 to r, in G. Such a path exists because G is strongly connected. We further choose a minimal path D such that $e \notin D$.

Suppose that $r = v_2$. Then, the path D is the empty path (by minimality), and the cycles $\mathcal{C}_1^{(e)} = P_1 \bullet e$ and $\mathcal{C}_2^{(e)} = lp$ satisfy the desired properties.

If $r \neq v_2$, we show that the cycles $\mathcal{C}_1^{(e)} = D \bullet P_1 \bullet e$ and $\mathcal{C}_2^{(e)} = D \bullet P_2$ are as desired. First, observe that they satisfy $\{f \in E(G) | n_f(\mathcal{C}_1^{(e)}) \neq n_f(\mathcal{C}_2^{(e)})\} \subseteq E(T) \cup \{e, lp\}$. Indeed, any $f \notin E(T) \cup \{e, lp\}$ is neither in P_1 nor in P_2, so $f \in D$ or $f \notin \mathcal{C}_1^{(e)} \cup \mathcal{C}_2^{(e)}$. In either case we have that $n_f(\mathcal{C}_1^{(e)}) = n_f(\mathcal{C}_2^{(e)})$. In addition, $e \notin \mathcal{C}_2^{(e)}$ and $e \in \mathcal{C}_1^{(e)}$. Finally, observe that the cycles obtained are non-empty. □

Before stating the next result on the cycle polytope, we take the following detour that is useful also for later purposes.

Lemma 6.2.6. Let G be a directed graph and w a walk on it. Then the multiset of edges of w can be decomposed into ℓ simple cycles (for some $\ell \geq 0$) $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ and a tail $\mathcal{T}$ that does not repeat vertices (but is possibly empty). Specifically, we have the following relation of multisets of edges of G:

$$w = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_\ell \sqcup \mathcal{T}.$$

When w is a cycle, then this decomposition can be further refined to include only simple cycles, that is we have the following relation of multisets of edges in G:

$$w = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_\ell,$$

for some $\ell \geq 0$.

Proof. This decomposition is obtained inductively on the number of edges. If w has no repeated vertices, the decomposition $w = \mathcal{T}$ satisfies the desired conditions. If w has repeated vertices, it has a simple cycle corresponding to the first repetition of a vertex. By pruning from the walk this simple cycle, we obtain a smaller walk which decomposes by the induction hypothesis. This gives us the first result.

If w is a cycle, apply to w the above decomposition for walks, and observe that the walk $w' = \mathcal{T}$ forms a smaller cycle or is the empty walk. However, $\mathcal{T}$ should not repeat vertices, so it is the empty walk, and we obtain the desired decomposition. □

Remark 6.2.7. The decomposition obtained above, of a walk w into cycles $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ and a tail $\mathcal{T}$, is a decomposition of the edge multiset. In particular, each of the cycles $\mathcal{C}_i$

or the tail $\mathcal{T}$ are *not* necessarily formed by consecutive sequences of edges of w. Explicit examples can be readily found.

Lemma 6.2.8. For a non-empty cycle $\mathcal{C}$ in G, we have that $\vec{e}_{\mathcal{C}} \in P(G)$.

Proof. We have seen in Lemma 6.2.6, that a cycle $\mathcal{C}$ has a decomposition into simple cycles as $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_\ell$. It follows that, for an edge $e \in E(G)$, we have

$$\vec{e}_{\mathcal{C}} = \sum_{j=1}^{\ell} \vec{e}_{\mathcal{C}_j} \frac{|\mathcal{C}_j|}{|\mathcal{C}|} \,. \tag{6.6}$$

Note that $\sum_{j=1}^{\ell} |\mathcal{C}_j| = |\mathcal{C}|$. Therefore, $\vec{e}_{\mathcal{C}}$ is a convex combination of the vertices of $P(G)$, as desired. ☐

For $e \in E(G) \setminus (E(T) \cup \{lp\})$, consider the cycles $\mathcal{C}_1^{(e)}, \mathcal{C}_2^{(e)}$ constructed in Lemma 6.2.5 and define the vector

$$\vec{v}^{(e)} := |\mathcal{C}_1^{(e)}|(\vec{e}_{\mathcal{C}_1^{(e)}} - \vec{e}_{lp}) - |\mathcal{C}_2^{(e)}|(\vec{e}_{\mathcal{C}_2^{(e)}} - \vec{e}_{lp}) \,.$$

In particular, observe that for $f \in E(G) \setminus (E(T) \cup \{lp\})$ we have

$$(\vec{v}^{(e)})_f = n_f(\mathcal{C}_1^{(e)}) - n_f(\mathcal{C}_2^{(e)}) \text{ is non-zero if and only if } e = f \,. \tag{6.7}$$

Lemma 6.2.9. The set $\{\vec{v}^{(e)} | e \in E(G) \setminus (E(T) \cup \{lp\})\}$ is linearly independent.

Proof. This follows immediately from Eq. (6.7). ☐

For a set $S \subseteq \mathbb{R}^{E(G)}$, recall that we defined the affine span as

$$\mathrm{Aff}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \text{ for } i = 1, \dots k \,, \sum_{i=1}^{k} \alpha_i = 1 \right\} .$$

In particular, note that if $\vec{0} \in S$, then $\mathrm{Aff}(S) = \mathrm{span}(S)$.

Proof of Theorem 6.2.2. We first assume that G has a loop lp. Then, from Lemma 6.2.4, we know that

$$\dim P(G) = \dim \mathrm{Aff}(P(G)) \leq |E(G)| - |V(G)| \,.$$

Define the translation $P(G)' = P(G) - \vec{e}_{lp}$. Observe that $\vec{0} \in P(G)'$, hence $\mathrm{Aff}(P(G)')$ is a linear space. Furthermore, observe that for each edge $e \in E(G) \setminus (E(T) \cup \{lp\})$, each vector $\vec{v}^{(e)}$ is a linear combination of $\vec{e}_{\mathcal{C}_1^{(e)}} - \vec{e}_{lp}$ and $\vec{e}_{\mathcal{C}_2^{(e)}} - \vec{e}_{lp}$, which are both in $P(G)'$, so $\vec{v}^{(e)} \in \mathrm{Aff}(P(G)')$.

Moreover, this is a linearly independent set of vectors, from Lemma 6.2.9, from which we conclude that $\dim P(G)' = \dim \mathrm{Aff}(P(G)') \geq |E(G) \setminus (E(T) \cup \{lp\})| = |E(G)| - |V(G)|$. The theorem follows, for the case where G has a loop, from $\dim P(G) = \dim P(G)'$.

Now suppose that G has no loops, and consider the graph $G \cup \{lp\}$ obtained from G by adding a loop lp to one of its vertices. By the above result, the polytope $P(G \cup \{lp\})$ has dimension $|E(G)| - |V(G)| + 1$. We can write

$$P(G \cup \{lp\}) = \mathrm{conv}(P(G) \cup \vec{e}_{lp})\,,$$

where $P(G) \subseteq \mathbb{R}^{E(G)} \setminus \{\vec{0}\}$ and $\vec{e}_{lp} \in (\mathbb{R}^{E(G)})^{\perp} \setminus \{\vec{0}\}$. It follows that

$$|E(G)| - |V(G)| + 1 = \dim P(G \cup \{lp\}) = \dim P(G) + 1\,,$$

concluding the proof of the theorem. □

We now generalize Theorem 6.2.2 to any graph. We start with the following technical result.

Proposition 6.2.10. Let $\mathfrak{a}_1 \subset \mathbb{R}^A, \mathfrak{a}_2 \subset \mathbb{R}^B$ be polytopes such that $\mathrm{Aff}(\mathfrak{a}_1), \mathrm{Aff}(\mathfrak{a}_2)$ do not contain the zero vector.

Then the dimension of the polytope $\mathfrak{c} = \mathrm{conv}(\mathfrak{a}_1 \times \{\vec{0}\}, \{\vec{0}\} \times \mathfrak{a}_2) \subset \mathbb{R}^{A \sqcup B}$ is

$$\dim(\mathfrak{c}) = \dim(\mathfrak{a}_1) + \dim(\mathfrak{a}_2) + 1.$$

For simplicity of notation, in this proof we let $d(\mathfrak{p})$ be the dimension of a polytope or affine space.

Proof. In this proof, for sake of simplicity, we will identify $\mathfrak{a}_1 \in \mathbb{R}^A$ and $\mathfrak{a}_2 \in \mathbb{R}^B$ with $\mathfrak{a}_1 \times \{\vec{0}\}$ and $\{\vec{0}\} \times \mathfrak{a}_2$, respectively. In particular, we will refer to points $\vec{x} \in \mathfrak{a}_i$ for $i = 1, 2$ as their suitable extensions $(\vec{x}, \vec{0})$ or $(\vec{0}, \vec{x})$, respectively, in $\mathbb{R}^{A \sqcup B}$ without further notice.

Suppose that $\mathrm{Aff}(\mathfrak{a}_1) = W_1 + \vec{x}_1$, $\mathrm{Aff}(\mathfrak{a}_2) = W_2 + \vec{x}_2$ and $\mathrm{Aff}(\mathfrak{c}) = W + \vec{x}_1$, where W_1, W_2, W are vector subspaces of $V := \mathbb{R}^{A \sqcup B}$ with dimension $d(\mathfrak{a}_1), d(\mathfrak{a}_2)$ and $d(\mathfrak{c})$ respectively. A choice of $\vec{x}_1, \vec{x}_2$ such $\vec{x}_i \in \mathfrak{a}_i$ for $i = 1, 2$ is always possible. Since $\vec{0} \notin \mathrm{Aff}(\mathfrak{a}_1), \vec{0} \notin \mathrm{Aff}(\mathfrak{a}_2)$, we have that $\vec{x}_1 \notin W_1, \vec{x}_2 \notin W_2$.

The dimension that we wish to compute is $d(\mathfrak{c}) = \dim(W)$. We will do this by establishing both underlying inequalities.

We start with a lower bound for $d(\mathfrak{c})$. Consider bases of W_1, W_2, $\{\vec{v}_1^{(1)}, \ldots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}\}$ and $\{\vec{v}_1^{(2)}, \ldots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}\}$, respectively. It is clear that $\vec{v}_i^{(1)} \in \mathrm{Aff}(\mathfrak{a}_1) - \vec{x}_1 \subseteq W$ for $i = 1, \ldots, d(\mathfrak{a}_1)$, and that $\vec{x}_2 - \vec{x}_1 \in W$. In addition we have that

$$\vec{v}_i^{(2)} \in \mathrm{Aff}(\mathfrak{a}_2) - \vec{x}_2 \subseteq \mathrm{Aff}(\mathfrak{c}) - \vec{x}_2 = \mathrm{Aff}(\mathfrak{c}) + (\vec{x}_2 - \vec{x}_1) - \vec{x}_2 = W, \qquad \text{for } i = 1, \ldots, d(\mathfrak{a}_2).$$

This proves that $\{\vec{v}_1^{(1)}, \ldots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}, \vec{v}_1^{(2)}, \ldots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}, \vec{x}_2 - \vec{x}_1\} \subseteq W$. We now show that this set is linearly independent.

Because $W_1 \cap W_2 = \{\vec{0}\}$, the set $\{\vec{v}_1^{(1)}, \ldots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}, \vec{v}_1^{(2)}, \ldots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}\}$ forms a basis of $W_1 \oplus W_2$. Because $\vec{x}_1 \in \mathbb{R}^A \setminus W_1$ and $\vec{x}_2 \in \mathbb{R}^B \setminus W_2$, adding the vectors $\vec{x}_1, \vec{x}_2$ extends this basis. It follows that

$$\{\vec{v}_1^{(1)}, \ldots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}, \vec{v}_1^{(2)}, \ldots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}, \vec{x}_2 - \vec{x}_1\}$$

is linearly independent.

Observe that we found a linearly independent set with $d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 1$ many vectors in W. This gives us a lower bound for $\dim \mathfrak{c}$.

For an upper bound, observe that $\mathrm{Aff}(\mathfrak{c}) \subseteq \operatorname{span} \mathfrak{c}$, and that

$$\dim(\operatorname{span} \mathfrak{c}) \leq \dim(\operatorname{span} \mathfrak{a}_1) + \dim(\operatorname{span} \mathfrak{a}_2) = d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 2\,.$$

We now prove that $0 \notin \mathrm{Aff}(\mathfrak{c})$ by contradiction. Assume otherwise that $\sum_i \alpha_i \vec{a}_i + \sum_j \beta_j \vec{b}_j = 0$, where $\vec{a}_i \in \mathfrak{a}_1$, $\vec{b}_j \in \mathfrak{a}_2$ and $\sum_i \alpha_i + \sum_j \beta_j = 1$. But $\sum_i \alpha_i \vec{a}_i \in \mathbb{R}^A$, and $-\sum_j \beta_j \vec{b}_j \in \mathbb{R}^B$, so $\sum_i \alpha_i \vec{a}_i = -\sum_j \beta_j \vec{b}_j \in \mathbb{R}^A \cap \mathbb{R}^B = \{\vec{0}\}$. Because $\sum_i \alpha_i + \sum_j \beta_j = 1$, without loss of generality we can assume that $\sum_i \alpha_i \neq 0$. Then we have $\frac{\sum_i \alpha_i \vec{a}_i}{\sum_i \alpha_i} = 0 \in \mathrm{Aff}(\mathfrak{a}_1)$, a contradiction.

Since $0 \in \operatorname{span}(\mathfrak{c})$, we conclude that $\mathrm{Aff}(\mathfrak{c}) \neq \operatorname{span} \mathfrak{c}$. It follows that

$$\dim \mathrm{Aff}(\mathfrak{c}) < d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 2\,.$$

This concludes the proof. □

With the help of Proposition 6.2.10, we can generalize Theorem 6.2.2 to the cycle polytope of any graph: We say that a graph $G = (V, E)$ is *full* if any edge $e \in E$ is part of a cycle of G. It is easy to see that if G is not full, then $P(G) = P(H)$, where $H \subseteq G$

is the largest full subgraph of G. Equivalently, H is obtained from G by removing all edges from G that are not part of a cycle.

