Homework Assignment 1 solution - The tensor product

Hopf algebras - Spring Semester 2018

What follows are proposed solutions that are not too thorough. Where some necessary details are lacking, a bold will be used. The student is encourage to fill in the details autonomously when a claim in bold does not seem to follow naturally.[1]

Exercise 1

Consider a module $X \in \mathcal{M}_R$ and two-sided ideals $I, J \subset R$.

a) Show that $X \otimes_R R/I \simeq X/XI$ in $\mathcal{M}_{R/I}$.

b) Show that $R/I \otimes_R R/J \simeq R/(I+J)$ in $\mathcal{M}_R$.

Sketch of proof of 1.a). We will construct two R-linear maps $\phi : X/XI \to X \otimes_R R/I$ and $\tilde{\psi} : X \otimes_R R/I \to X/XI$ that are each other inverses, which concludes the proof.

For the first map, take $\phi(x + XI) = x \otimes (1 + I)$, which **is well defined** (i.. does not depend on the choice of representative x) and R**-linear**, as one can easily show.

To construct $\tilde{\psi}$, recall the universal property for $X \otimes_R R/I$: For any middle linear map $\psi : X \times R/I \to M$ there exists a unique R-module morphism $\tilde{\psi}$ that makes the following commute:

$$\begin{array}{ccc} X \times R/I & \xrightarrow{\otimes} & X \otimes_R R/I \\ & \searrow^{\psi} & \downarrow{\scriptstyle \exists !\, \tilde{\psi}} \\ & & M \end{array} \tag{1}$$

Choose $M = X/XI$ and $\psi = ((x, r + I) \mapsto xr + I)$. After we show that **the map is well defined** and **middle linear**, we obtain an R-linear map $\tilde{\psi} : X \otimes_R R/I$.

Remains only to observe that $\tilde{\psi} \circ \phi = \mathrm{id}$, which is trivial, and $\phi \circ \tilde{\psi} = \mathrm{id}$, which can be computed for the generators of $X \otimes_R R/I$ as follows:

$$\phi \circ \tilde{\psi}(x \otimes (r + I)) = \phi(xr + I) = xr \otimes (1 + I) = x \otimes (r + I)\,.$$

This concludes the isomorphism. □

Sketch of proof of 1.b). This is very similar to 1.a), as we will find maps $\phi : R/(I+J) \to R/I \otimes_R R/J$ and $\tilde{\psi} : R/I \otimes_R R/J \to R/(I+J)$ that are R-linear and inverse of each other.

[1] If typos or incorrections are found please write to *raul.penaguiao@math.uzh.ch*

We define $\phi = (x + I + J \mapsto x + I \otimes 1 + J)$. Note that this is well defined, because if $x = x' + x_i + x_j$, with $x_i \in I$ and $x_j \in J$, then we have

$$\begin{aligned}\phi(x + I + J) &= x' + x_i + x_j + I \otimes 1 + J = (x' + I \otimes 1 + J) + (x_j + I \otimes 1 + J) \\ &= \phi(x' + I + J) + (1 + I \otimes x_j + J) = \phi(x' + I + J) + 0\,,\end{aligned} \tag{2}$$

so the definition of ϕ does not depend on the representative x. Additionally, this **is R-linear**.

To define $\tilde{\psi}$, let $\psi : R/I \times R/J$ be given by $(x_1 + I, x_2 + J) \mapsto x_1x_2 + I + J$. This **is well defined** and also **is middle linear**.

The fact that these maps are inverses of each other **is easy to check**. □

Exercise 2

a) Let $\iota : \mathbb{Z}/(2) \to \mathbb{Z}/(4)$ be the unique injective group homomorphism. Compute $\mathrm{id} \otimes_{\mathbb{Z}} \iota$ that maps $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4)$.

b) For two integers m, n, compute $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$.

c) Compute, for an abelian group G and an integer n, the tensor $G \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$.

d) An abelian group G is a torsion abelian group if for every element $g \in G$ there is a natural number n such that $ng = 0$. Show that for any torsion group G we have $G \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Solutions of exercise 2.

a) This is the zero map, as $\mathrm{id} \otimes_{\mathbb{Z}} \iota(\bar{1} \otimes \bar{1}) = \bar{1} \otimes \bar{2} = \bar{2} \otimes \bar{1} = 0$.

b) This is $\mathbb{Z}/(d)$, where $d = \gcd(m, n)$. Note exercise 1.b).

c) This is G/nG. Note exercise 1.a).

d) Any generator is the zero element, as $g \otimes q = g \otimes n\frac{q}{n} = gn \otimes \frac{q}{n} = 0$.

