The feasible regions for consecutive patterns of pattern avoiding permutations

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Abstract

In a recent article we introduce the feasible region for consecutive patterns of permutations and we show that this region is a cycle polytope. Here, we study its pattern avoiding counterpart. More precisely, given a permutation class $\mathcal{C}$, we study the limit of proportions of consecutive patterns on large permutations of $\mathcal{C}$. These limits form a region, called the *pattern avoiding feasible region for* $\mathcal{C}$. We show that, when $\mathcal{C}$ is the class of τ-avoiding permutations, with either τ of size three or τ a monotone pattern, the pattern avoiding feasible region for $\mathcal{C}$ is still a polytope. Finally, we also determine its dimension, developing, in the monotone pattern case, a new tool for this computation.

We further show some general results for the pattern avoiding feasible region for any permutation class $\mathcal{C}$ and we conjecture a general formula for its dimension.

1 Introduction

1.1 The pattern avoiding feasible regions

In [BP19] the authors (us :)) introduce the *feasible region* for consecutive patterns, that is defined for all $k \in \mathbb{Z}_{\geq 1}$ as

$$P_k := \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m \in \mathbb{Z}_{\geq 1}} \in \mathcal{S}^{\mathbb{Z}_{\geq 1}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\}, \quad (1)$$

where $\mathcal{S}_k$ denotes the set of permutations of size k, $\mathcal{S}$ the set of all permutations, and $\widetilde{\text{c-occ}}(\pi, \sigma)$ the proportion of consecutive occurrences of a pattern π in a permutation σ (see Section 1.5 for notation and basic definitions). We refer the reader to [BP19, Section 1.1] for motivations to investigate this region and to [BP19, Section 1.2] for a summary of the related literature.

In this paper we study a restricted version of the feasible region for consecutive patterns. Given a set of patterns $B \subset \mathcal{S}$, we denote by $\text{Av}_n(B)$ the set of B-avoiding permutations of size n and by $\text{Av}(B) := \bigcup_{n \in \mathbb{Z}_{\geq 1}} \text{Av}_n(B)$ the set of B-avoiding permutations of arbitrary finite size. We consider the *pattern avoiding feasible region for consecutive patterns for* $\text{Av}(B)$ defined by

$$P_k^{\text{Av}(B)} := \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m \in \mathbb{Z}_{\geq 1}} \in \text{Av}(B)^{\mathbb{Z}_{\geq 1}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\}.$$

For different choices of the class $\text{Av}(B)$, we refer to these regions as *pattern avoiding feasible regions*. Even though these are natural objects, we discuss here some motivations that lead us to investigate them.

The study of limits of pattern avoiding permutations is a very active field in combinatorics and discrete probability theory. There are two main ways of investigating those limits:

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- The most classical one is to look at the limits of various statistics for pattern avoiding permutations. For instance, the limit distribution of the longest increasing subsequences in uniform pattern avoiding permutations have been considered in [DHW03, MY19]. Another example is the general problem of studying the limiting distribution of the number of occurrences of a fixed pattern π in a uniform random permutation belonging to a fixed class when the size tends to infinity (see for instance Janson [Jan18a, Jan17, Jan18b], where the author studied this problem in the model of uniform permutations avoiding a fixed family of patterns of size three). Many other statistics have been considered, for instance in [BKL$^+$18] the authors studied the distribution of ascents, descents, peaks, valleys, double ascents, and double descents over pattern avoiding permutations.

- The second way is to look at the limiting shape of large pattern avoiding permutations. Two main notions of convergence for permutations have been defined: a global notion of convergence (called permuton convergence, [HKM$^+$13]) and a local notion of convergence (called Benjamini–Schramm convergence, [Bor20b]). For an intuitive explanation of them we refer the reader to our previous paper [BP19], where additional references can be found. We just mention here that permuton convergence is equivalent to the convergence of all *pattern density* statistics (see [BBF$^+$19, Theorem 2.5]); and Benjamini–Schramm convergence is equivalent to the convergence of all *consecutive pattern density* statistics (see [Bor20b, Theorem 2.19]), which are the objects of this paper.

The study of the pattern avoiding feasible regions is strongly related to both ways of studying limits of pattern avoiding permutations. For the first one, the statistic that we consider is the number of consecutive occurrences of a pattern. For the second one, the relation is with Benjamini–Schramm limits. In particular, having a precise description of the regions $P_k^{\mathrm{Av}(B)}$ for all $k \in \mathbb{Z}_{\geq 1}$ determines all the Benjamini–Schramm limits that can be obtained through sequences of permutations in $\mathrm{Av}(B)$.

Another orthogonal motivation for investigating the pattern avoiding feasible regions is the problem of *packing patterns* in pattern avoiding permutations. The classical question of *packing patterns* in permutations is to describe the maximum number of occurrences of a pattern π in any permutation of $\mathcal{S}_n$ (see for instance [AAH$^+$02, Bar04, Pri97]). More recently, the same question restricted to pattern avoiding permutations, i.e. to describe the maximum number of occurrences of a pattern π in any pattern avoiding permutation, has been addressed by Pudwell [Pud20]. Describing the pattern avoiding feasible region $P_k^{\mathrm{Av}(B)}$ gives an answer to the question of finding the asymptotic maximum number of consecutive occurrences of a pattern $\pi \in \mathcal{S}_k$ in large permutations of $\mathrm{Av}(B)$.

1.2 Previous results on the standard feasible region for consecutive patterns

Before presenting our results on the pattern avoiding feasible regions, we recall two key definitions from [BP19] and review some results presented in this work.

Definition 1.1. The *overlap graph* $\mathcal{O}v(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for every $\pi \in \mathcal{S}_k$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

For an example for $k = 3$ see the top-left side of Fig. 1. Given a permutation π, we denote the pattern induced by the first $k-1$ indices by $\mathrm{beg}_{k-1}(\pi)$ and the pattern induced by the last $k-1$ indices by $\mathrm{end}_{k-1}(\pi)$.

Definition 1.2. Let $G = (V, E)$ be a directed multigraph. For each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^E$ so that

$$(\vec{e}_{\mathcal{C}})_e \coloneqq \frac{\#\text{ of occurrences of } e \text{ in } \mathcal{C}}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E.$$

We define the *cycle polytope* of G to be the polytope $P(G) \coloneqq \mathrm{conv}\{\vec{e}_{\mathcal{C}} | \mathcal{C} \text{ is a simple cycle of } G\}$.