If G is a full graph, there are no *bridges*, that is an edge e connecting two distinct strongly connected components. Hence, we can decompose G as the disjoint union of strongly connected components and a set of isolated vertices V': $G = H_1 \sqcup \cdots \sqcup H_k \sqcup V'$. It can be seen that $P(G) = \operatorname{conv}\{P(H_i) | i = 1, \ldots k\}$, where identify $P(H_i)$ with its canonical image in $\mathbb{R}^{E(G)}$. Noting that $P(H_i) \subseteq \operatorname{Aff}(P(H_i)) \subseteq \mathbb{R}^{E(H_i)}$ and that $\operatorname{Aff}(P(H_i))$ does not contain the origin for any $i = 1, \ldots, k$, from Proposition 6.2.10 we have that

$$\dim P(G) = k - 1 + \sum_i \dim P(H_i) = k - 1 + |E| - |V \setminus V'| = |E| - |V| + |V'| + k - 1 \,.$$

Theorem 6.2.11. If G is a directed multigraph and $H \subseteq G$ its largest full subgraph, then the dimension of the polytope $P(G)$ is

$$\dim P(G) = |E(H)| - |V(G)| + |\{ \text{ connected components of } H\}| - 1.$$

6.2.3 Faces of the cycle polytope

We now focus on the faces of a cycle polytope $P(G)$. We prove two results: in Theorem 6.2.12 we describe the equations that define $P(G)$, then in Theorem 6.2.13 we find a bijection between faces of $P(G)$ and the subgraphs of G that are full.

Theorem 6.2.12. Let G be a directed graph. The polytope $P(G)$ is given by the equations in Lemma 6.2.4 together with the inequalities $\vec{x} \geq 0$. Specifically

$$P(G) = \left\{ \vec{x} \in \mathbb{R}^{E(G)} \middle| \sum_{e \in E(G)} x_e = 1 \,, \ \sum_{\mathrm{s}(e) = v} x_e = \sum_{\mathrm{a}(e) = v} x_e \,, \forall v \in V(G) \,, \quad \vec{x} \geq \vec{0} \right\} .$$

Proof. For simplicity of notation, let $H_1 = \{\vec{x} \in \mathbb{R}^{E(G)} | \sum_e x_e = 1\}$ and

$$P_+(G) = \left\{ \vec{x} \in \mathbb{R}^{E(G)} \middle| \sum_{\mathrm{s}(e) = v} x_e = \sum_{\mathrm{a}(e) = v} x_e \,, \forall v \in V(G) \,, \quad \vec{x} \geq \vec{0} \right\} .$$

We wish to show that

$$P(G) = P_+(G) \cap H_1 \,. \tag{6.8}$$

The inclusion $P(G) \subseteq P_+(G) \cap H_1$ is trivial. Suppose now, for the sake of contradiction that there is a point $\vec{x} \in (P_+(G) \cap H_1) \setminus P(G)$, and pick one that minimizes the size of the edge set $\mathcal{Z}(\vec{x}) := \{e \in E(G) | x_e \neq 0\}$. First observe that $\vec{0} \notin H_1$, so $\mathcal{Z}(\vec{x}) \neq \emptyset$.

We now show that any source v of the graph $(V(G), \mathcal{Z}(\vec{x}))$ is an isolated vertex. In fact, if no edge $e \in \mathcal{Z}(\vec{x})$ satisfies $\mathrm{a}(e) = v$, we have

$$\sum_{\mathrm{s}(e)=v} x_e = \sum_{\mathrm{a}(e)=v} x_e = 0\,,$$

and because $x_e \geq 0$ for any edge e, we have that $x_e = 0$ for any edge e such that $\mathrm{s}(e) = v$. Hence, $v \in V(G)$ is an isolated vertex in $(V(G), \mathcal{Z}(\vec{x}))$.

Because $(V(G), \mathcal{Z}(\vec{x}))$ is a graph with at least one oriented edge where any source is isolated, it must have a simple cycle $\mathcal{C}$. Let $e \in \mathcal{C}$ be the edge in $\mathcal{C}$ that minimizes x_e, and consider $\vec{y} \coloneqq \vec{x} - x_e|\mathcal{C}|\vec{e}_{\mathcal{C}}$.

In the case that we have $\vec{y} = \vec{0}$, then $\vec{x} = x_e|\mathcal{C}|\vec{e}_{\mathcal{C}} \in H_1$ by assumption. But $\vec{e}_{\mathcal{C}} \in H_1$, so then $\vec{x} = \vec{e}_{\mathcal{C}} \in P(G)$, which is a contradiction. In the case that $\vec{y}$ is a non-zero vector with non-negative coefficients, we have that $\sum_e y_e \neq 0$, and

$$\vec{z} \coloneqq \frac{\vec{y}}{\sum_e y_e} \in P_+(G) \cap H_1\,.$$

Now suppose that $\vec{z} \in P(G)$. Then $\vec{x} = \left(\sum_e y_e\right)\vec{z} + x_e|\mathcal{C}|\vec{e}_{\mathcal{C}}$ is a convex combination of points in $P(G)$, so $\vec{x} \in P(G)$, which contradicts our assumption that $\vec{x} \in (P_+(G) \cap H_1) \setminus P(G)$. Thus $\vec{z} \notin P(G)$. But, $\mathcal{Z}(\vec{z}) = \mathcal{Z}(\vec{y}) \subseteq \mathcal{Z}(\vec{x}) \setminus \{e\}$, contradicting the minimality of $\vec{x}$. We conclude that there is no $\vec{x} \in (P_+(G) \cap H_1) \setminus P(G)$, hence $P_+(G) \cap H_1 \subseteq P(G)$, as desired. □

Recall that a subgraph $H = (V, E')$ of a graph G is called a *full subgraph* if any edge $e \in E'$ is part of a cycle of H.

Theorem 6.2.13. The face poset of $P(G)$ is isomorphic to the poset of non-empty full subgraphs of G according to the following identification:

$$H \mapsto P(G)_H \coloneqq \{\vec{x} \in P(G) | x_e = 0 \text{ for } e \notin E(H)\}\,.$$

Further, if we identify $P(H)$ with its image under the canonical inclusion $\mathbb{R}^{E(H)} \hookrightarrow \mathbb{R}^{E(G)}$, we have that $P(H) = P(G)_H$.

In particular, $\dim P(G)_H = |E(H)| - |V| + |\{\text{connected components of } H\}| - 1$.

Proof. From Theorem 6.2.12, a face of $P(G)$ is given by setting some of the inequalities of $\vec{x} \geq 0$ as equalities. So, a face is of the form $P(G)_H$ for some subgraph $H = (V(G), E(H))$, where $E(G) \setminus E(H)$ corresponds to the inequalities of $\vec{x} \geq 0$ that become

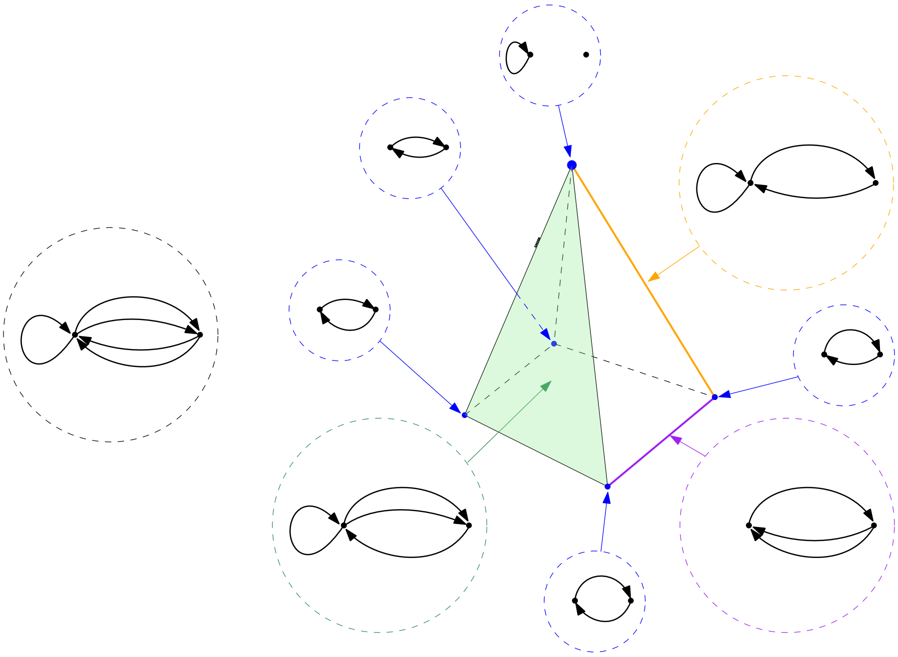


FIGURE 6.5: The face structure of the cycle polytope of a graph. On the left-hand side of the picture (inside the dashed black ball) we have a graph G with two vertices and five edges. On the right-hand side, we draw the associated cycle polytope $P(G)$ that is a pyramid with squared base. The blue dashed balls correspond to the simple cycles corresponding to the five vertices of the polytope. We also underline the relation between two edges of the polytope (in purple and orange respectively) and a face (in green) and the corresponding full subgraphs. Note that, for example, the graph corresponding to the green face is just the union of the three graphs corresponding to the vertices of that face.

equalities. It is immediate to observe that the identification $\mathbb{R}^{E(H)} \hookrightarrow \mathbb{R}^{E(G)}$ gives us that $P(H) = P(G)_H$.

We show that it suffices to take H a full subgraph: consider an edge $e_0 = v \to w$ in H that is not contained in any cycle in H. Then $(\vec{e}_{\mathcal{C}})_{e_0} = 0$ for any simple cycle $\mathcal{C}$ in H, and so $x_{e_0} = 0$ for any point $\vec{x} \in P(H) = P(G)_H$. It follows that $P(G)_H = P(G)_{H \setminus e_0}$.

Conversely, we can see that if $H_1 \neq H_2$ are two full subgraphs of G, then we have that $P(G)_{H_1} \neq P(G)_{H_2}$, that is, H_1, H_2 correspond to two different faces of $P(G)$. Indeed, without loss of generality we can assume that there is an edge $e \in E(H_1) \setminus E(H_2)$. This edge is, by hypothesis, contained in a simple cycle $\mathcal{C}$, so $\vec{e}_{\mathcal{C}} \in P(G)_{H_1} \setminus P(G)_{H_2}$, so $P(G)_{H_1} \neq P(G)_{H_2}$.

It is clear that if $H_1 \subseteq H_2$ then $P(G)_{H_1} \subseteq P(G)_{H_2}$, so the identification $H \mapsto P(G)_H$ preserves the poset structure. Finally, we obtained the dimension of $P(G)_H = P(H)$ in Theorem 6.2.11. □

Example 6.2.14 (Face structure of a specific cycle polytope)**.** Consider the graph G given on the left-hand side of Fig. 6.5, that has two vertices and five edges. It follows that the corresponding cycle polytope has dimension three, and its face structure is partially depicted in the right-hand side Fig. 6.5.

In fact, from Theorem 6.2.13, to each face of the polytope we can associate a full subgraph of G. Some of these correspondences are highlighted in Fig. 6.5 and described in its caption.

Given a simple cycle $\mathcal{C}$ in a graph G, a path P is a *chord* of $\mathcal{C}$ if it is edge-disjoint from $\mathcal{C}$ and it starts and arrives at vertices of $\mathcal{C}$. In particular, given two simple cycles sharing a vertex, any one of them forms a chord of the other.

Remark 6.2.15 (The skeleton of the polytope $P(G)$)**.** We want to characterize the pairs of vertices of $P(G)$ that are connected by an edge. The structure behind this is usually called the *skeleton of the polytope*. Suppose that we are given two vertices of the polytope $P(G)$, $\vec{e}_{\mathcal{C}_1}, \vec{e}_{\mathcal{C}_2}$ corresponding to the simple cycles $\mathcal{C}_1, \mathcal{C}_2$ of the graph G, according to Proposition 6.2.1.

With the description of the faces in Theorem 6.2.13, we have that a face $P(G)_H$ is an edge when it has dimension one, that is

$$|E(H)| - |V(G)| + |\{\text{connected components of } H\}| - 1 = 1\,.$$

This happens if and only if the undirected version of H is a forest with two edges added.

Because H is full, each connected component must contain a cycle, so H has either one or two connected components. Hence, it results either from the union of two vertex-disjoint simple cycles, or from the union of a simple cycle and one of its chords. Equivalently, $\vec{e}_{\mathcal{C}_1}, \vec{e}_{\mathcal{C}_2}$ are connected with an edge when $\mathcal{C}_1 \setminus \mathcal{C}_2$ forms a unique chord of $\mathcal{C}_2$, or when $\mathcal{C}_1, \mathcal{C}_2$ are vertex-disjoint.

For instance, in Fig. 6.5, there are two pairs of vertices of $P(G)$ that are not connected, and each pair corresponds to two cycles $\mathcal{C}_1, \mathcal{C}_2$ such that $\mathcal{C}_1 \setminus \mathcal{C}_2$ forms two chords of $\mathcal{C}_2$.

Remark 6.2.16 (Computing the volume of $P(G)$)**.** The problem of finding the volume of a polytope is a classical one in convex geometry. We propose an algorithmic approach that uses the face description of $P(G)$ in Theorem 6.2.13 and the following facts:

- Let A be a polytope and v a point in space. If $v \notin \operatorname{Aff}(A)$, then

$$\operatorname{vol}(\operatorname{conv}(A \cup \{v\})) = \operatorname{vol}(A) \operatorname{dist}(v, \operatorname{Aff}(A)) \frac{1}{\dim A + 1}\,.$$

- If v is vertex of the polytope $\mathfrak{p}$ of dimension d, then we have the following decomposition of the polytope $\mathfrak{p}$:

$$\mathfrak{p} = \bigcup_{v \notin \mathfrak{q} \subsetneq \mathfrak{p}} \operatorname{conv}(\mathfrak{q} \cup \{v\}) \,,$$

  where the union runs over all high-dimensional faces $\mathfrak{q}$ that do not contain the vertex v. This decomposition is such that the intersection of each pair of blocks has volume zero, and each block has a non-zero $d-1$ dimensional volume.

If $\mathcal{C}$ is a simple cycle of G, the following decomposition holds:

$$P(G) = \bigcup_{\mathcal{C} \nsubseteq H \subsetneq G} \operatorname{conv}(\vec{e}_{\mathcal{C}}, P(G)_H) \,,$$

where the union runs over all maximal full proper subgraphs of G that do not contain $\mathcal{C}$.

Hence, we obtain the volume of $P(G)$ as follows:

$$\operatorname{vol}(P(G)) = \sum_{\mathcal{C} \nsubseteq H \subsetneq G} \operatorname{conv}(\vec{e}_{\mathcal{C}}, P(G)_{G \setminus e}) = \sum_{\mathcal{C} \nsubseteq H \subsetneq G} \frac{\operatorname{vol}(P(H)) \operatorname{dist}(\vec{e}_{\mathcal{C}}, \operatorname{Aff}(P(G)_H))}{\dim P(G) + 1},$$

where the sum runs over all maximal full proper subgraphs of G that do not contain $\mathcal{C}$. This gives us a recursive way of computing the volume $\operatorname{vol}(P(G))$ by computing the volume of cycle polytopes of smaller graphs. We have unfortunately not been able to find a general formula for $\operatorname{vol} P(G)$, and leave this as an open problem.

