Feasible regions and permutation patterns

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(This talk is based on joint work with Jacopo Borga.)

Abstract

We study proportions of consecutive occurrences of permutation patterns of a given size by analysing the *feasible region*. We show that this feasible region is a polytope, more precisely the cycle polytope of a specific graph called *overlap graph*. This allows us to compute the dimension, vertices and faces of the polytope. We further develop the analysis of the feasible region by introducing the *consecutive patterns feasible reagion for* $\mathcal{C}$, where $\mathcal{C}$ is a permutation class. We give a precise description of this feasible regions whenever $\mathcal{C} = \mathrm{Av}(\tau)$ for τ a permutation of size 3 and for τ a monotone permutation. We also show that, whenever $\mathcal{C} = \mathrm{Av}(\tau)$, the resulting feasible region has dimension $|\mathrm{Av}_k(\tau)| - |\mathrm{Av}_{k-1}(\tau)|$. Finally, we conjecture that in these cases, the feasible region is a polytope.

1 Classical feasible regions

We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}$ the space of all permutations, and by $\widetilde{\text{occ}}(\pi, \sigma)$ (resp. $\widetilde{\text{c-occ}}(\pi, \sigma)$) the proportion of classical occurrences (resp. consecutive occurrences) of a permutation π in σ. In [**?**], the feasible region for **classical** patterns was defined:

$$
\begin{aligned}
clP_k &:= \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \quad (1) \\
&= \left\{ (\Delta_\pi(P))_{\pi\in\mathcal{S}_k} \middle| P \text{ is a permuton} \right\} ,
\end{aligned}
$$

The feasible region was first studied for some particular families of patterns instead of the whole $\mathcal{S}_k$. More precisely, given a list of finite sets of permutations $(\mathcal{P}_1, \dots, \mathcal{P}_\ell)$, the authors in [**?**] considered the *feasible region* for $(\mathcal{P}_1, \dots, \mathcal{P}_\ell)$:

$$
\left\{ \vec{v} \in [0,1]^\ell \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \sum_{\tau\in\mathcal{P}_i} \widetilde{\text{occ}}(\tau, \sigma^m) \to \vec{v}_i, \text{ for } i \in [\ell] \right\} .
$$

They first studied the simplest case when $\mathcal{P}_1 = \{12\}$ and $\mathcal{P}_2 = \{123, 213\}$ showing that the corresponding feasible region for $(\mathcal{P}_1, \mathcal{P}_2)$ is the region of the square $[0,1]^2$ bounded from below by the curve parameterized by $(2t - t^2, 3t^2 - 2t^3)_{t\in[0,1]}$ and from above by the curve parameterized by $(1 - t^2, 1 - t^3)_{t\in[0,1]}$ (see [**?**, Theorem 13]).

The set clP_k was also studied in [?], even though with a different goal. There, the notion of permutons was leveraged to establish a lower bound for the dimension of this feasible region. There is still no improvement on this bound, but an upper bound was indirectly established in [?] as the number of so-called *Lyndon permutations* of size at most k, whose set we denote $\mathcal{L}_k$. By analyzing smaller cases, we conjecture that this upper bound is tight:

Conjecture 1. *The feasible region clP_k is full-dimensional inside a manifold of dimension $|\mathcal{L}_k|$.*

2 Feasible regions for consecutive patterns

We introduce the parallel study of feasible regions using **consecutive** occurences as our statistic. This has several motivations and advantages. First, the study of consecutive occurrences led to a notion of permutation limit in [?], called *random infinite rooted shift-invariant permutations*, much in the same way as the study of classical permutation patterns led to the permuton limit of permutations, introduced in [?]. Thus, we expect to be able to leverage this new notion of permutation limit to extract results for the feasible region.

Second, the resulting feasible region is much more tractable, and arises as a suitable application of **cycle polytopes**, a polytope construction for graphs.

The feasible region for consecutive occurrences is thus defined in [?] as:

$$P_k := \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \tag{2}$$
$$= \left\{ (\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}_k} \middle| \sigma^\infty \text{ is a random infinite rooted } \textit{shift-invariant} \text{ permutation} \right\} .$$

Definition 2.1. The graph $\mathcal{O}v(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for every $\pi \in \mathcal{S}_k$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

The overlap graph $\mathcal{O}v(4)$ is displayed in **??**. Recall that a simple cycle on a directed graph is a cycle that dos not repeat vertices.

Definition 2.2. Let $G = (V, E)$ be a directed multigraph. For each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_\mathcal{C} \in \mathbb{R}^E$ so that

$$(\vec{e}_\mathcal{C})_e := \frac{\#\text{ of occurrences of } e \text{ in } \mathcal{C}}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E.$$

We define $P(G) := \text{conv}\{\vec{e}_\mathcal{C} | \mathcal{C} \text{ is a simple cycle of } G\}$, the *cycle polytope* of G.