We recall some results from [BP19].

Proposition 1.3 (Proposition 1.7 in [BP19])**.** *The cycle polytope of a strongly connected directed multigraph $G = (V, E)$ has dimension $|E| - |V|$.*

Theorem 1.4 (Theorem 1.6. in [BP19])**.** *P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. Its dimension is $k! - (k-1)!$ and its vertices are given by the simple cycles of $\mathcal{O}v(k)$.*

An instance of the result above is depicted on the top of Fig. 1.

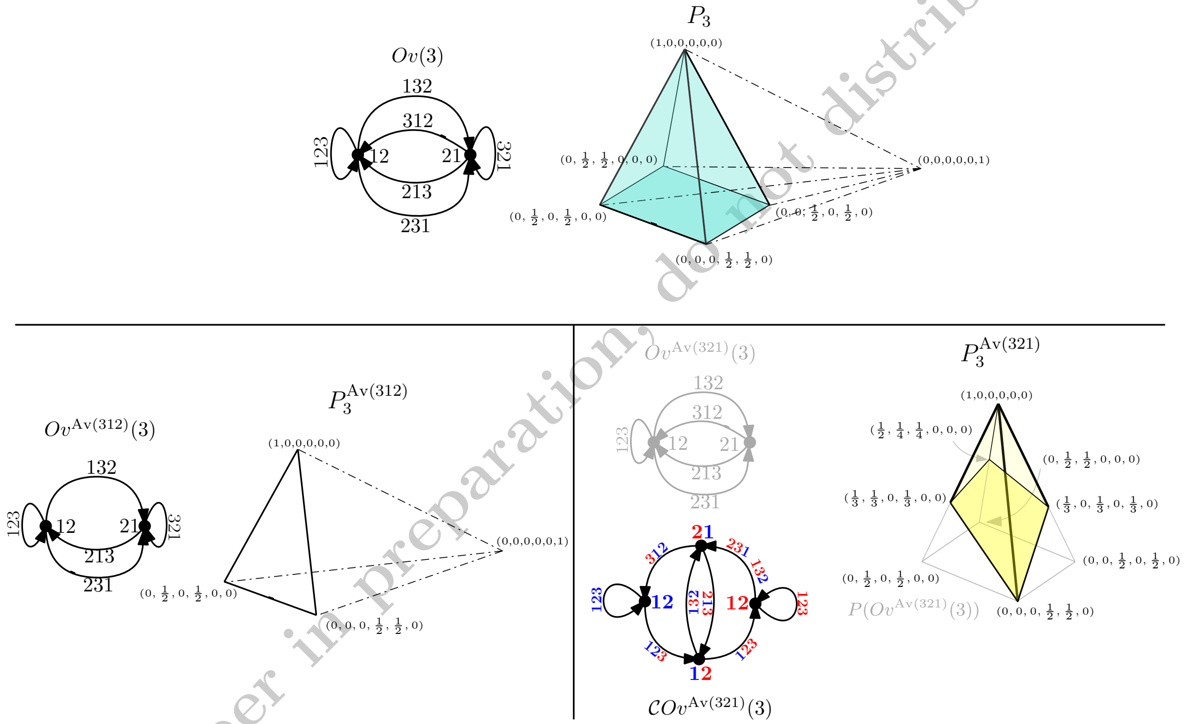


Figure 1: **Top:** The overlap graph $\mathcal{O}v(3)$ and the four-dimensional polytope P_3 given by the six patterns of size three. We highlight in light-blue one of the six three-dimensional faces of P_3. This face is a pyramid with a square base. The polytope itself is a four-dimensional pyramid, whose base is the highlighted face. The coordinates of the vertices correspond to the patterns $(123, 231, 312, 213, 132, 321)$ respectively. From Theorem 1.4 we have that P_3 is the cycle polytope of $\mathcal{O}v(3)$. **Bottom-left:** The overlap graph $\mathcal{O}v^{\mathrm{Av}(312)}(3)$ and the three-dimensional polytope $P_3^{\mathrm{Av}(312)}$. Note that $P_3^{\mathrm{Av}(312)} \subseteq P_3$. From Theorem 1.9 we have that $P_3^{\mathrm{Av}(312)}$ is the cycle polytope of $\mathcal{O}v^{\mathrm{Av}(312)}(3)$. **Bottom-right:** In grey the overlap graph $\mathcal{O}v^{\mathrm{Av}(321)}(3)$ and the corresponding three-dimensional cycle polytope $P(\mathcal{O}v^{\mathrm{Av}(321)}(3))$, that is strictly larger than $P_3^{\mathrm{Av}(321)}$. The latter feasible region is highlighted yellow. From Theorem 1.17 we have that $P_k^{\mathrm{Av}(321)}$ is the projection (defined precisely before Theorem 1.17) of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$ (see Definition 1.16 for a precise description). This graph is plotted on bottom-left side. Note that $P_3^{\mathrm{Av}(312)} \subseteq P_3$.

We also recall for later purposes the following construction related to the overlap graph $\mathcal{O}v(k)$. Given a permutation $\sigma \in \mathcal{S}_m$, for some $m \geq k$, we can associate to it a walk $W_k(\sigma) = (e_1, \dots, e_{m-k+1})$ in $\mathcal{O}v(k)$ of size $m-k+1$ defined by

$$\operatorname{lb}(e_i) \coloneqq \operatorname{pat}_{[i,i+k-1]}(\sigma), \quad \text{for all} \quad i \in [m-k+1]. \tag{2}$$

The map W_k is not injective, but in [BP19] we proved the following.

Lemma 1.5 (Lemma 3.8 in [BP19]). *Fix $k \in \mathbb{Z}_{\geq 1}$ and $m \geq k$. The map W_k, from the set $\mathcal{S}_m$ of permutations of size m to the set of walks in $\mathcal{O}v(k)$ of size $m-k+1$, is surjective.*

This lemma was a key step in the proof of Theorem 1.4.

1.3 Main results on the pattern avoiding feasible regions

We start with a natural generalization of Definition 1.1 to pattern avoiding permutations.

Definition 1.6. Fix a set of patterns $B \subset \mathcal{S}$ and $k \in \mathbb{Z}_{\geq 1}$. The *overlap graph* $\mathcal{O}v^{\operatorname{Av}(B)}(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\operatorname{Av}_{k-1}(B)$ and for every $\pi \in \operatorname{Av}_k(B)$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

Informally, $\mathcal{O}v^{\operatorname{Av}(B)}(k)$ arises simply as the restriction of $\mathcal{O}v(k)$ to all the edges and vertices in $\operatorname{Av}(B)$. We have the following result, that is proved in Section 2.