6.3 The feasible region P_k is a cycle polytope

Recall that we defined

$$P_k \coloneqq \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \,.$$

The goal of this section is to prove that P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$ (see Theorem 6.3.12). We first prove that P_k is closed and convex (see Proposition 6.3.2), then we use a correspondence between permutations and paths in $\mathcal{O}v(k)$ (see Definition 6.3.5) to prove the desired result.

6.3.1 The feasible region P_k is convex

We start with a preliminary result.

Lemma 6.3.1. The feasible region P_k is closed.

This is a classical consequence of the fact that P_k is a set of limit points. For completeness, we include a simple proof of this. Recall that $\widetilde{\text{c-occ}}_k(\sigma) := \big(\widetilde{\text{c-occ}}(\pi, \sigma)\big)_{\pi \in \mathcal{S}_k}$.

Proof. It suffices to show that, for any sequence $(\vec{v}_s)_{s\in\mathbb{N}}$ in P_k such that $\vec{v}_s \to \vec{v}$ for some $\vec{v} \in [0,1]^{\mathcal{S}_k}$, we have that $\vec{v} \in P_k$. For all $s \in \mathbb{N}$, consider a sequence of permutations $(\sigma_s^m)_{m\in\mathbb{N}}$ such that $|\sigma_s^m| \overset{m\to\infty}{\longrightarrow} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_s^m) \overset{m\to\infty}{\longrightarrow} \vec{v}_s$, and some index $m(s)$ of the sequence $(\sigma_s^m)_{m\in\mathbb{N}}$ such that for all $m \geq m(s)$,

$$|\sigma_s^m| \geq s \quad \text{and} \quad ||\widetilde{\text{c-occ}}_k(\sigma_s^m) - \vec{v}_s|| \leq \tfrac{1}{s}\,.$$

W.l.o.g. assume that $m(s)$ is increasing. For every $\ell \in \mathbb{N}$, define $\sigma^\ell := \sigma_\ell^{m(\ell)}$. It is easy to show that

$$|\sigma^\ell| \overset{\ell\to\infty}{\longrightarrow} \infty \quad \text{and} \quad \widetilde{\text{c-occ}}_k(\sigma^\ell) \overset{\ell\to\infty}{\longrightarrow} \vec{v}\,,$$

where we use that $\vec{v}_s \to \vec{v}$. Therefore $\vec{v} \in P_k$. □

We can now prove the first important result of this section.

Proposition 6.3.2. The feasible region P_k is convex.

Proof. Since P_k is closed (by Lemma 6.3.1) it is enough to consider rational convex combinations of points in P_k, *i.e.*, it is enough to establish that for all $\vec{v}_1, \vec{v}_2 \in P_k$ and all $s, t \in \mathbb{N}$, we have that

$$\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k.$$

Fix $\vec{v}_1, \vec{v}_2 \in P_k$ and $s, t \in \mathbb{N}$. Since $\vec{v}_1, \vec{v}_2 \in P_k$, there exist two sequences $(\sigma_1^m)_{m\in\mathbb{N}}$, $(\sigma_2^m)_{m\in\mathbb{N}}$ such that $|\sigma_i^m| \overset{m\to\infty}{\longrightarrow} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_i^m) \overset{m\to\infty}{\longrightarrow} \vec{v}_i$, for $i = 1, 2$.

Define $t_m := t \cdot |\sigma_1^m|$ and $s_m := s \cdot |\sigma_2^m|$.

We set $\tau^m := (\,\sigma_1^m)^{\oplus s_m} \oplus (\,\sigma_2^m)^{\oplus t_m}$. For a graphical interpretation of this construction we refer to Fig. 6.6. We note that for every $\pi \in \mathcal{S}_k$, we have

$$\text{c-occ}(\pi, \tau^m) = s_m \cdot \text{c-occ}(\pi, \sigma_1^m) + t_m \cdot \text{c-occ}(\pi, \sigma_2^m) + Er,$$

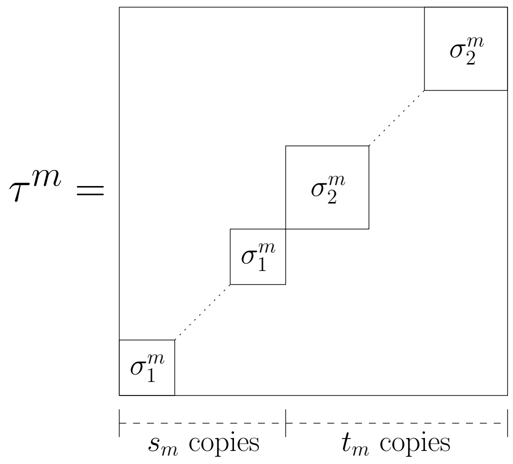


FIGURE 6.6: Schema for the definition of the permutation τ^m.

where $Er \leq (s_m + t_m - 1) \cdot |\pi|$. This error term comes from the number of intervals of size $|\pi|$ that intersect the boundary of some copies of σ_1^m or σ_2^m. Hence

$$\widetilde{\text{c-occ}}(\pi, \tau^m) = \frac{s_m \cdot |\sigma_1^m| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_1^m) + t_m \cdot |\sigma_2^m| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_2^m) + Er}{s_m \cdot |\sigma_1^m| + t_m \cdot |\sigma_2^m|}$$
$$= \frac{s}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_1^m) + \frac{t}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_2^m) + O\left(|\pi| \left(\tfrac{1}{|\sigma_1^m|} + \tfrac{1}{|\sigma_2^m|}\right)\right).$$

As m tends to infinity, we have

$$\widetilde{\text{c-occ}}_k(\tau^m) \to \frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2,$$

since $|\sigma_i^m| \stackrel{m\to\infty}{\longrightarrow} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_i^m) \stackrel{m\to\infty}{\longrightarrow} \vec{v}_i$, for $i = 1, 2$. Noting also that

$$|\tau^m| \to \infty,$$

we can conclude that $\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k$. This ends the proof. □

6.3.2 The feasible region P_k as the limit of random permutations

Using similar ideas to the ones used in the proof above, we can establish the equality between the sets in Eq. (6.2). We first recall the following.

Definition 6.3.3. For a total order $(\mathbb{Z}, \preccurlyeq)$, its *shift* $(\mathbb{Z}, \preccurlyeq')$ is defined by $i+1 \preccurlyeq' j+1$ if and only if $i \preccurlyeq j$. A random infinite rooted permutation, or equivalently a random total order on $\mathbb{Z}$, is said to be shift-invariant if it has the same distribution as its shift.

We refer to [Bor19, Section 2.6] for a full discussion on shift-invariant random permutations.

Proposition 6.3.4. The following equality holds

$$P_k = \left\{ (\Gamma_\pi(\sigma^\infty))_{\pi \in \mathcal{S}_k} \middle| \sigma^\infty \text{ is a random infinite rooted shift-invariant permutation} \right\} .$$

Proof. In [Bor19, Proposition 2.44 and Theorem 2.45] it was proved that a random infinite rooted permutation σ^∞ is shift-invariant if and only if it is the annealed Benjamini–Schramm limit of a sequence of random permutations[2]. Furthermore, we can choose this sequence of random permutations σ^n in such a way that $|\sigma^n| = n$ a.s., for all $n \in \mathbb{N}$.

This result and Theorem 1.7.3 immediately imply that

$$P_k \subseteq \left\{ (\Gamma_\pi(\sigma^\infty))_{\pi \in \mathcal{S}_k} \middle| \sigma^\infty \text{ is a random infinite rooted shift-invariant permutation} \right\} .$$

To show the other inclusion, it is enough to show that for every random infinite rooted shift-invariant permutation σ^∞, there exists a sequence of *deterministic* permutations that Benjamini–Schramm converges to σ^∞.

By the above mentioned result of [Bor19], there exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of *random* permutations such that $|\sigma^n| = n$ a.s., for all $n \in \mathbb{N}$, and $(\sigma^n)_{n\in\mathbb{N}}$ converges in the annealed Benjamini–Schramm sense to σ^∞. Using [Bor19, Theorem 2.24] we know that, for every $\pi \in \mathcal{S}$,

$$\mathbb{E}[\widetilde{\text{c-occ}}(\pi, \sigma^n)] \to \Gamma_\pi(\sigma^\infty)\,. \tag{6.9}$$

Let, for all $n \in \mathbb{N}$ and $\rho \in \mathcal{S}_n$,

$$p^n_\rho := \mathbb{P}(\sigma^n = \rho).$$

For every $n \in \mathbb{N}$, we can find $n!$ integers $\{q^n_\rho\}_{\rho\in\mathcal{S}_n}$ such that for every $\rho \in \mathcal{S}_n$,

$$\left| \frac{q^n_\rho}{\sum_{\theta\in\mathcal{S}_n} q^n_\theta} - p^n_\rho \right| \leq \frac{1}{n^n}. \tag{6.10}$$

Let us now consider the deterministic sequence of permutations of size $n\sum_{\theta\in\mathcal{S}_n} q^n_\theta$ defined as

$$\nu^n := \bigoplus_{\rho\in\mathcal{S}_n} \rho^{\oplus q^n_\rho},$$

where we fixed any order on $\mathcal{S}_n$. Using the same error estimates as in the proof of Proposition 6.3.2, it follows that

$$\widetilde{\text{c-occ}}(\pi, \nu^n) = \frac{\sum_{\rho\in\mathcal{S}_n} q^n_\rho \cdot \text{c-occ}(\pi, \rho) + Er}{n \cdot \sum_{\theta\in\mathcal{S}_n} q^n_\theta}, \quad \text{for all} \quad \pi \in \mathcal{S},$$

[2] The annealed Benjamini–Schramm convergence is an extension of the Benjamini–Schramm convergence to sequences of *random* permutations. For more details see [Bor19, Section 2.5.1].

with $Er \leq (-1 + \sum_{\theta \in \mathcal{S}_n} q^n_\theta) \cdot |\pi|$. Therefore

$$\begin{aligned}
&\left|\widetilde{\text{c-occ}}(\pi, \nu^n) - \mathbb{E}[\widetilde{\text{c-occ}}(\pi, \sigma^n)]\right| \\
&\quad \leq \left| \sum_{\rho \in \mathcal{S}_n} \frac{q^n_\rho}{\sum_{\theta \in \mathcal{S}_n} q^n_\theta} \cdot \widetilde{\text{c-occ}}(\pi, \rho) - \sum_{\rho \in \mathcal{S}_n} p^n_\rho \cdot \widetilde{\text{c-occ}}(\pi, \rho) \right| + \left| \frac{Er}{n \cdot \sum_{\theta \in \mathcal{S}_n} q^n_\theta} \right| \\
&\qquad \leq \frac{1}{n^n} \cdot \sum_{\rho \in \mathcal{S}_n} \widetilde{\text{c-occ}}(\pi, \rho) + \frac{|\pi|}{n},
\end{aligned}$$

where in the second inequality we used the bound in Eq. (6.10) and the bound for Er. Since the size of π is fixed and the term $\sum_{\rho \in \mathcal{S}_n} \widetilde{\text{c-occ}}(\pi, \rho)$ is bounded by $n!$, we can conclude that $\left|\widetilde{\text{c-occ}}(\pi, \nu^n) - \mathbb{E}[\widetilde{\text{c-occ}}(\pi, \sigma^n)]\right| \to 0$. Combining this with Eq. (6.9) we get

$$\widetilde{\text{c-occ}}(\pi, \nu^n) \to \Gamma_\pi(\sigma^\infty), \quad \text{for all} \quad \pi \in \mathcal{S}.$$

Therefore, using Theorem 1.7.3 we can finally deduce that the deterministic sequence $\{\nu^n\}_{n \in \mathbb{N}}$ converges to σ^∞ in the Benjamini–Schramm topology, concluding the proof. □

6.3.3 The overlap graph

We now want to study the way in which consecutive patterns of permutations can overlap.

We start by introducing some more notation. For a permutation $\pi \in \mathcal{S}_k$, with $k \in \mathbb{N}_{\geq 2}$, let $\text{beg}(\pi) \in \mathcal{S}_{k-1}$ (resp. $\text{end}(\pi) \in \mathcal{S}_{k-1}$) be the patterns generated by its first $k-1$ indices (resp. last $k-1$ indices). More precisely,

$$\text{beg}(\pi) := \pi|_{[1,k-1]} \quad \text{and} \quad \text{end}(\pi) := \pi|_{[2,k]}.$$

The following definition, introduced in [CDG92], is key in the description of the feasible region P_k.

Definition 6.3.5 (Overlap graph)**.** Let $k \in \mathbb{N}_{\geq 1}$. We define the *overlap graph* $\mathcal{O}v(k)$ of size k as a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for all $\pi \in \mathcal{S}_k$ we add the edge $\text{beg}(\pi) \to \text{end}(\pi)$ labeled by π.

This gives us a directed graph with $k!$ many edges, and $(k-1)!$ many vertices. Informally, the continuations of an edge τ in the overlap graph $\mathcal{O}v(k)$ records the consecutive patterns of size k that can appear after the consecutive pattern τ. More precisely, for a permutation $\sigma \in \mathcal{S}_{\geq k+1}$ and an interval $I \subseteq [|\sigma| - 1]$ of size k, let $\tau := \sigma|_I \in \mathcal{S}_k$, then we have that

$$\sigma|_{I+1} \in N_{\mathcal{O}v(k)}(\tau), \tag{6.11}$$

where $I+1$ denotes the interval obtained from I shifting all the indices by $+1$, and we recall that $N_{\mathcal{O}v(k)}(\tau)$ is the set of continuations of τ.

Example 6.3.6. We recall that the overlap graph $\mathcal{O}v(4)$ was displayed in Fig. 6.2 on page 140. The six vertices (in red) correspond to the six permutations of size three and the twenty-four oriented edges correspond to the twenty-four permutations of size four.

Given a permutation $\sigma \in \mathcal{S}_m$, for some $m \geq k$, we can associate to it a walk $W_k(\sigma) = (e_1, \dots, e_{m-k+1})$ in $\mathcal{O}v(k)$ of size $m-k+1$ defined by

$$\mathrm{lb}(e_i) \coloneqq \sigma|_{[i,i+k-1]}, \quad \text{for all} \quad i \in [m-k+1]. \tag{6.12}$$

Note that Eq. (6.11) justifies that this sequence of edges is indeed a walk in the overlap graph.

Example 6.3.7. Take the graph $\mathcal{O}v(4)$ from Fig. 6.2 on page 140, and consider the permutation $\sigma = 628451793 \in \mathcal{S}_9$. The corresponding walk $W_4(\sigma)$ in $\mathcal{O}v(4)$ is

$$(3142, 1423, 4231, 2314, 2134, 1342)$$

and it is highlighted in green in Fig. 6.2.

Note that the map W_k is not injective (see for instance the Example 6.3.9 below) but the following holds.