Our first main result relates the feasible region for consecutive occurrences P_k with the cycle polytope of the overlap graph. An example for $k = 3$ is displayed in **??**.

Theorem 2. *P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. Its dimension is $k! - (k-1)!$ and its vertices are given by the simple cycles of $\mathcal{O}v(k)$.*

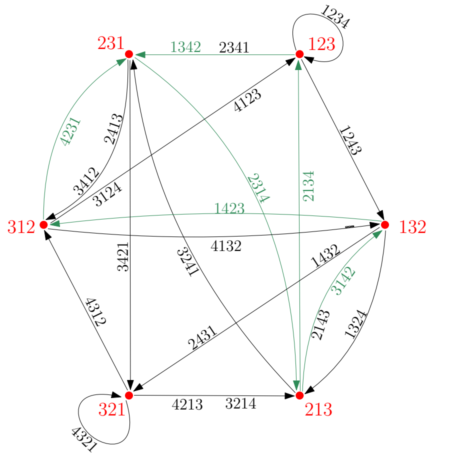

Figure 1: The overlap graph $\mathcal{O}v(4)$. The six vertices are painted in red and the edges are drawn as labeled arrows. Note that in order to obtain a clearer picture we did not draw multiple edges, but we use multiple labels.

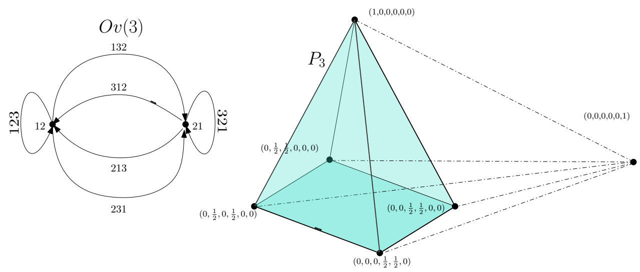

Figure 2: **Left:** The overlap graph $\mathcal{O}v(3)$ (see **??**). **Right:** The four-dimensional polytope P_3 given by the six patterns of size three (see (**??**) for a precise definition). We highlight in light-blue one of the six three-dimensional faces of P_3. This face is a pyramid with square base. The polytope itself is a four-dimensional pyramid, whose base is the highlighted face. From **??** we have that P_3 is the cycle polytope of $\mathcal{O}v(3)$.

3 Pattern avoidance

We propose here a further development of the study of feasible regions, where the feasible points are now obtained from a sequence of permutations in a specified permutation class. The feasible regions thus obtained, called *consecutive patterns feasible region*, are smaller.

Let us now consider B a collection of permutations. We introduced the *consecutive patterns feasible region for* $\mathrm{Av}(B)$, defined by

$$P_k^B := \{\vec{v} \in [0,1]^{\mathcal{S}_k} \big| \exists (\sigma^m)_{m \in \mathbb{Z}_{\geq 1}} \in \mathrm{Av}(B)^{\mathbb{Z}_{\geq 1}} \text{ such that} \\ |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\}.$$

In words, the region P_k^B is formed by the $k!$-dimensional vectors $\vec{v}$ for which there exists a sequence of permutations in $\mathrm{Av}(B)$ whose size tends to infinity and whose proportion of consecutive patterns of size k tends to $\vec{v}$. For simplicity, whenever $B = \{\tau\}$ we simply write P_k^τ for $P_k^{\{\tau\}}$ (and we use the same convention for related notation). Recall that $\oplus$ and $\ominus$ are classical binary operations on permutations.

Theorem 3.1. *[?, Theorem 1.1] Fix $k \in \mathbb{Z}_{\geq 1}$ and a set of patterns $B \subset \mathcal{S}$ such that the family* $\mathrm{Av}(B)$ *is closed either for the $\oplus$ operation or $\ominus$ operation. The feasible region P_k^B is closed and convex. Moreover,*

$$\dim(P_k^B) = |\operatorname{Av}_k(B)| - |\operatorname{Av}_{k-1}(B)|.$$

Remark that, whenever $B = \{\tau\}$ is a singleton, this permutation class is either closed for the $\oplus$ operation (whenever τ is $\oplus$ indecomposible) or closed for the $\ominus$ operation (whenever τ is $\ominus$ indecomposible). Therefore, this and the following theorems apply when B is a singleton.