Theorem 1.7. *Fix $k \in \mathbb{Z}_{\geq 1}$. For all sets of patterns $B \subset \mathcal{S}$, the feasible region $P_k^{\operatorname{Av}(B)}$ is a closed set and satisfies $P_k^{\operatorname{Av}(B)} \subseteq P(\mathcal{O}v^{\operatorname{Av}(B)}(k))$.*

Moreover, if $\operatorname{Av}(B)$ is closed either for the direct or skew sum then the feasible region $P_k^{\operatorname{Av}(B)}$ is convex and $\dim(P_k^{\operatorname{Av}(B)}) \leq |\operatorname{Av}_k(B)| - |\operatorname{Av}_{k-1}(B)|$.

In particular, $P_k^{\operatorname{Av}(\tau)}$ is always convex for any pattern $\tau \in \mathcal{S}$, as $\operatorname{Av}(\tau)$ is either closed for the $\oplus$ operation (whenever τ is $\oplus$-indecomposable) or closed for the $\ominus$ operation (whenever τ is $\ominus$-indecomposable).

We will show later (see Theorem 1.17) that sometimes $P_k^{\operatorname{Av}(B)} \neq P(\mathcal{O}v^{\operatorname{Av}(B)}(k))$ (see also the bottom-right side of Fig. 1) but we believe that the bound on the dimension of the feasible regions given above is tight whenever $|B| = 1$.

Conjecture 1.8. *Fix $k \in \mathbb{Z}_{\geq 1}$. For all patterns $\tau \in \mathcal{S}$, we have that*

$$\dim(P_k^{\operatorname{Av}(\tau)}) = |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|.$$

We state this conjecture for the case $|B| = 1$, but it is natural to wonder what happens for for $|B| \geq 2$. Indeed, for some permutation classes, the space is not even convex. For instance, if $B = \{132, 213, 231, 312\}$, then $\operatorname{Av}(B)$ is the set of monotone permutations. Therefore, the resulting pattern avoiding feasible region is not connected (it is formed by two distinct points).

The main goal of this paper is to prove that Conjecture 1.8 is true when $|\tau| = 3$ or τ is a monotone pattern, i.e. $\tau = n \cdots 1$ or $\tau = 1 \cdots n$, for $n \in \mathbb{Z}_{\geq 2}$. More precisely, we will completely describe the feasible regions $P_k^{\operatorname{Av}(\tau)}$ for such patterns τ. By symmetry, we only need to study the cases $\tau = 312$ and $\tau = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$.

1.3.1 312-avoiding permutations

The overlap graph that we have to consider for $\tau = 312$ is exactly the one given in Definition 1.6 as shown in the following theorem.

Theorem 1.9. *Fix $k \in \mathbb{Z}_{\geq 1}$. The feasible region $P_k^{\mathrm{Av}(312)}$ is the cycle polytope of the overlap graph $\mathcal{O}v^{\mathrm{Av}(312)}(k)$. Its dimension is $C_k - C_{k-1}$, where C_k is the k-th Catalan number, and its vertices are given by the simple cycles of $\mathcal{O}v^{\mathrm{Av}(312)}(k)$.*

An instance of the result above is depicted on the bottom-left side of Fig. 1.

Differently, the overlap graph that we have to consider for $\tau = n \cdots 1$ is defined using an alternative construction based on permutation colourings (see Definition 1.16 below). The main results for the feasible region $P_k^{\mathrm{Av}(n\cdots 1)}$ are given in Theorem 1.17 and Theorem 1.18 below. We now introduce the notions needed to state these results.

1.3.2 Monotone-avoiding permutations

In this section we fix $\tau = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$, the decreasing pattern of size n, and an integer $k \in \mathbb{Z}_{\geq 1}$.

We start with the following observation: by considering the overlap graph $\mathcal{O}v^{\mathrm{Av}(\tau)}(k)$, the corresponding cycle polytope $P(\mathcal{O}v^{\mathrm{Av}(\tau)}(k))$ is strictly larger than the feasible region $P_k^{\mathrm{Av}(\tau)}$ (see for instance the bottom-right side of Fig. 1). This is so because there are some paths in the graph $\mathcal{O}v^{\mathrm{Av}(\tau)}(k)$ that do not correspond to a suitable permutation, see for instance Example 1.10 below. Thus, the feasible region $P_k^{\mathrm{Av}(\tau)}$ cannot be directly described as the cycle polytope of the overlap graph $\mathcal{O}v^{\mathrm{Av}(\tau)}(k)$. We will consider an enriched overlap graph with colourings (see Definition 1.11 below).

Example 1.10. Consider $k = 3$ and $n = 3$. We present a walk in $\mathcal{O}v^{\mathrm{Av}(\tau)}(k)$ that cannot be inverted via the map W_k presented at the end of Section 1.2.

In $\mathcal{O}v^{\mathrm{Av}(\tau)}(k)$ take the following walk $w = (312, 231)$. Notice that any permutation σ that has the consecutive occurrences 312 and 231 arising in consecutive intervals $\{i, i+1, i+2\}$ and $\{i+1, i+2, i+3\}$ respectively, necessarily has that $\mathrm{pat}_{\{i,i+1,i+3\}}(\sigma) = 321$, therefore $\sigma \notin \mathrm{Av}(321)$. This means that it does not exist any permutation $\sigma \in \mathrm{Av}(321)$ such that $W_k(\sigma) = w$, and so the map W_k is not surjective in this case.

In the following we will see that by considering a coloured version of the graph $\mathcal{O}v^{\mathrm{Av}(\tau)}(k)$ we can overcome this problem. We start by introducing colourings of permutations.