Lemma 6.3.8. Fix $k \in \mathbb{N}_{\geq 2}$ and $m \geq k$. The map W_k, from the set $\mathcal{S}_m$ of permutations of size m to the set of walks in $\mathcal{O}v(k)$ of size $m-k+1$, is surjective.

Proof. We exhibit a greedy procedure that, given a walk $w = (e_1, \dots, e_s)$ in $\mathcal{O}v(k)$, constructs a permutation σ of size $s+k-1$ such that $W_k(\sigma) = w$. Specifically, we construct a sequence of s permutations $(\sigma_i)_{i \leq s}$, with $|\sigma_i| = i+k-1$, in such a way that σ is equal to σ_s. For this proof, it is useful to consider permutations as diagrams.

The first permutation is defined as $\sigma_1 = \mathrm{lb}(e_1)$. To construct σ_{i+1} we add to the diagram of σ_i a final additional point on the right of the diagram between two rows, in such a way that the last k points induce the consecutive pattern $\mathrm{lb}(e_{i+1})$ (the choice for this final additional point may not be unique, but exists). Setting $\sigma \coloneqq \sigma_s$ we have by construction that $W_k(\sigma) = w$. $\square$

We illustrate the construction above in a concrete example.

Example 6.3.9. Consider the walk $w = (3142, 1423, 4231, 2314, 2134, 1342)$ obtained in Example 6.3.7 and construct, as explained in the previous proof, a permutation σ such that $W_k(\sigma) = w$. We set $\sigma_1 = 3142$. Then, since $e_2 = 1423$, we add a point between the second and the third row of σ_1 (see Fig. 6.7 for the diagrams of the considered permutations), obtaining $\sigma_2 = 41523$. Note that the pattern induced by the last 4 points of σ_2 is exactly $e_2 = 1423$. We highlight that we could also add the point between the third and the fourth row of σ_1 obtaining the same induced pattern. However, in this example, we always chose to add the points in the bottommost possible place. We iterate this procedure constructing $\sigma_3 = 516342$, $\sigma_4 = 6173425$, $\sigma_5 = 71834256$, $\sigma_6 = 819452673$. Setting $\sigma := \sigma_6 = 819452673$ we obtain that $W_4(\sigma) = w$. Note that this is not the same permutation considered in Example 6.3.7, indeed the map W_k is not injective.

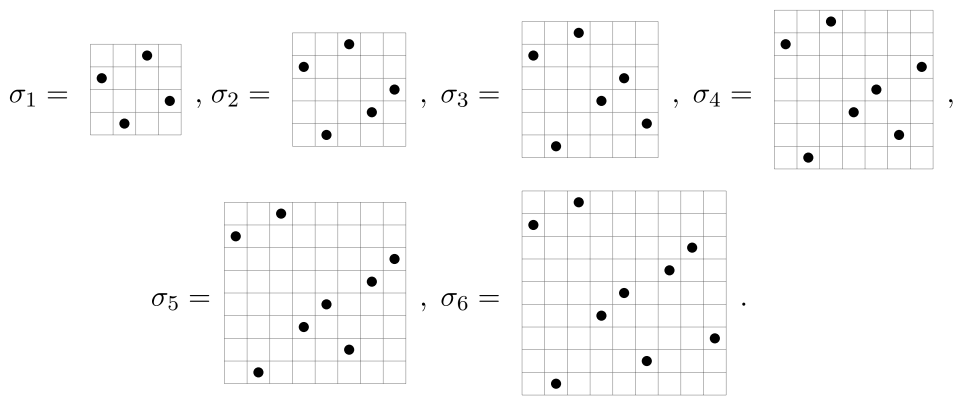


FIGURE 6.7: The diagrams of the six permutations considered in Example 6.3.9. Note that every permutation is obtained by adding a new final point to the previous one.

We conclude this section with two simple results useful for the following sections.

Lemma 6.3.10. If σ is a permutation and $w = W_k(\sigma) = (e_1, \dots, e_s)$ is its corresponding walk on $\mathcal{O}v(k)$, then

$$\text{c-occ}(\pi, \sigma) = |\{i \leq s | \operatorname{lb}(e_i) = \pi\}|.$$

Proof. This is a trivial consequence of the definition of the map W_k. See in particular Eq. (6.12). □

Observation 6.3.11. Let π_1 and π_2 be two permutations of size $k - 1 \geq 1$, and take $\tau = \pi_1 \oplus \pi_2$. Then the path $W_k(\tau)$ goes from π_1 to π_2. Consequently, $\mathcal{O}v(k)$ is strongly connected.

6.3.4 A description of the feasible region P_k

The goal of this section is to prove the following result.

Theorem 6.3.12. The feasible region P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$, *i.e.*,

$$P_k = P(\mathcal{O}v(k)). \tag{6.13}$$

As a consequence, the vertices of P_k are precisely $\{\vec{e}_{\mathcal{C}} | \mathcal{C}$ is a simple cycle of $\mathcal{O}v(k)\}$ and the dimension of P_k is $k! - (k-1)!$. Moreover, the polytope P_k is described by the equations

$$P_k = \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \sum_{\pi \in \mathcal{S}_k} v_\pi = 1 \, , \sum_{\mathrm{beg}(\pi)=\rho} v_\pi = \sum_{\mathrm{end}(\pi)=\rho} v_\pi \, , \forall \rho \in \mathcal{S}_{k-1} \, , \quad \vec{v} \geq \vec{0} \right\} .$$

Proof. The first step is to show that, for any simple cycle $\mathcal{C}$ of $\mathcal{O}v(k)$, the vector $\vec{e}_{\mathcal{C}}$ is in P_k. This, together with Proposition 6.3.2 implies that $P(\mathcal{O}v(k)) \subseteq P_k$.

According to Lemma 6.3.8, for every $m \in \mathbb{N}$, there is a permutation σ^m such that $W_k(\sigma^m)$ is the walk resulting from the concatenation of m copies of $\mathcal{C}$. We claim that $\widetilde{\text{c-occ}}_k(\sigma^m) \to \vec{e}_{\mathcal{C}}$ and $|\sigma^m| \to \infty$. The latter affirmation is trivial since, by Lemma 6.3.8, $|\sigma^m| = |\mathcal{C}|m + k - 1$. For the first claim, according to Lemma 6.3.10, we have that $\text{c-occ}(\pi, \sigma^m) = m$ for any π that is the label of an edge in the simple cycle $\mathcal{C}$, and $\text{c-occ}(\pi, \sigma^m) = 0$ otherwise. Hence

$$\widetilde{\text{c-occ}}_k(\sigma^m) = \vec{e}_{\mathcal{C}} \frac{m|\mathcal{C}|}{|\sigma^m|} \to \vec{e}_{\mathcal{C}} \, .$$

as desired.

On the other hand, suppose that $\vec{v} \in P_k$, so we have a sequence σ^m of permutations such that $|\sigma^m| \to \infty$ and $\widetilde{\text{c-occ}}_k(\sigma^m) \to \vec{v}$. We will show that $\mathrm{dist}(\widetilde{\text{c-occ}}_k(\sigma^m), P(\mathcal{O}v(k))) \to 0$. It is then immediate, since $P(\mathcal{O}v(k))$ is closed, that $\vec{v} \in P(\mathcal{O}v(k))$, proving that $P_k = P(\mathcal{O}v(k))$. We consider the walk $w^m = W_k(\sigma^m)$. Using Lemma 6.2.6, the edge multiset of the walk w^m can be decomposed into simple cycles and a tail (that does not repeat vertices and may be empty) as follows

$$w^m = \mathcal{C}_1^m \sqcup \cdots \sqcup \mathcal{C}_\ell^m \sqcup \mathcal{T}^m.$$

Then from Lemma 6.3.10 we can compute $\widetilde{\text{c-occ}}_k(\sigma^m)$ as a convex combination of $\vec{e}_{\mathcal{C}}$ for some simple cycles $\mathcal{C}$, plus a small error term. Specifically,

$$\widetilde{\text{c-occ}}_k(\sigma^m) = \vec{e}_{\mathcal{C}_1^m} \frac{|\mathcal{C}_1^m|}{|\sigma^m|} + \cdots + \vec{e}_{\mathcal{C}_\ell^m} \frac{|\mathcal{C}_\ell^m|}{|\sigma^m|} + \vec{Er}^m,$$

where $|\vec{Er}^m| \leq \frac{(k-1)!}{|\sigma^m|}$ since there are $(k-1)!$ distinct vertices in $\mathcal{O}v(k)$ and the path $\mathcal{T}^m$ does not contain repeated vertices. In particular, $|\vec{Er}^m| \to 0$ since k is constant and $|\sigma^m| \to \infty$.

Noting that

$$\widetilde{\text{c-occ}}_k(\sigma^m) = \vec{Er}^m + \frac{\sum_i |\mathcal{C}_i^m|}{|\sigma^m|}\vec{w}_m,$$

where $\vec{w}_m = \frac{1}{\sum_i |\mathcal{C}_i^m|}\left(\vec{e}_{\mathcal{C}_1^m}|\mathcal{C}_1^m| + \cdots + \vec{e}_{\mathcal{C}_\ell^m}|\mathcal{C}_\ell^m|\right) \in P(\mathcal{O}v(k))$, we can conclude that

$$\text{dist}\left(\widetilde{\text{c-occ}}_k(\sigma^m), P(\mathcal{O}v(k))\right) \leq \text{dist}\left(\vec{Er}^m + \frac{\sum_i |\mathcal{C}_i^m|}{|\sigma^m|}\vec{w}_m, \vec{w}_m\right) \to 0\,,$$

since $|\vec{Er}^m| \to 0$, $\frac{\sum_i |\mathcal{C}_i^m|}{|\sigma^m|} \to 1$ and $\vec{w}_m$ is uniformly bounded. This concludes the proof of Eq. (6.13).

The characterization of the vertices is a trivial consequence of Proposition 6.2.1. For the dimension, it is enough to note that $\mathcal{O}v(k)$ is strongly connected from Observation 6.3.11. So by Theorem 6.2.2 it has dimension $|E(\mathcal{O}v(k))| - |V(\mathcal{O}v(k))| = k! - (k-1)!$, as desired. Finally, the equations for P_k are determined using Theorem 6.2.12 and the definition of overlap graph. □

Remark 6.3.13. Since for two different simple cycles $\mathcal{C}_1, \mathcal{C}_2$ we have that $\vec{e}_{\mathcal{C}_1} \neq \vec{e}_{\mathcal{C}_2}$, enumerating the vertices corresponds to enumerating simple cycles of $\mathcal{O}v(k)$ (which seems to be a difficult problem). This problem was partially investigated in [AF18]. There, all the cycles of size one and two are enumerated.

6.4 Mixing classical patterns and consecutive patterns

In Section 6.1.5 we explained that a natural question is to describe the feasible region when we mix classical and consecutive patterns.

More generally, let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ be two finite sets of permutations. We consider the following sets of points

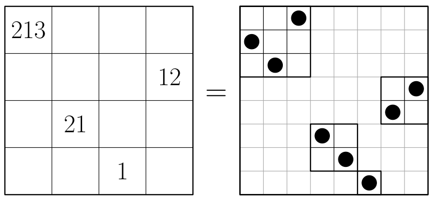
$$\begin{aligned} A &= \left\{ \vec{v} \in [0,1]^{\mathcal{A}} \middle| \exists\, (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \left(\widetilde{\text{c-occ}}(\pi, \sigma^m)\right)_{\pi\in\mathcal{A}} \to \vec{v} \right\}, \\ B &= \left\{ \vec{v} \in [0,1]^{\mathcal{B}} \middle| \exists\, (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } (\widetilde{\text{occ}}(\pi, \sigma^m))_{\pi\in\mathcal{B}} \to \vec{v} \right\}. \end{aligned} \tag{6.14}$$

We want to investigate the set

$$\begin{aligned} C = \Big\{ \vec{v} \in [0,1]^{\mathcal{A}\sqcup\mathcal{B}} \Big| \exists\, (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty, \\ \left(\widetilde{\text{c-occ}}(\pi, \sigma^m)\right)_{\pi\in\mathcal{A}} \to (\vec{v})_{\mathcal{A}} \text{ and } (\widetilde{\text{occ}}(\pi, \sigma^m))_{\pi\in\mathcal{B}} \to (\vec{v})_{\mathcal{B}} \Big\}. \end{aligned} \tag{6.15}$$

For the statement of the next theorem we need to recall the definition of the *substitution operation* on permutations. For $\theta, \nu^{(1)}, \ldots, \nu^{(d)}$ permutations such that $d = |\theta|$, the substitution $\theta[\nu^{(1)}, ..., \nu^{(d)}]$ is defined as follows: for each i, we replace the point $(i, \theta(i))$ in the diagram of θ with the diagram of $\nu^{(i)}$. Then, rescaling the rows and columns yields the diagram of a larger permutation $\theta[\nu^{(1)}, ..., \nu^{(d)}]$. Note that $|\theta[\nu^{(1)}, ..., \nu^{(d)}]| = \sum_{i=1}^{d} |\nu^{(i)}|$ (see Fig. 6.8 for an example).

$4213[213, 21, 1, 12] =$



$= 76832145\,.$

FIGURE 6.8: Example of substitution of permutations.

Theorem 6.4.1. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ be finite sets of permutations, and A, B, C be defined as in Eqs. (6.14) and (6.15). It holds that

$$A \times B = C\,. \tag{6.16}$$

Specifically, given two points $\vec{v}_{\mathcal{A}} \in A, \vec{v}_{\mathcal{B}} \in B$, consider two sequences $(\sigma_A^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ and $(\sigma_B^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ such that

$$\begin{aligned} &|\sigma_A^m| \to \infty \text{ and } \left(\widetilde{\text{c-occ}}(\pi, \sigma_A^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}_A, \\ &|\sigma_B^m| \to \infty \text{ and } (\widetilde{\text{occ}}(\pi, \sigma_B^m))_{\pi\in\mathcal{B}} \to \vec{v}_B, \end{aligned} \tag{6.17}$$

then the sequence $(\sigma_C^m)_{m\in\mathbb{N}}$ defined by

$$\sigma_C^m := \sigma_B^m[\sigma_A^m, \ldots, \sigma_A^m], \quad \text{for all} \quad m \in \mathbb{N}, \tag{6.18}$$

satisfies

$$|\sigma_C^m| \to \infty, \quad \left(\widetilde{\text{c-occ}}(\pi, \sigma_C^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}_A \quad \text{and} \quad (\widetilde{\text{occ}}(\pi, \sigma_C^m))_{\pi\in\mathcal{B}} \to \vec{v}_B. \tag{6.19}$$

Proof. Let $(\sigma_C^m)_{m\in\mathbb{N}}$ be defined as in Eq. (6.18). The fact that the size of σ_C^m tends to infinity follows from $|\sigma_C^m| = |\sigma_A^m||\sigma_B^m| \to \infty$. For the second limit in Eq. (6.19), note that for every pattern $\pi \in \mathcal{A}$,

$$\widetilde{\text{c-occ}}(\pi, \sigma_C^m) = \frac{\text{c-occ}(\pi, \sigma_C^m)}{|\sigma_B^m| \cdot |\sigma_A^m|} = \frac{\text{c-occ}(\pi, \sigma_A^m) \cdot |\sigma_B^m| + Er}{|\sigma_B^m| \cdot |\sigma_A^m|} = \widetilde{\text{c-occ}}(\pi, \sigma_A^m) + \frac{Er}{|\sigma_B^m| \cdot |\sigma_A^m|},$$

where $Er \leq |\sigma_B^m| \cdot |\pi|$. This error term comes from intervals of $[|\sigma_C^m|]$ that intersect more than one copy of σ_A^m. Since $|\pi|$ is fixed and $|\sigma_A^m| \to \infty$ we can conclude the desired limit, using the assumption in Eq. (6.17) that $\left(\widetilde{\text{c-occ}}(\pi, \sigma_A^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}_A$.