□

Exercise 3

Let X, Y be vector spaces over k, and $U \subseteq X$, $V \subseteq Y$ be subspaces. Let $p_U : U \to X$ and $p_V : V \to Y$ be the canonical inclusions.

Show that

$$\ker p_U \otimes_k p_V = X \otimes_k V + U \otimes_k Y\,.$$

Sketch of proof. Let $Z = X \otimes_k V + U \otimes_k Y$. Note that $p_U \otimes_R p_V$ is surjective, so it suffices to show that the map $\overline{p_U \otimes_R p_V} : (X \otimes_R Y)/Z \to X/U \otimes_R Y/V$ is well defined and injective.

The fact that it is well defined is trivial, since **we can observe** that $Z \subset \ker p_U \otimes_R p_V$. To show that it is injective, it is enough to show that $\tilde{\phi}$ defined via **the middle linear map**

$$\phi : (x + U, y + V) \mapsto x \otimes y + Z\,,$$

is the left inverse of $\overline{p_U \otimes_R p_V}$, i.e. $\phi \circ \overline{p_U \otimes p_V} = \mathrm{id}$.

It immediately follows that $\overline{p_U \otimes p_V}$ is injective, as desired. □

Exercise 4

Find modules M, N over a ring R such that $M \otimes_{\mathbb{Z}} N \not\simeq M \otimes_R N$ as $\mathbb{Z}$-modules.

Proof. Take $R = \mathbb{Z}[i]$, the ring of Gaussian integers, and take $M = N = R$. Clearly $M \otimes_{\mathbb{Z}} N \simeq \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}^2 \simeq \mathbb{Z}^4$.

However $M \otimes_R N \simeq R$ as R-modules, so $M \otimes_R N \simeq \mathbb{Z}^2$ as $\mathbb{Z}$-modules. □

Exercise 5

Let X, Y be k-vector spaces

a) Show that the map $(x, f) \mapsto (y \mapsto f(y)x)$ defines a linear map from $X \times Y^*$, which gives rise to a linear map $\phi_{X,Y} : X \otimes_k Y^* \to \mathrm{Hom}_k(Y, X)$. Additionally, show that if either X or Y are finite dimensional, then $\phi_{X,Y}$ is an isomorphism.

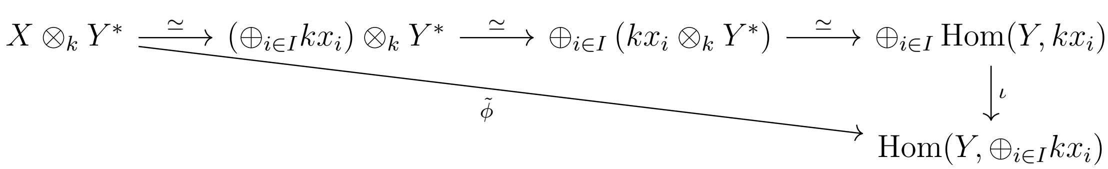
b) Show that the map $(x, f) \mapsto f(x)$ defines a bilinear map from $X \times X^*$, which gives rise to a linear map $e_X : X \otimes_k X^* \to k$.

c) Define $\mathrm{Tr}_X = e_X \circ \phi^{-1}_{X,X} : \mathrm{End}_k(X) \to k$, for X finite dimensional. Show that if we take $F \in \mathrm{End}_k(X)$ and $G \in \mathrm{End}_k(Y)$ then

$$\mathrm{Tr}_{X\otimes Y}(F \otimes G) = \mathrm{Tr}_X(F)\,\mathrm{Tr}_Y(G)\,.$$

Sketch of a). That the map $\phi = (x, f) \mapsto (y \mapsto f(y)x)$ **satisfies the middle linear property** implies that it lifts to an k-linear map $\tilde{\phi} : X \otimes_R Y^* \to \mathrm{Hom}_R(Y, X)$.