Definition 1.11 (Colourings and RITMO colourings)**.** Fix an integer $m \in \mathbb{Z}_{\geq 1}$. For a permutation σ, an m-colouring of σ is a map $\mathfrak{c} : [|\sigma|] \to [m]$, which is to be interpreted as a mapping from the set of *indices* of σ to $[m]$. An m-colouring $\mathfrak{c}$ is said to be *surjective* when $\mathrm{im}(\mathfrak{c}) = [m]$. For any permutation σ there is a unique *right-top monotone colouring* (simply RITMO colouring henceforth), which we describe now, and we denote as $\mathbb{C}(\sigma)$. We construct this colouring iteratively, starting with the highest value of the permutation and going down while assigning the lowest possible colour that avoids an occurrence of a monochromatic 21. To be more precise, start by defining $k_i = \sigma^{-1}(|\sigma| - i + 1)$ for all $i \in [|\sigma|]$, so that k_i is the index of the i-th largest value of the permutation σ, and so $\sigma(k_1) = |\sigma|$ and $\sigma(k_{|\sigma|}) = 1$. Define $\mathbb{C}(\sigma)(k_1) = 1$. On the i-th step, we define $\mathbb{C}(\sigma)(k_i)$ as

$$\mathbb{C}(\sigma)(k_i) = \min\{j \in \mathbb{Z}_{\geq 1} | \text{ if } \mathbb{C}(\sigma)(k_\iota) = j \text{ then } k_\iota > k_i \text{ for } \iota < i\}\,.$$

If a permutation is coloured with its RITMO colouring, its highest and leftmost increasing subsequence (i.e. the one formed by the left-to-right maxima) will correspond to the elements coloured by 1, and iteratively for the remaining elements. We suggest the reader to keep in mind both point of views (the one given in the definition and the one described now) on RITMO colourings.

Example 1.12. One can see in Fig. 2 an example of the RITMO colourings for permutations 312, 1427536 and 124376985. In all our examples, we paint in red the values coloured by one, in blue the ones coloured by two, and in green the ones coloured by three.

Consider for instance the permutation $\sigma = 1427536$, which avoids the permutation 4321. Let us compute its RITMO colouring $\mathbb{C}(\sigma)$ using both methods explained above:

$$1427536 \to 1427536 \to 1427536 \to 1427536 \to 1427536 \to 1427536 \to 1427536 \to 1427536\,,$$

$$1427536 \to 1427536 \to 1427536 \to 1427536\,.$$

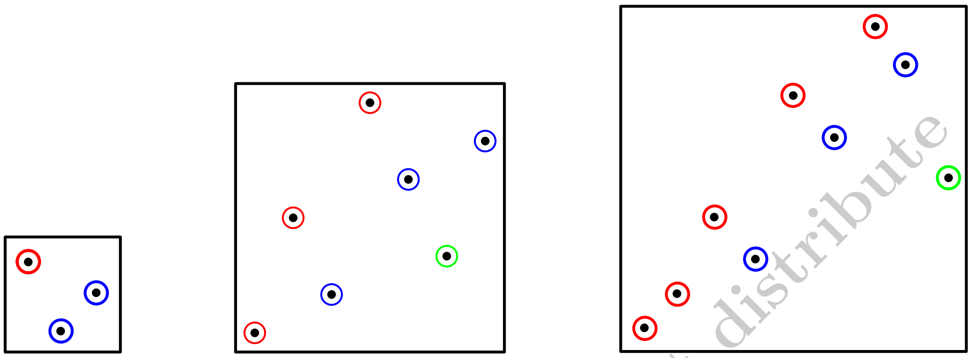


Figure 2: The RITMO colourings of 312, 1427536 and 124376985.

For the pair $(\sigma, \mathbb{C}(\sigma))$ we simply write $\mathbb{S}(\sigma)$. If σ avoids the permutation τ, it is known that its RITMO colouring is an $(n-1)$-colouring (the origins of this result are hard to trace, but it goes back at least to [Gre74] where it is already noted as something that is not hard to prove; see also [B12́, Chapter 4.3]).

We furthermore allow for taking *restrictions of colourings*. Given a permutation σ of size k, a colouring $\mathfrak{c}$ of σ and a subset $I = \{i_1, \dots, i_j\} \subseteq [k]$, we consider the restriction $\mathrm{pat}_I(\mathfrak{c})$ to be the colouring of the permutation $\mathrm{pat}_I(\sigma)$. Observe that it may be the case that $\mathrm{pat}_I(\mathbb{C}(\sigma))$ and $\mathbb{C}(\mathrm{pat}_I(\sigma))$ are distinct colorings of the permutation $\mathrm{pat}_I(\sigma)$. For instance, if $\sigma = 2134$ and $I = \{2,3,4\}$ then $\mathrm{pat}_I(\mathbb{C}(\sigma)) = \mathrm{pat}_{\{2,3,4\}}(2134) = 123$ but $\mathbb{C}(\mathrm{pat}_I(\sigma)) = \mathbb{C}(123) = 123$. The following definition is fundamental in our results.

Definition 1.13. We say that an m-coloring $\mathfrak{c}$ of a permutation π of size k is *inherited* if there is some permutation σ of size $\ell \geq k$ such that $\mathrm{end}_k(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$.

Let $\mathcal{C}_m(\pi)$ be the set of all inherited m-colourings of a permutation π. We also set $\mathcal{C}_m(k) = \{(\pi, \mathfrak{c}) | \pi \in \mathcal{S}_k,\ \mathfrak{c} \text{ is an inherited } m\text{-colouring of } \pi\}$, that is the set of all inherited m-colourings of permutations of size k.

Example 1.14. In Table 1 we present all the inherited 2-colorings of permutations of size three. Thus,

$$\mathcal{C}_2(3) = \{123, 123, 123, 123, 132, 132, 213, 231, 312\}\ .$$

The following simple result is a key step for the next definition.

Observation 1.15. For all permutations $\sigma \in \mathrm{Av}(n \cdots 1)$ and all $j \leq |\sigma|$, we have that

$$\mathrm{beg}_j(\mathbb{C}(\sigma)) = \mathbb{C}(\mathrm{beg}_j(\sigma))\,.$$

Definition 1.16. The *coloured overlap graph* $\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k)$ is defined with the vertex set

$$V := \mathcal{C}_{n-1}(k-1) = \{(\pi, \mathfrak{c}) | \pi \in \mathcal{S}_{k-1},\ \mathfrak{c} \text{ is an inherited } (n-1)\text{-colouring of } \pi\},$$

123	$123 = \mathbb{C}(123)$, $123 = \mathrm{pat}_{\{2,3,4\}}(2134)$, $123 = \mathrm{pat}\,\{2,3,4\}(3124)$, $123 = \mathrm{pat}_{\{2,3,4\}}(4123)$
132	$132 = \mathbb{C}(132)$, $132 = \mathrm{pat}_{\{2,3,4\}}(3142)$
213	$213 = \mathbb{C}(213)$
231	$231 = \mathbb{C}(231)$
312	$312 = \mathbb{C}(312)$

Table 1: The permutations of size three, and their corresponding inherited 2-colourings. Note that the permutations of size four are coloured according to their RITMO colouring.

and the edge set

$$E \coloneqq \mathcal{C}_{n-1}(k) = \{(\pi, \mathfrak{c}) | \tau \in \mathcal{S}_k,\ \mathfrak{c} \text{ is an inherited } (n-1)\text{-colouring of } \pi\}\,,$$

where the edge $(\pi, \mathfrak{c})$ connects $v_1 \to v_2$ with $v_1 = \mathrm{beg}_{k-1}(\pi, \mathfrak{c})$ and $v_2 = \mathrm{end}_{k-1}(\pi, \mathfrak{c})$.