Finally, for the third limit in Eq. (6.19) we note that setting $n = |\sigma_C^m|$ and $k = |\pi|$,

$$\widetilde{\text{occ}}(\pi, \sigma_C^m) = \frac{\mathbf{p}_\pi(\sigma_C^m)}{\binom{n}{k}} = \mathbb{P}\left(\sigma_C^m|_{\boldsymbol{I}} = \pi\right), \tag{6.20}$$

where $\boldsymbol{I}$ is a random set, uniformly chosen among the $\binom{n}{k}$ subsets of $[n]$ with k elements (we denote random quantities in **bold**). Let now E^m be the event that the random set $\boldsymbol{I}$ contains two indices $\boldsymbol{i}, \boldsymbol{j}$ of $[|\sigma_C^m|]$ that belong to the same copy of σ_A^m in σ_C^m. Denote by $(E^m)^C$ the complement of the event E^m. We have

$$\mathbb{P}\left(\sigma_C^m|_{\boldsymbol{I}} = \pi\right) = \mathbb{P}\left(\sigma_C^m|_{\boldsymbol{I}} = \pi | E^m\right) \cdot \mathbb{P}\left(E^m\right) + \mathbb{P}\left(\sigma_C^m|_{\boldsymbol{I}} = \pi | (E^m)^C\right) \cdot \mathbb{P}\left((E^m)^C\right). \tag{6.21}$$

We claim that

$$\mathbb{P}\left(E^m\right) \leq \binom{k}{2} \frac{1}{|\sigma_B^m|} \to 0. \tag{6.22}$$

Indeed, the factor $\binom{k}{2}$ counts the number of pairs i, j in a set of cardinality k and the factor $\frac{1}{|\sigma_B^m|}$ is an upper bound for the probability that given a uniform two-element set $\{\boldsymbol{i}, \boldsymbol{j}\}$ then $\boldsymbol{i}, \boldsymbol{j}$ belong to the same copy of σ_A^m in σ_C^m (recall that there are $|\sigma_B^m|$ copies of σ_A^m in σ_C^m). Note also that

$$\mathbb{P}\left(\sigma_C^m|_{\boldsymbol{I}} = \pi | (E^m)^C\right) = \widetilde{\text{occ}}(\pi, \sigma_B^m) \to \vec{v}_B, \tag{6.23}$$

where the last limit comes from Eq. (6.17). Using Eqs. (6.20) to (6.23), we obtain that

$$(\widetilde{\text{occ}}(\pi, \sigma_C^m))_{\pi\in\mathcal{B}} \to \vec{v}_B.$$

This concludes the proof of Eq. (6.19). The result in Eq. (6.16) follows from the fact that we trivially have $C \subseteq A \times B$, and for the other inclusion we use the construction above, which proves that $(\vec{v}_A, \vec{v}_B) \in C$, for every $\vec{v}_{\mathcal{A}} \in A, \vec{v}_{\mathcal{B}} \in B$. $\square$

Chapter 7

Open problems and further work

In this section we collect some of the open problems indicated in this thesis, and additionally remarks on future research directions in the topics are presented.

7.1 On combinatorial presheaves and pattern algebras

We have observed that some of the usual Hopf algebras in combinatorics arise as pattern Hopf algebras. This contributes to the recognition of pattern Hopf algebras as an object of interest. The central example is the Hopf algebra of symmetric functions (see Section 2.3.4). Some other Hopf algebras seem to arise also in this way, for instance:

Problem 7.1.1. Show that the pattern Hopf algebra on set compositions is isomorphic to $QSym$. Specifically, that the coproduct structure of the free generators of $\mathcal{A}(\texttt{SComp})$ constructed via Theorem 2.3.9 match the coproduct structure of the free generators of $QSym$ constructed in [BZ09].

We now address the freeness question on pattern algebras.

Problem 7.1.2. Does any associative presheaf generate a free pattern Hopf algebra?

This question is motivated by the several combinatorial presheaves that we have observed in this thesis to have a free pattern Hopf algebra. Specifically, it is motivated by the pattern Hopf algebra on permutations, as well as pattern Hopf algebras resulting from the presheaf on marked permutations and the commutative presheaves presented in Chapter 3 and Section 2.3, respectively.

Ditters' conjecture, proven in [Haz01], strengthens the question of freeness of an algebra. Specifically, it asks about freeness over the integers and over $\mathbb{Z}_p$ of the Hopf algebra of quasi-symmetric functions.

Problem 7.1.3. Is the pattern Hopf algebra on permutations, $\mathcal{A}(\texttt{Per})$, free over the integers? How about the pattern Hopf algebra on marked permutations $\mathcal{A}(\texttt{MPer})$? Additionally, can the methods used in [Haz01] to construct a family of free generators over the integers $\mathbb{Z}_p$ be applied to these pattern algebras?

A factorization theorem can be used not only to establish the freeness of a pattern algebra (or of an algebra in general), but also to enumerate a basis of the primitive space of a pattern Hopf algebra. We already know that the pattern algebra on the presheaf of graphs with the inflation product is free. However, with a factorization theorem we can additionally enumerate a basis of the primitive space, as done for marked permutations.

Problem 7.1.4. Does the inflation product on graphs have a factorization theorem?

7.2 On chromatic invariants

In Chapter 4, we have discussed the Hopf algebra morphisms $\Psi_{\mathbf{G}} : \mathbf{G} \to Sym$ and $\Psi_{\mathbf{G}} : \mathbf{HGP} \to QSym$. Specifically, we gave a set of generators for the kernel of these maps. A direct consequence of this description is a structured strategy to solve the tree conjecture. The first part of this strategy is the following:

Problem 7.2.1. Find other invariants of graphs Γ that satisfy $\ker \Gamma = \ker \Psi_{\mathbf{G}}$.

With this, the tree conjecture on $\Psi_{\mathbf{G}}$ is equivalent to the tree conjecture on Γ. Remark that the invariant $\tilde{\Psi}$, introduced in Section 4.3.1, is a possible answer to this result. With new invariants like $\tilde{\Psi}$, we have access to new information that can be used to solve the tree conjecture.

We also wish to study the chromatic invariant on other Hopf algebras. The description of the kernel of $\Psi_{\mathbf{HGP}}$ is only a partial result in the following project:

Problem 7.2.2. Find generators for the kernel of $\Psi_{\mathbf{GP}}$. Specifically, can we use the description of the image of $\mathbf{\Upsilon}_{\mathbf{GP}}$, given in Section 4.5, to find a family of generators of the kernel of $\Psi_{\mathbf{GP}}$?

We single out two other interesting cases where the kernel problem is of interest. First, we introduce the matroid Hopf algebra, introduced in [Sch94], that we denote by $\mathbf{M}$. This is a Hopf algebra spanned by matroids, which inherits a Hopf algebra structure via the embedding $Z : \mathbf{M} \to \mathbf{GP}$ defined as

$$Z(M) = \text{conv}\{\vec{e}_{\mathcal{B}} | \mathcal{B} \text{ is a basis of } M\}.$$

For more on this Hopf algebra, see [GR14].

Problem 7.2.3. Consider the Malvenuto–Reutenauer Hopf algebra **MR**, and the Hopf algebra on matroids **M**. What are the corresponding “modular relations” of $\Psi_{\mathbf{M}}$ and $\Psi_{\mathbf{MR}}$? Find generators for the kernel of these chromatic invariants.

7.3 On the antipode of pattern algebras

In a preliminary result, the author found a cancellation-free and grouping-free formula for the antipode of the pattern Hopf algebra on permutations. This is a desirable result in many Hopf algebras in combinatorics, and the methods therein seem to depend on the particular factorization theorem of $\oplus$ in permutations.

Problem 7.3.1. Given an associative presheaf $(h, *, 1)$ that satisfies a given **factorization theorem**, find a cancellation-free and grouping-free formula for the antipode of $\mathcal{A}(h)$.

With the discovery of a cancellation-free and grouping-free formula for the antipode of a Hopf algebra comes a collection of questions that are interesting to study. These are, for instance, describing chromatic polynomials or finding reciprocity results. Unfortunately these questions only make sense if the Hopf algebra at hand is **graded**.

Problem 7.3.2. Can we adapt the filtered Hopf algebra structure of $\mathcal{A}(\mathtt{Per})$ to a graded Hopf algebra structure?

In [Hof00], a method for transforming quasi-shuffle algebras into shuffle algebras is described, and is a candidate procedure to give a satisfactory answer to this problem. Once we have a graded Hopf algebra, we can further endow $\mathcal{A}(\mathtt{Per})$ with a character, and study its **chromatic invariants**.

Problem 7.3.3. Find characters in the pattern Hopf algebra $\mathcal{A}(\mathtt{Per})$, and the corresponding chromatic polynomials $\chi_a(n)$. Does the antipode formula give a simple description of $\chi_a(-1)$?

We remark that, in pattern algebras, a large family of characters can be immediately constructed by design. Specifically, given a combinatorial presheaf h and a coinvariant $a \in \mathcal{G}(h)$, the following map is a character:

$$\zeta_a : \mathbf{p}_b \mapsto \mathbf{p}_b(a) \,.$$

7.4 On feasible regions

Regarding feasible regions of classical patterns in permutations, we have already referred to the dimension problem in Conjecture 1.7.1. This is a problem that started with [GHK+17], where a lower bound was established. Latter on, in a seamingly unrelated work in the area of algebraic combinatorics, an upper bound was given in [Var14] by establishing the freeness of $\mathcal{A}(\texttt{Per})$. We conjecture that this upper bound is tight:

Conjecture 7.4.1. Show that the feasible region clP_k is full dimensional in a variety of dimension $|\mathcal{L}_k|$.

We now return to the topic of consecutive occurrences. Specifically, we talk about the feasible region in the setting of consecutive occurrences, where we impose some sort of restriction on the sequences of permutations along which we take the limit. For that, we introduce *permutation classes*. Recall that we denote by $\mathcal{S}$ the set of all permutations. A permutation class $\mathcal{P} \subseteq \mathcal{S}$ is a family of permutations with the following property: if π, σ are permutations such that π is a pattern of σ, and $\pi \in \mathcal{P}$, then $\sigma \in \mathcal{P}$. Given a permutation π write $\text{Av}(\pi)$ for the set of permutations that have no pattern matching π. These are examples of permutation classes.

In this way, we can study the asymptotic behavior of sequences of permutations that belong to a specific permutation class, resulting in what we call **restricted feasible regions**.

Problem 7.4.2. Given a permutation class $\mathcal{C}$, what is $P_k^{\mathcal{C}}$, the feasible region of consecutive occurrences for permutations, when we consider only sequences of permutations on a permutation class. Specifically, is

$$P_k^{\mathcal{C}} = \{\vec{v} \in [0,1]^{\mathcal{S}_k} | \exists \{\sigma^m\}_{m\geq 1} \subseteq \mathcal{C} \text{ s.t. } \lim_{n\to\infty} |\sigma^n| = +\infty, \text{ and } \lim_{n\to\infty} \widetilde{\text{c-occ}}(\pi, \sigma^n) = \vec{v}_\pi \text{ for } \pi \in \mathcal{S}_k\} . \tag{7.1}$$

a polytope? Furthermore, can we describe it as the cycle polytope of a particular graph?

The cases of permutation classes $\text{Av}(1\dots n)$, and $\text{Av}(\pi)$ for each $|\pi| = 3$ are preliminary results in an upcomming manuscript, [BP20].

Observe that, if $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $P_k^{\mathcal{P}_1} \subseteq P_k^{\mathcal{P}_2}$. Thus, for any two permutations π, τ and integer k, we have that $P_k^{Av(\pi,\tau)} \subseteq P_k^{Av(\pi)} \cap P_k^{Av(\tau)}$.

Problem 7.4.3. Do we have that $P_k^{Av(\pi,\tau)} = P_k^{Av(\pi)} \cap P_k^{Av(\tau)}$ for any integer k and any permutation π, τ with $|\pi|, |\tau| \leq 3$?

Other combinatorial objects also admit a notion of "consecutive" pattern, for instance rooted graphs. Given a graph $G = (V, E)$ and a vertex v of it, we can consider the ball around v of radius δ, that is,

$$B_\delta^G(v) = \{w \in V | \text{ there is a path connecting } v \text{ and } w \text{ of lenght at most } \delta\}.$$

In this way, a consecutive pattern with radius δ of a marked graph G^* on a graph H is simply a vertex v of H such that G^* is isomorphic to $B_\delta^H(v)$ with the vertex v marked.

Problem 7.4.4. What is the feasible region for consecutive occurrences of rooted graphs. Is it a polytope?

In [BLP20], a preliminary manuscript, the problem of describing the feasible region on planar rooted trees is addressed. There, for a planar rooted tree T and a vertex v, we define its ball of radius δ, or its **fringe**, inductively as $B_0^T(v) = \{v\}$, and

$$B_{\delta+1}^T(v) = \{w \in T | \text{ is the offspring of some } w' \in B_\delta(v)\}.$$

For two planar rooted trees T_1, T_2 we define an occurrence of T_2 in T_1 to be a vertex v of T_1 such that T_2 is isomorphic to $B_\delta^{T_1}(v)$ with v as its rooted vertex.

Problem 7.4.5. What is the feasible region for consecutive occurrences of rooted graphs. Is it a polytope?

Appendix A

Hopf Algebras

In this section we will define the most important concepts in Hopf algebras. We introduce **algebras**, **coalgebras** and **Hopf algebras**. Finally, we summarise some simple constructions on Hopf algebras, like the primitive space and the coradical filtration. Finally, we present a structure theorem on Hopf algebras called the *Milnor-Moore theorem*.