This map, once chosen a basis $\{x_I\}_{i\in I}$ of X, **can be easily seen** to be the following composition

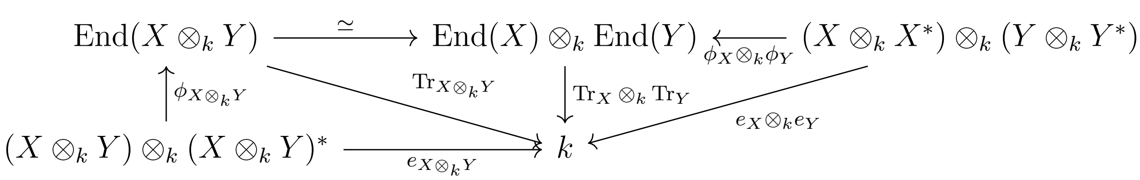


(3)

where ι is the canonical inclusion. If X is finite dimensional, then $f \mapsto (x_i^* \circ f)_{i\in I}$ is an inverse of ι. If Y is finite dimensional we can see that ι **is also invertible**. In both cases, we conclude that $\tilde{\phi}$ is an isomorphism. □

In b), we need only to check the middle linear property of the given map and recall the universal property.

Sketch of c). The following diagram commutes for X, Y finite dimensional:



(4)

by simply checking that the lower triangle and the leftmost triangle commute by definition of Tr, whereas the pentagram commutes because by picking basis for X and Y, and the corresponding dual basis via $X^* \otimes_k Y^* \simeq (X \otimes_k Y)^*$, direct computations **imply directly the commutativity**. It follows that the inner triangle commutes, as envisaged. $\square$

Exercise 6

Given M, N left modules over a ring R, show that the functors $\mathrm{Hom}(-, M)$ and $\mathrm{Hom}(N, -)$ are both left exact. I.e. whenever $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ and $0 \to X' \to Y' \to Z$ are exact sequences of left R-modules, then the following are exact:

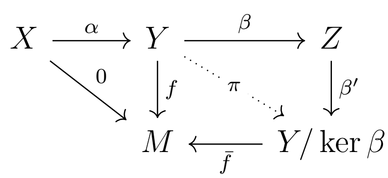
$$0 \to \mathrm{Hom}_R(Z, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Y, M) \xrightarrow{\alpha_*} \mathrm{Hom}_R(X, M)\,,$$

$$0 \to \mathrm{Hom}_R(N, X) \to \mathrm{Hom}_R(N, Y) \to \mathrm{Hom}_R(N, Z)\,.$$

Proof. For the first exactness, it suffices to show the following three properties:

- The composition relation $\alpha_* \circ \beta_* = 0$ holds, which **is trivial**.
- The map β_* is injective, which follows from the fact that if $\beta_*(f) = 0$ then $\beta \circ f = 0$, which implies that $f = 0$ because β is epimorphism.
- The arguably hardest part of this exercise, which is to show that there is no $f \in \ker \alpha_* \setminus \mathrm{im}\, \beta_*$, thereby showing, together with $\alpha_* \circ \beta_* = 0$, that $\ker \beta_* = \mathrm{im}\, \alpha_*$.

  To show the last item, note that $\ker \beta = \mathrm{im}\, \alpha \subset \ker f$ so we can find $\bar{f} : Y/\ker \beta \to M$ that makes the following diagram commute:

  (5)

  where β' is the left inverse of the injective map $\bar{\beta} : Y/\ker \beta \to Z$.

  But $f = \beta_*(\bar{f} \circ \beta')$, contradicting the fact that $f \notin \mathrm{im}\, \beta_*$.

the second exactness **follows similarly**. $\square$