In Fig. 3 we present the coloured overlap graph corresponding to $k = 3$ and $n = 2$.

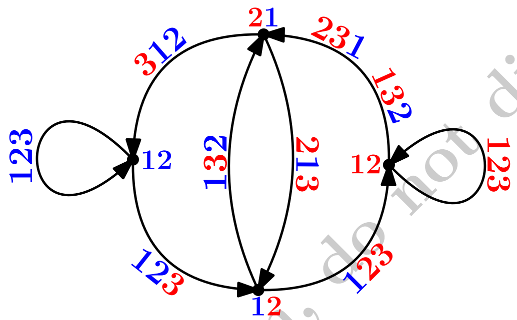


Figure 3: The coloured overlap graph for $k = 3$ and $m = 2$, that also appears in the bottom-right side of Fig. 1. Note that in order to obtain a clearer picture we do not draw multiple edges, but we use multiple labels (for example the edge $12 \to 21$ is labeled with the permutations 231 and 132 and should be thought of as two distinct edges labeled with 231 and 132 respectively). Observe also that the coloured permutation 213 is not an edge of the coloured overlap graph because it is not inherited.

We now justify that the coloured overlap graph is well-defined, i.e. that for any edge $(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k)$, then both $\mathrm{beg}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$ and $\mathrm{end}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$. Equivalently, we have to show that given an inherited $(n-1)$-colouring $(\pi, \mathfrak{c})$ of size k, both $\mathrm{beg}_{k-1}(\pi, \mathfrak{c})$ and $\mathrm{end}_{k-1}(\pi, \mathfrak{c})$ are inherited $(n-1)$-colourings of size $k-1$.

By definition of inherited colouring, there exists a permutation σ such that $\mathrm{end}_k(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$. Then, naturally, we have that $\mathrm{end}_{k-1}(\mathbb{S}(\sigma)) = \mathrm{end}_{k-1}(\pi, \mathfrak{c})$, and therefore $\mathrm{end}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$. On the other hand, from Observation 1.15 we have that

$$\mathrm{beg}_{k-1}(\pi, \mathfrak{c}) = \mathrm{beg}_{k-1}(\mathrm{end}_k(\mathbb{S}(\sigma))) = \mathrm{end}_{k-1}(\mathrm{beg}_{|\sigma|-1}(\mathbb{S}(\sigma))) \overset{1.15}{=} \mathrm{end}_{k-1}(\mathbb{S}(\mathrm{beg}_{|\sigma|-1}(\sigma)))\,, \quad (3)$$

and so $\mathrm{beg}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$.

In the following, we denote $\Pi : \mathbb{R}^{\mathcal{C}_{n-1}(k)} \to \mathbb{R}^{\mathrm{Av}_k(\tau)}$ for the projection mapping that sends the basis elements $\vec{e}_{(\pi,\mathfrak{c})} \coloneqq (\delta_{(\pi,\mathfrak{c})}(x))_{x \in \mathcal{C}_{n-1}(k)}$ to $\vec{e}_\pi \coloneqq (\delta_\pi(x))_{x \in \mathrm{Av}_k(\tau)}$, i.e. the mapping that "forgets" colourings. The main results for the monotone patterns case are the following two.

Theorem 1.17. *The pattern avoiding feasible region $P_k^{\mathrm{Av}(\tau)}$ is the Π-projection of the cycle polytope of the overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$. That is,*

$$P_k^{\mathrm{Av}(\tau)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))\,.$$

An instance of this result above is depicted on the bottom-right side of Fig. 1.

Theorem 1.18. *The dimension of* $P_k^{\mathrm{Av}(\tau)}$ *is* $|\mathrm{Av}_k(\tau)| - |\mathrm{Av}_{k-1}(\tau)|$.

For more information on the numbers $A(n,k) = |\mathrm{Av}_k(\tau)|$ we refer to [Slo96, A214015]. We just recall that a closed formula for these numbers is not available. However, thanks to the Robinson-Schensted correspondence, $A(n,k)$ is equal to $\sum_\lambda f_\lambda^2$, where the sum runs over all partitions λ of k with length at most $n-1$ and f_λ is the number of standard Young tableaux with shape λ.

1.4 Future projects and open questions

We collect here some ideas for future projects and some open questions.

- Theorem 1.9 and Theorem 1.17 give a description of the feasible regions $P_k^{\mathrm{Av}(\tau)}$ for all patterns τ of size three. Can we describe the feasible regions $P_k^{\mathrm{Av}(B)}$ for all subsets $B \subseteq \mathcal{S}_3$? It is easy to see that $P_k^{\mathrm{Av}(B)} \subseteq \bigcap_{\tau \in B} P_k^{\mathrm{Av}(\tau)}$, but the other inclusion is not trivial and does not hold in general. We believe that it would be interesting to investigate it.

- We think that the feasible region $P_k^{\mathrm{Av}(\tau)}$ can be precisely described for other specific patterns τ different from the ones already considered in this paper. In particular, we believe that a good choice would be the patterns τ for which the corresponding classes $\mathrm{Av}(\tau)$ have been enumerated through generating trees. Indeed, the first author of this article has recently shown in [Bor20a] that generating trees behave well in the analysis of consecutive patterns of permutations in the classes $\mathrm{Av}(\tau)$. We belive that generating trees would be particularly helpful to prove some analogues of Lemma 4.4 - that is the key lemma in the proof of Theorem 1.17 - for other classes of permutations.

- The main open question of this article is obviously the Conjecture 1.8.

1.5 Notation

Permutations and patterns

We recall that we denoted by $\mathcal{S}_n$ the set of permutations of size n, and by $\mathcal{S}$ the set of all permutations.