Fix a field $\mathbb{K}$, that we assume to have characteristic zero.

A.1 Algebras and coalgebas

We start by introducing algebras over a field. The claims presented here that are not proven can be found in any introductory book in algebras and Lie algebras, for instance [Bou03, Hal15, Stu18, GR14].

Definition A.1.1 (Algebra)**.** An *algebra* over $\mathbb{K}$ is a triple (A, μ, ι) where A is a $\mathbb{K}$-vector space, and $\mu : A \otimes A \to A$, $\iota : \mathbb{K} \to A$ are linear maps such that the following diagrams commute

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}_A} & A \otimes A \\ \Big\downarrow{\scriptstyle \mathrm{id}_A \otimes \mu} & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \tag{A.1}$$

$$\begin{array}{ccccc} A \otimes \mathbb{K} & \xrightarrow{\mathrm{id}_A \otimes \iota} & A \otimes A & \xleftarrow{\iota \otimes \mathrm{id}_A} & \mathbb{K} \otimes A \\ & {\scriptstyle \cong}\nwarrow & \Big\downarrow{\scriptstyle \mu} & \nearrow{\scriptstyle \cong} & \\ & & A & & \end{array} \tag{A.2}$$

where we denote by id_A the identity map on A.

We require μ to be a map $\mu : A \otimes A \to A$, instead of a more usual way of thinking about a multiplication as $\mu : A \times A \to A$. Indeed, by requiring μ to have the vector space $A \otimes A$ as source, we are by design making the product a distributive one with respect to the addition in A.

It is probably unfamiliar when phrased in this way, but (A.1) is simply the associativity axiom on the product μ.

Definition A.1.2 (Coalgebra)**.** A *coalgebra* over k is a triple (C, Δ, ε) where C is a $\mathbb{K}$-vector space, and $\Delta : C \to C \otimes C$, $\varepsilon : C \to \mathbb{K}$ are maps such that

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow[\Delta \otimes \mathrm{id}_C]{} & C \otimes C \\ {\scriptstyle \mathrm{id}_C \otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow[\Delta]{} & C \end{array} \tag{A.3}$$

$$\begin{array}{ccccc} C \otimes \mathbb{K} & \xleftarrow[\mathrm{id}_C \otimes \varepsilon]{} & C \otimes C & \xrightarrow{\varepsilon \otimes \mathrm{id}_C} & \mathbb{K} \otimes C \\ & {\scriptstyle \cong}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \cong} & \\ & & C & & \end{array} \tag{A.4}$$

If $(A, \mu_A, \iota_A), (B, \mu_B, \iota_B)$ are $\mathbb{K}$-algebras, then $A \otimes B$ inherits an algebra structure from the composition:

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes \mathbf{twist} \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B \tag{A.5}$$

Symilarly, if $(A, \Delta_A, \varepsilon_A), (B, \Delta_B, \varepsilon_B)$ are $\mathbb{K}$-coalgebras, then $A \otimes B$ inherits a coalgebra structure.

Definition A.1.3 (Bialgebra and Hopf algebra)**.** A *bialgebra* over a field $\mathbb{K}$ is a 5-tuple $(B, \mu, \iota, \Delta, \varepsilon)$ where (B, μ, ι) is an algebra, (B, Δ, ε) is a coalgebra and the following diagrams commute:

$$\begin{array}{ccccc} B \otimes B & \xrightarrow{\Delta \otimes \Delta} & B \otimes B \otimes B \otimes B & \xrightarrow{\mathrm{id}_B \otimes \mathbf{twist} \otimes \mathrm{id}_B} & B \otimes B \otimes B \otimes B \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu \otimes \mu} \\ B & & \xrightarrow{\quad\quad\quad \Delta \quad\quad\quad} & & B \otimes B \end{array} \tag{A.6}$$

$$
\begin{array}{ccc}
B \otimes B & \xrightarrow{\varepsilon \otimes \varepsilon} & \mathbb{K} \\
\downarrow{\scriptstyle \mu} & \nearrow{\scriptstyle \varepsilon} & \\
B & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\iota \otimes \iota} & B \otimes B \\
& \searrow{\scriptstyle \iota} & \uparrow{\scriptstyle \Delta} \\
& & B
\end{array}
$$

$$
\begin{array}{ccc}
B & \xrightarrow{\varepsilon} & \mathbb{K} \\
& \nwarrow{\scriptstyle \iota} & \uparrow{\scriptstyle \cong} \\
& & \mathbb{K}
\end{array}
\tag{A.7}
$$

A *Hopf algebra* over k is a 6-tuple $(H, \mu, \iota, \Delta, \varepsilon, S)$ such that $(H, \mu, \iota, \Delta, \varepsilon)$ is a bialgebra, and $S : H \to H$ a linear map that satisfies the antipode property:

$$
\begin{array}{ccccccc}
& & H \otimes H & \xrightarrow{S \otimes id_H} & H \otimes H & & \\
& \nearrow{\scriptstyle \Delta} & & & & \searrow{\scriptstyle \mu} & \\
H & & \xrightarrow{\quad \varepsilon \quad} & \mathbb{K} & \xrightarrow{\quad \iota \quad} & & H \\
& \searrow{\scriptstyle \Delta} & & & & \nearrow{\scriptstyle \mu} & \\
& & H \otimes H & \xrightarrow{\mathrm{id}_H \otimes S} & H \otimes H & &
\end{array}
\tag{A.8}
$$

We recover here a simple example of a Hopf algebra from Chapter 1. Consider a finite group G, and let $\mathbb{K}[G]$ be a vector space, with a basis $\{e_g\}_{g \in G}$ indexed by G. The product is defined as

$$e_g e_h = e_{g \cdot h} \,,$$

and the coproduct is defined as

$$\Delta(e_g) = g \otimes g \,.$$

In this way, the linear map S defined on a basis as $S(e_g) = e_{g^{-1}}$ is the antipode map and endows $\mathbb{K}[G]$ with a Hopf algebra structure.

If 1 is the identity of G, then e_1 is the unit of $\mathbb{K}[G]$. The counit map ε is defined in the basis elements as $\varepsilon(e_g) = 1$.

A.2 Connected Hopf algebras, filtered Hopf algebras and Takeuchi's formula

It is often the case that, for simple Hopf algebras, finding the antipode is just a matter of finding a suitable inductive construction. In this section, we will see what is the Takeuchi's formula, and that this formula gives us an expression for the antipode whenever the bialgebra H is graded or filtered.

The main use of this formula is that it allows us to introduce Hopf algebras without explicitly presenting an antipode, a task that is often times dificult.

Observation A.2.1. A *graded bialgebra* is a bialgebra $(H, \mu, \iota, \Delta, \varepsilon)$ such that $H = \oplus_{n \geq 0} H_n$ and

- $H_i \subseteq H_{i+1}$,
- $\mu(H_i \otimes H_j) \subseteq H_{i+j}$,
- $\Delta(H_m) \subseteq \oplus_{i+j=m} H_i \otimes H_j$,
- $\iota(1) \in H_0$,
- $\varepsilon(H_k) = 0$ for any $k > 0$.

Observe that $(H_0, \mu, \iota, \Delta, \varepsilon)$ is also a bialgebra. A graded bialgebra is said to be *connected* if $\dim_{\mathbb{K}} H_0 = 1$.

Theorem A.2.2 ([GR14, Proposition 1.4.22.])**.** Let $H = \oplus_{n \geq 0} H_n$ be a graded connected bialgebra.

Then, the following map S is an antipode of H

$$S = \sum_{k \geq 0} (-1)^k \mu^{\circ k-1} \circ (\mathrm{id}_H - \iota \circ \varepsilon)^{\otimes k} \circ \Delta^{\circ k-1} \,,$$

where we use the convention that $\Delta^{\circ -1} = \varepsilon$ and $\mu^{\circ -1} = \iota$. In particular, H is a Hopf algebra.

The proof of this claim can be found in [GR14, Proposition 1.4.22.]. The key observation for the proof of this theorem is that the sum is always finite. In fact, if $h \in H : n$, the sum above computed in $S(h)$ has at most $n+1$ terms.

It is worth to point out that Takeushi's formula can be extended to a result in *filtered* bialgebras:

Definition A.2.3. A *filtered bialgebra* is a bialgebra $(H, \mu, \iota, \Delta, \varepsilon)$ such that $H = \bigcup_{n \geq 0} H_n$ is the union of vector spaces H_n such that:

- $\mu(H_i \otimes H_j) \subseteq H_{i+j}$,
- $\Delta(H_m) \subseteq \sum_{i+j=m} H_i \otimes H_j$,
- $\iota(1) \in H_0$.

Observe that $(H_0, \mu, \iota, \Delta, \varepsilon)$ is also a bialgebra. A filtered bialgebra is said to be *connected* if $\dim_{\mathbb{K}} H_0 = 1$.

Theorem A.2.4. Let $H = \bigcup_{n\geq 0} H_n$ be a filtered connected bialgebra such that H_0 is a Hopf algebra. Then H is a Hopf algebra.

We present a proof of this fact after introducing the *convolution product.*

A.3 The convolution product and characters

Definition A.3.1 (Convolution product)**.** Consider now an algebra A and a coalgebra C. Then we define a convolution product on the linear maps $\mathrm{Hom}(C, A)$ as follows. If $f, g \in \mathrm{Hom}(C, A)$, define

$$f \star g = \mu_A \circ (f \otimes g) \circ \Delta_C \,.$$

Note that this is an associative operation with unit $\iota_A \circ \varepsilon_C$. Observe that a Hopf algebra is a bialgebra H such that $\mathrm{id}_H \in \mathrm{Hom}(H, H)$ has an inverse under the convolution product, which is S, the antipode. This is the two sided inverse by definition.

Observe that if H is either commutative or cocommutative, then $\mathrm{Hom}(H, H)$ is a commutative group.

In general, we can see that if $f_1, \dots, f_k \in \mathrm{Hom}(C, A)$, then

$$f_1 \star \cdots \star f_k = \mu_A \circ (f_1 \otimes \cdots \otimes f_k) \circ \Delta_C \,.$$

In this way, we define the shorthand notation $f^{\star k} = f \star \cdots \star f$. Here, we assume the convention that $f^{\star 0} = \iota \circ \varepsilon$.

The following proof is a technique due to [Tak71].

Proof of Theorem A.2.4. Let $f \in \mathrm{Hom}(H, H)$ be given as $f = \mathrm{id}_H - \iota \circ \varepsilon$. We claim that

$$S = \sum_{k\geq 0} (-1)^k f^{\star k} = \sum_{k\geq 0} (-1)^k \mu^{\circ k-1} \circ (\mathrm{id}_H - \iota \circ \varepsilon)^{\otimes k} \circ \Delta^{\circ k-1} \,,$$

is an antipode in a filtered connected bialgebra.

First, we observe that f is zero in H_0. Indeed, because H_0 is one dimensional, and $\iota(1)$ is a non-zero element of H_0, we have that $H_0 = \iota(1)\mathbb{K}$. But $f(\iota(1)) = \iota(1) - \iota(\varepsilon(\iota(1)))$, thus $f(\iota(1)) = \iota(1) - \iota(1) = 0$, see the last diagram in (A.7).

Therefore, we have the following for $f^{\star k}(h)$, if $h \in H_n$: after applying successively the filtration assumption and disregarding the terms on H_0 we get

$$f^{\star k}(h) \in \sum_{\substack{\alpha \models n \\ l(\alpha)=k}} H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_k} \,, \tag{A.9}$$

where the sum runs over compositions of n. Remarkably, if $k \geq n+1$, the sum is empty and we have that $f^{\star k}(h) = 0$.

Then, this function S is well defined, that is that for each h, $S(h)$ is given by a finite sum.

Now we observe the telescopic sum

$$\begin{aligned} S \star \mathrm{id}_H(h) &= \sum_{k=0}^{n} (-1)^k f^{\star k} \star \mathrm{id}_H(h) \\ &= \sum_{k=0}^{n} (-1)^k \left(f^{\star k} \star f + f^{\star k} \star (\iota \circ \varepsilon) \right) (h) \\ &= \sum_{k=0}^{n} (-1)^k \left(f^{\star k+1} + f^{\star k} \right) (h) \\ &= (-1)^n f^{\star n+1}(h) + f^{\star 0}(h) = \iota \circ \varepsilon(h) \,. \end{aligned} \tag{A.10}$$

and similarly we obtain that $\mathrm{id}_H \star S(h) = \iota \circ \varepsilon(h)$, so S is an antipode. □

We remark that this technique can be used to compute inverses in the convolution group of filtered bialgebras. That is, if $\phi \in \mathrm{Hom}(H, H)$ such that $(\phi - \iota \circ \varepsilon)|_{H_0}$ is the zero map, then:

$$\phi^{\star -1} = \sum_{k \geq 0} (-1)^k \mu^{\circ k-1} \circ (\phi - \iota \circ \varepsilon)^{\otimes k} \circ \Delta^{\circ k-1} \,.$$

A.4 Structures on Hopf algebras

In this section we introduce important structures in Hopf algebras. These are the group-like elements, the primitive space and the coradical filtration. The structure theorem known as Milnor Moore's theorem uses the primitive space to present tight restrictions on Hopf algebras that are graded connected and cocommutative.

Definition A.4.1 (Group-like elements and primitive space)**.** Given a coalgebra C, the group-like elements of C, $G(C)$ is the set

$$G(C) = \{g \in H | \, \Delta g = g \otimes g\} \,.$$

The primitive space of C, written $P(C)$, is the set

$$P(C) = \{x \in C | \, \Delta x = 1 \otimes x + x \otimes 1\} \, .$$

Group-like elements and primitive spaces in a Hopf algebra posess unique properties. For instance, if H is a Hopf algebra, $G(H)$ is a group, and the inverse of any element $g \in G(H)$ is precisely $S(g)$. Furthermore, the elements of $G(H)$ are all linearly independent. The space of primitive elements is a Lie algebra, under the usual bracket $[a, b] = ab - ba$.

To introduce the coradical filtration, we first introduce the *wedge power* of subspaces of a coalgebra. Let C be a coalgebra. For any linear subspace $X \subseteq C$ we define $\wedge^0 X = 0$ and

$$\wedge^{n+1} X = \Delta^{-1}(X^n \otimes C + C \otimes X) = \Delta^{-1}(X \otimes C + C \otimes X^n) \, .$$

Definition A.4.2 (Coradical filtration)**.** Given a coalgebra C, a simple subcoalgebra $D \subseteq C$ is a coalgebra that has no non-trivial subcoalgebra $D' \subseteq D$. The **coradical** of a coalgebra C is the sum of all simple subcoalgebras of C:

$$C_0 = \sum_{D \subseteq C \text{ simple}} D \, .$$

For any $n \geq 1$, let $C_n = \wedge^n C_0$. Then $C_0 \subseteq C_1 \subseteq \ldots$, and we call $(C_i)_{i \geq 0}$ the coradical filtration of C. The fact that $C = \bigcup_{n \geq 0} C_n$ is shown in [Stu18].