Given that we have a dimension, it is natural to wonder if the restricted feasible regions can be described geometrically. Thus, we get the following conjecture:

Conjecture 3.2. *Fix $k \in \mathbb{Z}_{\geq 1}$ and a sets of patterns $B \subset \mathcal{S}$ such that the family* $\mathrm{Av}(B)$ *is closed either for the $\oplus$ operation or $\ominus$ operation. The feasible region P_k^B is a polytope.*

The following section will be dedicated to establishing this conjecture for some particular cases. As we will see latter in **??**, even these particular cases do not enjoy of a simple polytopal description so this conjecture is rather ambitious.

4 Pattern avoidance: 312 and monotone patterns

We now introduce a new overlap graph in order to study the restricted feasible regions. Of particular interest are the feasible regions that arise for $B = \{\tau\}$.

Definition 4.1. Fix a set of patterns $B \subset \mathcal{S}$ and $k \in \mathbb{Z}_{\geq 1}$. The *overlap graph* $\mathcal{O}v^B(k)$ is a directed multigraph with labelled edges, where the vertices are elements of $\mathrm{Av}_{k-1}(B)$ and for every $\pi \in \mathrm{Av}_k(B)$ there is an edge labelled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

For an example with $k = 3$ see the top of **??**.

Theorem 4.2. *[?, Theorem 1.14] Fix $k \in \mathbb{Z}_{\geq 1}$. The feasible region P_k^{312} is the cycle polytope of the overlap graph $\mathcal{O}v^{312}(k)$.*

To help us with the case where τ is a monotone permutation, a different overlap graph $\mathfrak{C}\mathcal{O}v^{\nu_n}(k)$ was introduced in [?]. This new graph is the one that helps us establish our main result for the monotone patterns case, which differs slightly from the result on P_k^{312} above. Indeed, in [?] we observe that in general $P_k^{\nu_n}$ is different from the cycle polytope of the overlap graph $\mathcal{O}v^{\nu_n}(k)$.

Theorem 4.3. *[?, Theorem 1.16] Fix $\nu_n = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$. There exists a projection map Π, explicitly described in [?], such that the consecutive patterns feasible region $P_k^{\nu_n}$ is the Π-projection of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\nu_n}(k)$. That is,*

$$P_k^{\nu_n} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\nu_n}(k)))\,.$$

An instance of the result stated in **??** is depicted on the bottom part of **??**. We remark that **??** highlights what kind of difficulties can be encountered in proving **??**, because it shows that we cannot hope for a general description of the feasible reagion via cyclic polytopes.

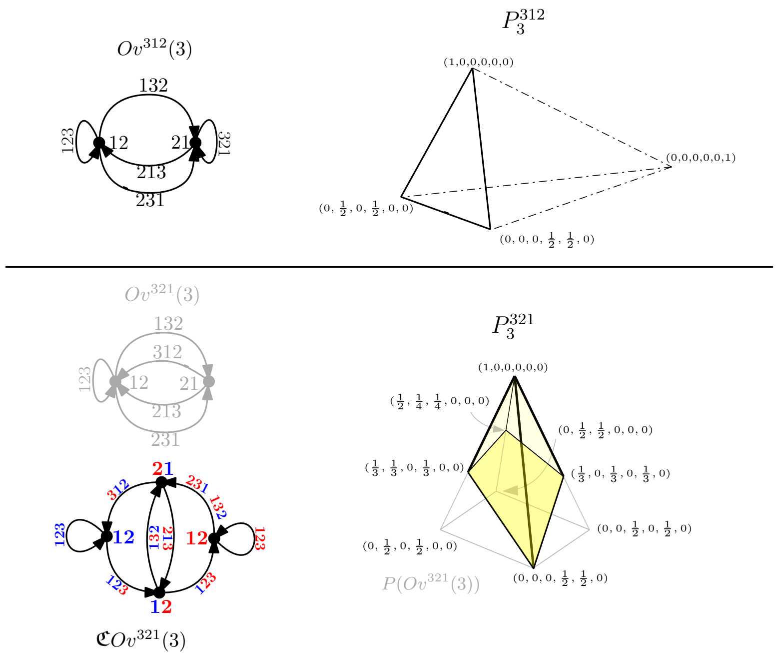

Figure 3: **Top:** The overlap graph $\mathcal{O}v^{312}(3)$ and the three-dimensional polytope P_3^{312}. Note that $P_3^{312} \subset P_3$. From **??** we have that P_3^{312} is the cycle polytope of $\mathcal{O}v^{312}(3)$. **Bottom:** In light grey the overlap graph $\mathcal{O}v^{321}(3)$ and the corresponding three-dimensional cycle polytope $P(\mathcal{O}v^{321}(3))$, that is strictly larger than P_3^{321}. The latter feasible region is highlighted in yellow. From **??** we have that P_k^{321} is the projection of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{321}(3)$. This graph is plotted in the bottom-left side. Note that $P_3^{321} \subset P_3$.

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