If $x_1, \dots, x_n$ is a sequence of distinct numbers, let $\mathrm{std}(x_1, \dots, x_n)$ be the unique permutation π in $\mathcal{S}_n$ whose elements are in the same relative order as $x_1, \dots, x_n$, i.e. $\pi(i) < \pi(j)$ if and only if $x_i < x_j$. Given a permutation $\sigma \in \mathcal{S}_n$ and a subset of indices $I \subseteq [n]$, let $\mathrm{pat}_I(\sigma)$ be the permutation induced by $(\sigma(i))_{i\in I}$, namely, $\mathrm{pat}_I(\sigma) \coloneqq \mathrm{std}\left((\sigma(i))_{i\in I}\right)$. We recall that we extend this notation also to coloured permutations in Section 1.3.2. For example, if $\sigma = 24637185$ and $I = \{2,4,7\}$, then $\mathrm{pat}_{\{2,4,7\}}(24637185) = \mathrm{std}(438) = 213$. In two particular cases, we use the following more compact notation: for $k \leq |\sigma|$, $\mathrm{beg}_k(\sigma) \coloneqq \mathrm{pat}_{\{1,2,\dots,k\}}(\sigma)$ and $\mathrm{end}_k(\sigma) \coloneqq \mathrm{pat}_{\{|\sigma|-k+1,|\sigma|-k+2,\dots,|\sigma|\}}(\sigma)$.

Given two permutations, $\sigma \in \mathcal{S}_n$ for some $n \in \mathbb{Z}_{\geq 1}$ and $\pi \in \mathcal{S}_k$ for some $k \leq n$, and a set of indices $I = \{i_1 < \ldots < i_k\}$, we say that $\sigma(i_1)\ldots\sigma(i_k)$ is an *occurrence* of π in σ if $\mathrm{pat}_I(\sigma) = \pi$ (we will also say that π is a *pattern* of σ). If the indices $i_1, \dots, i_k$ form an interval, then we say that $\sigma(i_1)\ldots\sigma(i_k)$ is a *consecutive occurrence* of π in σ (we will also say that π is a *consecutive pattern* of σ). We denote intervals of integers as $[n,m] = \{n, n+1, \dots, m\}$ for $n, m \in \mathbb{Z}_{\geq 1}$ with $n \leq m$.

Example 1.19. The permutation $\sigma = 1532467$ contains an occurrence of 1423 (but no such consecutive occurrences) and a consecutive occurrence of 321. Indeed $\mathrm{pat}_{\{1,2,3,5\}}(\sigma) = 1423$ but no interval of indices of σ induces the permutation 1423. Moreover, $\mathrm{pat}_{[2,4]}(\sigma) = \mathrm{pat}_{\{2,3,4\}}(\sigma) = 321$.

We denote by $\mathrm{occ}(\pi,\sigma)$ the number of occurrences of a pattern π in σ, more precisely

$$\mathrm{occ}(\pi,\sigma) \coloneqq \left|\left\{ I \subseteq [n] | \mathrm{pat}_I(\sigma) = \pi \right\}\right|.$$

We denote by $\text{c-occ}(\pi,\sigma)$ the number of consecutive occurrences of a pattern π in σ, more precisely

$$\text{c-occ}(\pi,\sigma) \coloneqq \left|\left\{ I \subseteq [n] |\ I \text{ is an interval, } \mathrm{pat}_I(\sigma) = \pi \right\}\right|.$$

Moreover, we denote by $\widetilde{\mathrm{occ}}(\pi,\sigma)$ (resp. by $\widetilde{\text{c-occ}}(\pi,\sigma)$) the proportion of occurrences (resp. consecutive occurrences) of a pattern π in σ, that is,

$$\widetilde{\mathrm{occ}}(\pi,\sigma) \coloneqq \frac{\mathrm{occ}(\pi,\sigma)}{\binom{n}{k}} \in [0,1], \qquad \widetilde{\text{c-occ}}(\pi,\sigma) \coloneqq \frac{\text{c-occ}(\pi,\sigma)}{n} \in [0,1]\,.$$

Remark 1.20. The natural choice for the denominator of the expression in the right-hand side of the equation above should be $n-k+1$ and not n, but we make this choice for later convenience. Moreover, for every fixed k, there is no difference in the asymptotics when n tends to infinity.

For a fixed $k \in \mathbb{Z}_{\geq 1}$ and a permutation $\sigma \in \mathcal{S}_{\geq k}$, we let $\widetilde{\mathrm{occ}}_k(\sigma), \widetilde{\text{c-occ}}_k(\sigma) \in [0,1]^{\mathcal{S}_k}$ be the vectors

$$\widetilde{\mathrm{occ}}_k(\sigma) \coloneqq \left(\widetilde{\mathrm{occ}}(\pi,\sigma)\right)_{\pi\in\mathcal{S}_k}, \qquad \widetilde{\text{c-occ}}_k(\sigma) \coloneqq \left(\widetilde{\text{c-occ}}(\pi,\sigma)\right)_{\pi\in\mathcal{S}_k}.$$

We say that σ *avoids* π if σ does not contain any occurrence of π. We point out that the definition of π-avoiding permutations refers to occurrences and not to consecutive occurrences. Given a set of patterns $B \subset \mathcal{S}$, we say that σ *avoids* B if σ avoids π, for all $\pi \in B$. We denote by $\mathrm{Av}_n(B)$ the set of B-avoiding permutations of size n and by $\mathrm{Av}(B) \coloneqq \bigcup_{n\in\mathbb{Z}_{\geq 1}} \mathrm{Av}_n(B)$ the set of B-avoiding permutations of arbitrary finite size. The set $\mathrm{Av}(B)$ is often called a *permutation class*.

We also introduce two classical operations on permutations. We denote with $\oplus$ the *direct sum* of two permutations, i.e. for $\tau \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$,

$$\tau \oplus \sigma = \tau(1)\dots\tau(k)(\sigma(1)+m)\dots(\sigma(n)+m)\,,$$

and we denote with $\oplus_\ell\, \sigma$ the direct sum of ℓ copies of σ (we remark that the operation $\oplus$ is associative). A similar definition holds for the *skew sum* $\ominus$, "gluing" permutations along the decreasing diagonal instead of the increasing one as done for the operation $\oplus$.