We now start describing some preliminary objects in order to present the Milnor Moore theorem.

Definition A.4.3 (Universal enveloping algebra of a Lie algebra)**.** Given a Lie algebra $\mathfrak{G}$, we define the **universal enveloping algebra** $U(\mathfrak{G})$ as the quocient

$$T(\mathfrak{G})/_I \, ,$$

where $T(\mathfrak{G})$ is the *tensor algebra* on $\mathfrak{G}$, and I is the two sided ideal generated by all the relations of the form

$$a \otimes b - b \otimes a - [a, b] \, ,$$

for $a, b \in \mathfrak{G}$.

The universal enveloping algebra is usually regarded as the correct algebra over which one should study representations of a Lie algerba. A universal property of this construction can be found in [Hal15]. The following theorem is from [MM65] and deals with the structure of a Hopf algebra.

Theorem A.4.4 (Milnor Moore theorem)**.** If $H = \oplus_{n \geq 0} H_n$ is a graded connected co-commutative Hopf algebra, such that $\dim H_n < +\infty$, then it is isomorphic to $U(P(H))$. In particular, the isomorphism is $U(P(H)) \to H$ is the natural one, mapping $h \in P(H)$ to $h \in H$.

Appendix B

Monoidal categories

In the following we will provide some details on monoidal categories. The reader can also find this material in [AM10].

B.1 Monoids, comonoids, Hopf monoids

The notion of monoidal categories is driven by the motication of describing categories that have a notion of a product in its objects. Most famously, the category of sets admits the Cartesian product $\times$, and indeed this leads to a notion of a monoidal category. Other examples are the categories of $\mathbb{K}$-vector spaces with the tensor product $\otimes$, and the category of combinatorial presheaves together with the Cauchy product $\odot$.

Definition B.1.1 (Monoidal categories)**.** A **monoidal category** is a pair $(\mathcal{C}, \bullet)$ where $\mathcal{C}$ is a category and $\bullet : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor such that there is a natural isomorphism $\alpha : \mathrm{id}_{\mathcal{C}} \times \bullet \Rightarrow \bullet \times \mathrm{id}_{\mathcal{C}}$ that satisfies the following pentagon rule:

$$\begin{array}{ccccc} & & (A \bullet B) \bullet (C \bullet D) & & \\ & \overset{\alpha_{A\bullet B,C,D}}{\nearrow} & & \overset{\alpha_{A,B,C\bullet D}}{\searrow} & \\ ((A \bullet B) \bullet C) \bullet D & & & & A \bullet (B \bullet (C \bullet D)) \\ \downarrow{\scriptstyle \alpha_{A,B,C} \bullet \mathrm{id}_D} & & & & \uparrow{\scriptstyle \mathrm{id}_A \bullet \alpha_{B,C,D}} \\ A \bullet ((B \bullet C) \bullet D) & & \xrightarrow{\alpha_{A,B\bullet C,D}} & & (A \bullet (B \bullet C)) \bullet D \end{array} \tag{B.1}$$

and a distinguisehd object $\mathcal{E}$ of $\mathcal{C}$ that is equiped with natural isomorphisms

$$\lambda_A : A \to \mathcal{E} \bullet A, \qquad \rho_A : A \to A \bullet \mathcal{E}\,,$$

such that the following commutes

$$\begin{array}{ccc} (A \bullet \mathcal{E}) \bullet B & \xrightarrow{\alpha_{A,I,B}} & A \bullet (I \bullet B) \\ & {\scriptstyle \rho_A \bullet \mathrm{id}_B} \nwarrow \quad \nearrow {\scriptstyle \mathrm{id}_A \bullet \lambda_B} & \\ & A \bullet B & \end{array} \tag{B.2}$$

The pentagon diagram in (B.1) is a strengthening of the usual notion of associativity. Not only we require that there is an isomorphism $\alpha_{A,B,C} : (A \bullet B) \bullet C \to A \bullet (B \bullet C)$, but we also require this isomorphism to the natural in the variables A, B, C, as well as self-coherent in that it satisfies the pentagon rule.

On monoidal categories, we will be able to define a **monoid** and a **comonoid**, respectively generalizations of the notion of algebra and coalgebra.

A *braided monoidal category* is a monoidal category $(\mathcal{C}, \bullet)$ together with a *braiding* β. A braiding is a natural isomorphism $\beta : \bullet \Leftrightarrow (\bullet \circ \mathbf{twist})$ between the functors $\bullet : (A, B) \mapsto A \bullet B$ and $\bullet \circ \mathbf{twist} : (A, B) \mapsto B \bullet A$.

On a braided monoidal category is where we are able to define a **bimonoid** and a **Hopf monoid**.

So, for instance, the category of vector spaces over $\mathbb{K}$ is a braided monoidal category, where we take $\bullet$ to be the tensor product. The category of species is also a monoidal category under the *Cauchy product*: given two species h_1, h_2, we can define the species

$$h_1 \odot h_2[I] = \oplus_{I = A \sqcup B} h_1[A] \otimes h_2[B] \, .$$

Importantly, the category of combinatorial presheaves, defined in Definition 1.5.1, is also a braided monoidal category, with the Cauchy product $\odot$. In all these cases, the braiding is the expected natural isomorphism.

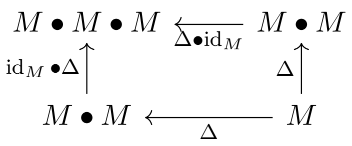
A **monoid** in $(\mathcal{C}, \bullet)$ is a triple (M, μ, ι) where $M \in \mathrm{Obj}(\mathcal{C})$, $\mu : M \bullet M \to M$ and $\iota : \mathcal{E} \to M$ are maps that make the following associativity and unit diagrams commute.

$$\begin{array}{ccc} M \bullet M \bullet M & \xrightarrow{\mu \bullet \mathrm{id}_A} & M \bullet M \\ \downarrow {\scriptstyle \mathrm{id}_M \bullet \mu} & & \downarrow {\scriptstyle \mu} \\ M \bullet M & \xrightarrow{\mu} & M \end{array} \tag{B.3}$$

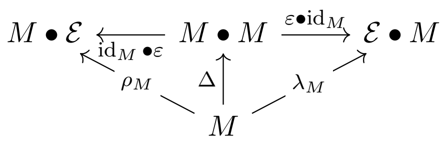
$$\begin{array}{ccccc} M \bullet \mathcal{E} & \xrightarrow{\mathrm{id}_M \bullet \iota} & M \bullet M & \xleftarrow{\iota \bullet \mathrm{id}_M} & \mathcal{E} \bullet M \\ & {\scriptstyle \rho_M} \nwarrow & \downarrow {\scriptstyle \mu} & \nearrow {\scriptstyle \lambda_M} & \\ & & M & & \end{array} \tag{B.4}$$

If we are ginve a monoidal category $\mathcal{C}$, then we write $\mathrm{Mon}(\mathcal{C})$ for the category of monoids of $\mathcal{C}$, where we only consider the morphisms that preserve the monoid structure maps.

A **comonoid** in $(\mathcal{C}, \bullet)$ is a triple (M, Δ, ε) where $M \in \mathrm{Obj}(\mathcal{C})$, $\Delta : M \to M \bullet M$ and $\varepsilon : M \to \mathcal{E}$ are maps that satisfy make the following coassociativity and counit diagrams commute



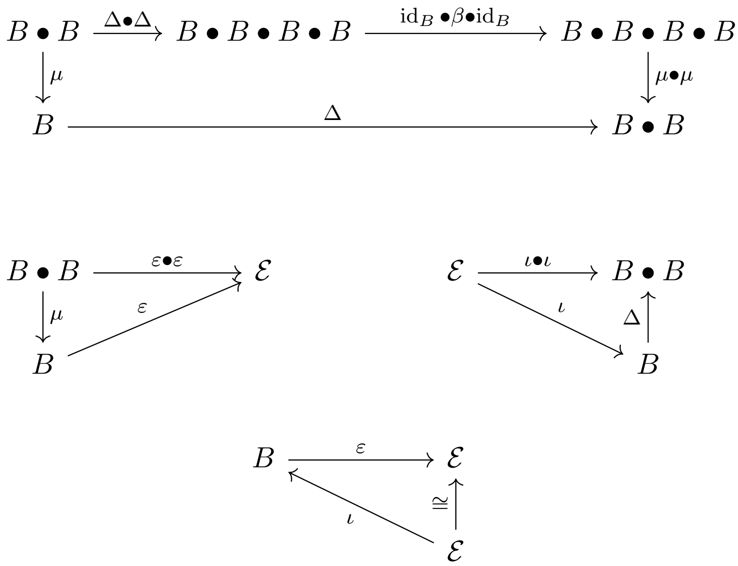
(B.5)



(B.6)

In this way, monoids and comonoids in the monoidal category $(\mathrm{Vec}_{\mathbb{K}}, \bullet)$ correspond to algebras and coalgebras over $\mathbb{K}$. A monoid in the category of finite sets, `Set`, is a finite monoid in the usual sense, a set with an associative map and a unit.

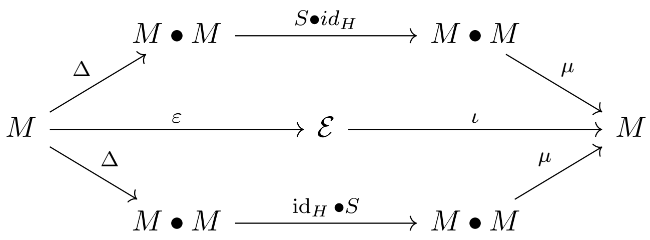
A **bimonoid** in the braided monoidal category $(\mathcal{C}, \bullet)$ is a 5-tuple $(M, \mu, \iota, \Delta, \varepsilon)$ where (M, μ, ι) is a monoid, (M, Δ, ε) is a comonoid, and the maps satisfy the following bimonoid diagrams:



(B.7)

Note that in the first diagram, we use the braiding in order to describe the compatibility between μ and Δ.

A **Hopf monoid** in the braided monoidal category $(\mathcal{C}, \bullet)$ is a 6-triple $(M, \mu, \iota, \Delta, \varepsilon, S)$ such that $(M, \mu, \iota, \Delta, \varepsilon)$ is a bimonoid, and $S : M \to M$ satisfies the following antipode diagram:



(B.8)

Bibliography

[AA17] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra. *preprint arXiv:1709.07504*, 2017.

[AAK03] Michael H Albert, Mike D Atkinson, and Martin Klazar. The enumeration of simple permutations. *J. Integer Seq*, 6(03.4):4, 2003.

[ABD10] Federico Ardila, Carolina Benedetti, and Jeffrey Doker. Matroid polytopes and their volumes. *Discrete & Computational Geometry*, 43(4):841–854, 2010.

[ABF18] Michael Albert, Mathilde Bouvel, and Valentin Féray. Two first-order logics of permutations. *preprint arXiv:1808.05459*, 2018.

[ABS06] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. Combinatorial Hopf algebras and generalized dehn–sommerville relations. *Compositio Mathematica*, 142(1):1–30, 2006.

[AF18] John Asplund and N. Bradley Fox. Enumerating cycles in the graph of overlapping permutations. *Discrete Mathematics*, 341(2):427–438, 2018.

[Agu00] Marcelo Aguiar. Infinitesimal hopf algebras. *Contemporary Mathematics*, 267:1–30, 2000.

[AM10] Marcelo Aguiar and Swapneel Arvind Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29. American Mathematical Society Providence, RI, 2010.

[APZ14] José Aliste-Prieto and José Zamora. Proper caterpillars are distinguished by their chromatic symmetric function. *Discrete Mathematics*, 315:158–164, 2014.

[Bar80] Jean-Pierre Barthélémy. An asymptotic equivalent for the number of total preorders on a finite set. *Discrete Mathematics*, 29(3):311–313, 1980.

[Bau97] Petra Bauer. The circuit polytope: Facets. *Mathematics of Operations Research*, 22(1):110–145, 1997.

[BBF+17] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, Mickaël Maazoun, and Adeline Pierrot. Universal limits of substitution-closed permutation classes. *preprint arxiv:1706.08333*, 2017.

[BBF+18] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, and Adeline Pierrot. The Brownian limit of separable permutations. *Ann. Probab.*, 46(4):2134–2189, 2018.

[BBF+19] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, Mickaël Maazoun, and Adeline Pierrot. Scaling limits of permutation classes with a finite specification: a dichotomy. *preprint arxiv:1903.07522*, 2019.

[BBFS19] Jacopo Borga, Mathilde Bouvel, Valentin Féray, and Benedikt Stufler. A decorated tree approach to random permutations in substitution-closed classes. *preprint arxiv:1904.07135*, 2019.

[BCL+08] Christian Borgs, Jennifer T Chayes, László Lovász, Vera T Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs i: Subgraph frequencies, metric properties and testing. *Advances in Mathematics*, 219(6):1801–1851, 2008.

[Bev19] David Bevan. Permutations with few inversions are locally uniform. *preprint arxiv:1908.07277*, 2019.

[Bir12] George D Birkhoff. A determinant formula for the number of ways of coloring a map. *The Annals of Mathematics*, 14(1/4):42–46, 1912.

[BJR09] Louis J Billera, Ning Jia, and Victor Reiner. A quasisymmetric function for matroids. *European Journal of Combinatorics*, 30(8):1727–1757, 2009.

[BK16] Felix Breuer and Caroline J Klivans. Scheduling problems. *Journal of Combinatorial Theory, Series A*, 139:59–79, 2016.

[BLP20] Jacopo Borga, Baptiste Louf, and Raul Penaguiao. The feasible region for fringe subtrees is a polytope. To be published, 2020.

[BO00] Egon Balas and Maarten Oosten. On the cycle polytope of a directed graph. *Networks: An International Journal*, 36(1):34–46, 2000.

[Bón16] Miklós Bóna. *Combinatorics of permutations.* Chapman and Hall/CRC, 2016.

[Bor19] Jacopo Borga. Local convergence for permutations and local limits for uniform ρ-avoiding permutations with $|\rho| = 3$. *Probability Theory and Related Fields, to appear (https://doi.org/10.1007/s00440-019-00922-4)*, 2019.

[Bou03] Nicolas Bourbaki. *Elements of mathematics: Algebra*. Springer, 2003.

[BP19] Jacopo Borga and Raul Penaguiao. The feasible region for consecutive patterns of permutations is a cycle polytope. *preprint arXiv:1910.02233*, 2019.