If A, B are two disjoint sets, equipped with the partial orders $\leq_A, \leq_B$, respectively, we denote by $\leq_A \bullet \leq_B$ the partial order on $A \uplus B$ that restricts to $\leq_A$ in A, that restricts to $\leq_B$ in B and that has $a \leq_A \bullet \leq_B b$ for any $a \in A, b \in B$.

Directed graphs

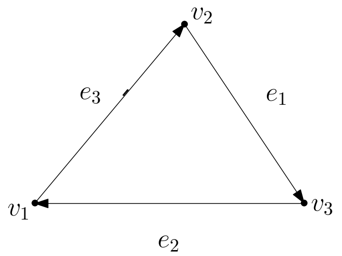
All graphs, their subgraphs and their subtrees are considered to be directed multigraphs in this paper (and we often refer to them as directed graphs or simply as graphs). In a directed multigraph $G = (V(G), E(G))$, the set of edges $E(G)$ is a multiset, allowing for loops and parallel edges. An edge $e \in E(G)$ is an oriented pair of vertices, (v,u), often denoted by $e = v \to u$. We write $\mathrm{s}(e)$ for the starting vertex v and $\mathrm{a}(e)$ for the arrival vertex u. We often consider directed graphs G with labelled edges, and write $\mathrm{lb}(e)$ for the label of the edge $e \in E(G)$. In a graph with labelled edges we refer to edges by using their labels. Given an edge $e = v \to u \in E(G)$, we denote by $C_G(e)$ (for "set of *continuations* of e") the set of edges $e' \in E(G)$ such that $e' = u \to w$ for some $w \in V(G)$, i.e. $C_G(e) = \{e' \in E(G) |\, \mathrm{s}(e') = \mathrm{a}(e)\}$.

A *walk* of size k on a directed graph G is a sequence of k edges $(e_1,\dots,e_k) \in E(G)^k$ such that for all $i \in [k-1]$, $\mathrm{a}(e_i) = \mathrm{s}(e_{i+1})$. We also denote this walk by $w = (e_1,\dots,e_k)$ to avoid a heavy use of parenthesis. A walk is a *cycle* if $\mathrm{s}(e_1) = \mathrm{a}(e_k)$. A walk is a *path* if all the edges are distinct, as well as its vertices, with a possible exception that $\mathrm{s}(e_1) = \mathrm{a}(e_k)$ may happen. A cycle that is a path is called a *simple cycle*. Given two walks $w = (e_1, \dots, e_k)$ and $w' = (e'_1, \dots, e'_{k'})$ such that $\mathrm{a}(e_k) = \mathrm{s}(e'_1)$, we write $w \bullet w'$ for the concatenation of the two walks, i.e. $w \bullet w' = (e_1, \dots, e_k, e'_1, \dots, e'_{k'})$. For a walk w, we denote by $|w|$ the number of *edges* in w.

Given a walk $w = (e_1, \dots, e_k)$ and an edge e, we denote by $n_e(w)$ the number of times the edge e is traversed in w, i.e. $n_e(w) \coloneqq |\{i \leq k | e_i = e\}|$.

The *incidence matrix* of a directed graph G is the matrix $L(G)$ with rows indexed by $V(G)$, and columns indexed by $E(G)$, such that for any edge $e = v \to u$ with $v \neq u$, the corresponding column in $L(G)$ has $(L(G))_{v,e} = 1$, $(L(G)))_{u,e} = -1$ and is zero everywhere else. Moreover, if $e = v \to v$ is a loop, the corresponding column in $L(G)$ has zero everywhere.

For instance, we show in Fig. 4 a graph G with its incidence matrix $L(G)$.

$G =$ 

$$, \quad L(G) = \begin{array}{cc} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} & \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} & \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \end{array} .$$

Figure 4: A graph G with its incidence matrix $L(G)$.

2 Topological properties of the pattern avoiding feasible regions and an upper-bound on their dimensions

This section is devoted to the proof of Theorem 1.7.

Proposition 2.1. *Fix $k \in \mathbb{Z}_{\geq 1}$. For any set of patterns $B \subseteq \mathcal{S}$, the feasible region $P_k^{\mathrm{Av}(B)}$ is a closed set.*

This is a classical consequence of the fact that $P_k^{\mathrm{Av}(B)}$ is a set of limit points. For completeness, we include a simple proof of the statement. Recall that we defined $\widetilde{\text{c-occ}}_k(\sigma) \coloneqq \big(\widetilde{\text{c-occ}}(\pi, \sigma)\big)_{\pi \in \mathcal{S}_k}$.

Proof. It suffices to show that, for any sequence $(\vec{v}_s)_{s \in \mathbb{Z}_{\geq 1}}$ in $P_k^{\mathrm{Av}(B)}$ such that $\vec{v}_s \to \vec{v}$ for some $\vec{v} \in [0,1]^{\mathcal{S}_k}$, we have that $\vec{v} \in P_k^{\mathrm{Av}(B)}$. For all $s \in \mathbb{Z}_{\geq 1}$, consider a sequence of permutations $(\sigma_s^m)_{m \in \mathbb{Z}_{\geq 1}}$ such that $|\sigma_s^m| \xrightarrow{m \to \infty} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_s^m) \xrightarrow{m \to \infty} \vec{v}_s$, and some index $m(s)$ of the sequence $(\sigma_s^m)_{m \in \mathbb{Z}_{\geq 1}}$ such that for all $m \geq m(s)$,

$$|\sigma_s^m| \geq s \quad \text{and} \quad ||\widetilde{\text{c-occ}}_k(\sigma_s^m) - \vec{v}_s|| \leq \tfrac{1}{s} \,.$$

Without loss of generality, assume that $m(s)$ is increasing. For every $\ell \in \mathbb{Z}_{\geq 1}$, define $\sigma^\ell \coloneqq \sigma_\ell^{m(\ell)}$. It is easy to show that

$$|\sigma^\ell| \xrightarrow{\ell \to \infty} \infty \quad \text{and} \quad \widetilde{\text{c-occ}}_k(\sigma^\ell) \xrightarrow{\ell \to \infty} \vec{v} \,,$$

where we use the fact that $\vec{v}_s \to \vec{v}$. Furthermore, by assumption we have that $\sigma^\ell \in \mathrm{Av}(B)$. Therefore $\vec{v} \in P_k^{\mathrm{Av}(B)}$. $\square$

The following result is an analogue of [BP19, Proposition 3.2], where it was proved that the feasible region P_k is convex.