[BP20] Jacopo Borga and Raul Penaguiao. The feasible region for consecutive patterns in pattern avoiding permutations. To be published, 2020.

[Bro09] Christian Brouder. Quantum field theory meets hopf algebra. *Mathematische Nachrichten*, 282(12):1664–1690, 2009.

[BS09] Egon Balas and Rüdiger Stephan. On the cycle polytope of a directed graph and its relaxations. *Networks: An International Journal*, 54(1):47–55, 2009.

[BS17] Carolina Benedetti and Bruce E Sagan. Antipodes and involutions. *Journal of Combinatorial Theory, Series A*, 148:275–315, 2017.

[BS19] Jacopo Borga and Erik Slivken. Square permutations are typically rectangular. *preprint arxiv:1904.03080*, 2019.

[BV08] Welleda Baldoni and Michèle Vergne. Kostant partitions functions and flow polytopes. *Transformation Groups*, 13(3-4):447–469, 2008.

[BZ09] Nantel Bergeron and Mike Zabrocki. The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and co-free. *Journal of Algebra and Its Applications*, 8(04):581–600, 2009.

[CDG92] Fan Chung, Persi Diaconis, and Ron Graham. Universal cycles for combinatorial structures. *Discrete Math.*, 110(1-3):43–59, 1992.

[CFL58] Kuo Tsai Chen, Ralph H Fox, and Roger C Lyndon. Free differential calculus, iv. The quotient groups of the lower central series. *Annals of Mathematics*, pages 81–95, 1958.

[Cha00] Frédéric Chapoton. Algebres de hopf des permutahedres, associahedres et hypercubes. *Advances in Mathematics*, 150(2):264–275, 2000.

[CK99] Alain Connes and Dirk Kreimer. Hopf algebras, renormalization and noncommutative geometry. In *Quantum field theory: perspective and prospective*, pages 59–109. Springer, 1999.

[CP89] Collette R. Coullard and William R. Pulleyblank. On cycle cones and polyhedra. *Linear Algebra Appl.*, 114/115:613–640, 1989.

[CvW15] Soojin Cho and Stephanie van Willigenburg. Chromatic bases for symmetric functions. *preprint arXiv:1508.07670*, 2015.

[DG04] Geir Dahl and Luis Gouveia. On the directed hop-constrained shortest path problem. *Oper. Res. Lett.*, 32(1):15–22, 2004.

[Dit72] EJ Ditters. Curves and formal (co) groups. *Inventiones mathematicae*, 17(1):1–20, 1972.

[DK92] Gérard Duchamp and Daniel Krob. On the partially commutative shuffle product. *Theoretical computer science*, 96(2):405–410, 1992.

[Dok11] Jeffrey Samuel Doker. *Geometry of generalized permutohedra.* PhD thesis, UC Berkeley, 2011.

[DR00] Geir Dahl and Bjørnar Realfsen. The cardinality-constrained shortest path problem in 2-graphs. *Networks*, 36(1):1–8, 2000.

[Dri86] Vladimir Gershonovich Drinfeld. Quantum groups. *Zapiski Nauchnykh Seminarov POMI*, 155:18–49, 1986.

[Fér15] Valentin Féray. Cyclic inclusion-exclusion. *SIAM Journal on Discrete Mathematics*, 29(4):2284–2311, 2015.

[Foi12] Loïc Foissy. Free and cofree Hopf algebras. *Journal of Pure and Applied Algebra*, 216(2):480–494, 2012.

[FPT16] Loïc Foissy, Frédéric Patras, and Jean-Yves Thibon. Deformations of shuffles and quasi-shuffles. In *Annales de l'Institut Fourier*, volume 66, pages 209–237, 2016.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics.* Cambridge University press, 2009.

[Ges84] Ira M Gessel. Multipartite p-partitions and inner products of skew schur functions. *Contemp. Math*, 34(289-301):101, 1984.

[GHK$^+$17] Roman Glebov, Carlos Hoppen, Tereza Klimošová, Yoshiharu Kohayakawa, Daniel Král', and Hong Liu. Densities in large permutations and parameter testing. *European J. Combin.*, 60:89–99, 2017.

[GKL$^+$94] Israel Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S Retakh, and J-Y Thibon. Noncommutative symmetric functions. *preprint arxiv hep-th/9407124*, 1994.

[GP02] Gregory Gutin and Abraham P. Punnen. *The traveling salesman problem and its variations*, volume 12 of *Combinatorial Optimization.* Kluwer Academic Publishers, Dordrecht, 2002.

[GP13] Mathieu Guay-Paquet. A modular law for the chromatic symmetric functions of (3+ 1)-free posets. *preprint arXiv:1306.2400*, 34(289-301):101, 2013.

[GPS19] Vladimir Grujić, Marko Pešović, and Tanja Stojadinović. Weighted quasisymmetric enumerator for generalized permutohedra. *Journal of Algebraic Combinatorics*, pages 1–26, 2019.

[GR14] Darij Grinberg and Victor Reiner. Hopf algebras in combinatorics. *preprint arXiv:1409.8356*, 2014.

[Grö04] Martin Grötschel. *The sharpest cut: The impact of Manfred Padberg and his work*. SIAM, 2004.

[Gru16] Vladimir Grujić. Faces of graphical zonotopes. *preprint arXiv:1604.06931*, 2016.

[GS01] David D Gebhard and Bruce E Sagan. A chromatic symmetric function in noncommuting variables. *Journal of Algebraic Combinatorics*, 13(3):227–255, 2001.

[Hal15] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[Hat05] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, U.K., 2005.

[Haz01] Michiel Hazewinkel. The algebra of quasi-symmetric functions is free over the integers. *Advances in Mathematics*, 164(2):283–300, 2001.

[Hiv00] Florent Hivert. Hecke algebras, difference operators, and quasi-symmetric functions. *Advances in Mathematics*, 155(2):181–238, 2000.

[HKM$^+$13] Carlos Hoppen, Yoshiharu Kohayakawa, Carlos Gustavo Moreira, Balázs Ráth, and Rudini Menezes Sampaio. Limits of permutation sequences. *Journal of Combinatorial Theory, Series B*, 103(1):93–113, 2013.

[HLN$^+$14] Hao Huang, Nati Linial, Humberto Naves, Yuval Peled, and Benny Sudakov. On the 3-local profiles of graphs. *J. Graph Theory*, 76(3):236–248, 2014.

[HLN$^+$16] Hao Huang, Nati Linial, Humberto Naves, Yuval Peled, and Benny Sudakov. On the densities of cliques and independent sets in graphs. *Combinatorica*, 36(5):493–512, 2016.

[HM12] Brandon Humpert and Jeremy L Martin. The incidence Hopf algebra of graphs. *SIAM Journal on Discrete Mathematics*, 26(2):555–570, 2012.

[Hof00] Michael E Hoffman. Quasi-shuffle products. *Journal of Algebraic Combinatorics*, 11(1):49–68, 2000.

[Jam99] Ioan Mackenzie James. *History of topology*. Elsevier, 1999.

[Joh09] J. Robert Johnson. Universal cycles for permutations. *Discrete Math.*, 309(17):5264–5270, 2009.

[Joy81] André Joyal. Une théorie combinatoire des séries formelles. *Advances in mathematics*, 42(1):1–82, 1981.

[JR79] SA Joni and G-C Rota. Coalgebras and bialgebras in combinatorics. *Studies in Applied Mathematics*, 61(2):93–139, 1979.

[KKRW15] Richard Kenyon, Daniel Král', Charles Radin, and Peter Winkler. Permutations with fixed pattern densities. *preprint arxiv:1506.02340*, 2015.

[Knu68] Donald E Knuth. The art of computer programming, vol 1: Fundamental. *Algorithms. Reading, MA: Addison-Wesley*, 1968.

[KPV19] Sergey Kitaev, Vladimir N. Potapov, and Vincent Vajnovszki. On shortening u-cycles and u-words for permutations. *Discrete Appl. Math.*, 260:203–213, 2019.

[LdCS13] Abilio Lucena, Alexandre Salles da Cunha, and Luidi Simonetti. A new formulation and computational results for the simple cycle problem. *Electronic Notes in Discrete Mathematics*, 44:83–88, 2013.

[Lov12] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[LR98] Jean-Louis Loday and María O Ronco. Hopf algebra of the planar binary trees. *Advances in Mathematics*, 139(2):293–309, 1998.

[LS81] Alain Lascoux and Marcel-P Schützenberger. Le monoıde plaxique. *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, 109:129–156, 1981.

[Mac63] Saunders MacLane. Natural associativity and commutativity. *Rice Institute Pamphlet-Rice University Studies*, 49(4), 1963.

[Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

[ML14] Miguel A Méndez and Jean Carlos Liendo. An antipode formula for the natural Hopf algebra of a set operad. *Advances in Applied Mathematics*, 53:112–140, 2014.

[MM65] John W Milnor and John C Moore. On the structure of hopf algebras. *Annals of Mathematics*, pages 211–264, 1965.

[MP15] Frédéric Menous and Frédéric Patras. Right-handed hopf algebras and the prelie forest formula. *preprint arXiv:1511.07403*, 2015.

[MR95] Clauda Malvenuto and Christophe Reutenauer. Duality between quasi-symmetrical functions and the solomon descent algebra. *Journal of Algebra*, 177(3):967–982, 1995.

[NT06] Jean-Christophe Novelli and Jean-Yves Thibon. Polynomial realizations of some trialgebras. In *Formal power series and algebraic combinatorics*. Citeseer, 2006.

[OS14] Rosa Orellana and Geoffrey Scott. Graphs with equal chromatic symmetric functions. *Discrete Mathematics*, 320:1–14, 2014.

[Pat15] Rebecca Patrias. Antipode formulas for some combinatorial hopf algebras. *preprint arXiv:1501.00710*, 2015.

[Pen18] Raul Penaguiao. The kernel of chromatic quasisymmetric functions on graphs and hypergraphic polytopes. *preprint arXiv:1803.08824*, 2018.

[Pen20] Raul Penaguiao. Pattern Hopf algebras. *preprint arXiv:2001.09800*, 2020.

[Pil17] Vincent Pilaud. Which nestohedra are removahedra? *Revista Colombiana de Matemáticas*, 51(1):21–42, 2017.

[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *International Mathematics Research Notices*, 2009(6):1026–1106, 2009.

[PRW08] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Doc. Math*, 13(207-273):51, 2008.

[Rad79] David E Radford. A natural ring basis for the shuffle algebra and an application to group schemes. *Journal of Algebra*, 58(2):432–454, 1979.

[Raz07] Alexander A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[Raz08] Alexander A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17(4):603–618, 2008.

[Rom06] Dan Romik. Permutations with short monotone subsequences. *Adv. in Appl. Math.*, 37(4):501–510, 2006.

[RS06] Mercedes Rosas and Bruce Sagan. Symmetric functions in noncommuting variables. *Transactions of the American Mathematical Society*, 358(1):215–232, 2006.

[RVV16] Mustazee Rahman, Balint Virag, and Mate Vizer. Geometry of permutation limits. *preprint arxiv:1609.03891*, 2016.

[San19] Mario Sanchez. Möbius inversion as duality for Hopf monoids. In *Formal power series and algebraic combinatorics*. Citeseer, 2019.

[Saw96] Stephen Sawin. Links, quantum groups and tqfts. *Bulletin of the American Mathematical Society*, 33(4):413–445, 1996.

[Sch93] William R Schmitt. Hopf algebras of combinatorial structures. *Canadian Journal of Mathematics*, 45(2):412–428, 1993.

[Sch94] William R Schmitt. Incidence Hopf algebras. *Journal of Pure and Applied Algebra*, 96(3):299–330, 1994.

[Sch96] Astrid Christina Jeanne Scholtens. *S-typical curves in non-commutative Hopf algebras*. Vrije Universiteit Amsterdam, 1996.

[SS93] Richard P Stanley and John R Stembridge. On immanants of jacobi-trudi matrices and permutations with restricted position. 1993.

[SST15] Isaac Smith, Zane Smith, and Peter Tian. Symmetric chromatic polynomial of trees. *preprint arXiv:1505.01889*, 2015.

[Sta72] Richard P Stanley. *Ordered structures and partitions*, volume 119. American Mathematical Soc., 1972.

[Sta75] Richard P Stanley. Combinatorial reciprocity theorems. In *Combinatorics*, pages 307–318. Springer, 1975.

[Sta95] Richard P Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111(1):166–194, 1995.

[Sta00] Richard P Stanley. Enumerative combinatorics. vol. ii, the Wadsworth & Brooks/Cole Mathematics series, Wadsworth & Brooks, 2000.

[Sta07] Richard P Stanley. *Combinatorics and commutative algebra*, volume 41. Springer Science & Business Media, 2007.

[Sta09] Shannon Starr. Thermodynamic limit for the Mallows model on S_n. *J. Math. Phys.*, 50(9):095208, 15, 2009.

[Ste09] Rüdiger Stephan. Facets of the (s,t)-p-path polytope. *Discrete Appl. Math.*, 157(14):3119–3132, 2009.

[Sto93] Christopher R Stover. The equivalence of certain categories of twisted lie and Hopf algebras over a commutative ring. *Journal of pure and applied Algebra*, 86(3):289–326, 1993.

[Stu18] Benedikt Stufler. Hopf algebras. 2018.

[Sup19] Mariel Supina. The Hopf monoid of orbit polytopes. *preprint arxiv:1904.08437*, 2019.

[Swe69] M.E. Sweedler. *Hopf algebras*. Mathematics lecture note series. W. A. Benjamin, 1969.

[Tak71] Mitsuhiro Takeuchi. Free Hopf algebras generated by coalgebras. *Journal of the Mathematical Society of Japan*, 23(4):561–582, 1971.

[Var14] Yannic Vargas. Hopf algebra of permutation pattern functions. In *Discrete Mathematics and Theoretical Computer Science*, pages 839–850. Discrete Mathematics and Theoretical Computer Science, 2014.

[VdL19] Tim Van der Linden. On split extensions of bialgebras. Presentation, 2019.

[W+36] Margarete C Wolf et al. Symmetric functions of non-commutative elements. *Duke Mathematical Journal*, 2(4):626–637, 1936.

[Whi32] Hassler Whitney. The coloring of graphs. *Annals of Mathematics*, pages 688–718, 1932.

[Whi16] Jacob White. The Hopf monoid of coloring problems. *preprint arXiv:1611.04079*, 2016.

[Wil02] Herbert S Wilf. The patterns of permutations. *Discrete Mathematics*, 257(2-3):575–583, 2002.

[Zei99] Doron Zeilberger. Proof of a conjecture of Chan, Robbins, and Yuen. *Electron. Trans. Numer. Anal*, 9(147-148):1–2, 1999.

[Zie12] Günter M Ziegler. *Lectures on polytopes*, volume 152. Springer Science & Business Media, 2012.