Proposition 2.2. *Fix $k \in \mathbb{Z}_{\geq 1}$. Consider a set of patterns $B \subset \mathcal{S}$ such that the class $\mathrm{Av}(B)$ is closed for one of the two operations $\oplus, \ominus$. Then, the feasible region $P_k^{\mathrm{Av}(B)}$ is convex.*

Proof. We will present a proof for the case where $\mathrm{Av}(B)$ is closed for the $\oplus$ operation, however the arguments hold equally for the $\ominus$ operation.

Since $P_k^{\mathcal{C}}$ is a closed set (by Lemma 2.1) it is enough to consider rational convex combinations of points in $P_k^{\mathrm{Av}(B)}$, i.e. it is enough to establish that for all $\vec{v}_1, \vec{v}_2 \in P_k^{\mathrm{Av}(B)}$ and all $s, t \in \mathbb{Z}_{\geq 1}$, we have that

$$\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k^{\mathrm{Av}(B)}.$$

Fix $\vec{v}_1, \vec{v}_2 \in P_k$ and $s, t \in \mathbb{Z}_{\geq 1}$. Since $\vec{v}_1, \vec{v}_2 \in P_k^{\mathrm{Av}(B)}$, there exist two sequences $(\sigma_1^\ell)_{\ell \in \mathbb{Z}_{\geq 1}}$, $(\sigma_2^\ell)_{\ell \in \mathbb{Z}_{\geq 1}}$ such that $|\sigma_i^\ell| \xrightarrow{\ell \to \infty} \infty$, $\sigma_i^\ell \in \mathrm{Av}(B)$ and $\widetilde{\text{c-occ}}_k(\sigma_i^\ell) \xrightarrow{\ell \to \infty} \vec{v}_i$, for $i = 1, 2$.

Define $t_\ell := t \cdot |\sigma_1^\ell|$ and $s_\ell := s \cdot |\sigma_2^\ell|$.

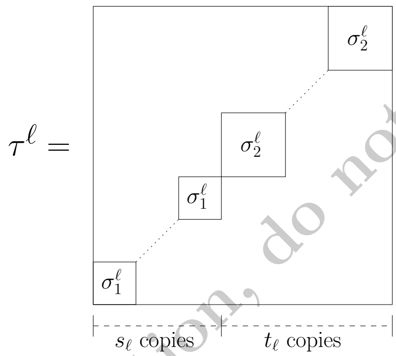


Figure 5: Schema for the definition of the permutation τ^ℓ.

We set $\tau^\ell := \left(\oplus_{s_\ell} \sigma_1^\ell\right) \oplus \left(\oplus_{t_\ell} \sigma_2^\ell\right)$. For a graphical interpretation of this construction we refer to Fig. 5. We note that for every $\pi \in \mathcal{S}_k$, we have

$$\widetilde{\text{c-occ}}(\pi, \tau^\ell) = s_\ell \cdot \widetilde{\text{c-occ}}(\pi, \sigma_1^\ell) + t_\ell \cdot \widetilde{\text{c-occ}}(\pi, \sigma_2^\ell) + Er,$$

where $Er \leq (s_\ell + t_\ell - 1) \cdot |\pi|$. This error term comes from the number of intervals of size $|\pi|$ that intersect the boundary of some copies of σ_1^ℓ or σ_2^ℓ. Hence

$$\begin{aligned}\widetilde{\text{c-occ}}(\pi, \tau^\ell) &= \frac{s_\ell \cdot |\sigma_1^\ell| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_1^\ell) + t_\ell \cdot |\sigma_2^\ell| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_2^\ell) + Er}{s_\ell \cdot |\sigma_1^\ell| + t_\ell \cdot |\sigma_2^\ell|} \\ &= \frac{s}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_1^\ell) + \frac{t}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_2^\ell) + O\left(|\pi|\left(\tfrac{1}{|\sigma_1^\ell|} + \tfrac{1}{|\sigma_2^\ell|}\right)\right).\end{aligned}$$

As ℓ tends to infinity, we have

$$\widetilde{\text{c-occ}}_k(\tau^\ell) \to \frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2,$$

since $|\sigma_i^\ell| \xrightarrow{\ell \to \infty} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_i^\ell) \xrightarrow{m \to \infty} \vec{v}_i$, for $i = 1, 2$. Noting also that $|\tau^\ell| \to \infty$, we can conclude that $\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k^{\mathrm{Av}(B)}$. This ends the proof. □

Proposition 2.3. *Fix $k \in \mathbb{Z}_{\geq 1}$. For all set of patterns $B \subset \mathcal{S}$, we have that*

$$P_k^{\mathrm{Av}(B)} \subseteq P(\mathcal{O}v^{\mathrm{Av}(B)}(k)).$$

Recall that the map W_k was defined at the end of Section 1.2.

Proof. Consider any point $\vec{v} \in P_k^{\mathrm{Av}(B)}$, and a corresponding sequence $\left(\sigma^\ell\right)_{\ell \geq 0} \in \mathrm{Av}(B)^{\mathbb{Z}_{\geq 0}}$ such that $\widetilde{\text{c-occ}}_k(\sigma^\ell) \to \vec{v}$. Because $\sigma^\ell \in \mathrm{Av}(B)$, we know that for each ℓ, $W_k(\sigma^\ell)$ is a walk in $\mathcal{O}v^{\mathrm{Av}(B)}(k)$. Using the same method as in the proof of $P_k \subseteq P(\mathcal{O}v(k))$ in [BP19, Theorem 3.12], we can deduce that $\widetilde{\text{c-occ}}_k(\sigma^\ell)$ converges to a point in $P(\mathcal{O}v^{\mathrm{Av}(B)}(k))$, and so $\vec{v} \in P(\mathcal{O}v^{\mathrm{Av}(B)}(k))$. Because $\vec{v}$ is generic, it follows that $P_k^{\mathrm{Av}(B)} \subseteq P(\mathcal{O}v^{\mathrm{Av}(B)}(k))$. $\square$

Proposition 2.4. *Fix $k \in \mathbb{Z}_{\geq 1}$ and a set of patterns $B \subset \mathcal{S}$ such that the class $\mathrm{Av}(B)$ is closed for one of the two operations $\oplus, \ominus$. Then the graph $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ is strongly connected and* $\dim(P(\mathcal{O}v^{\mathrm{Av}(B)}(k))) = |\mathrm{Av}_k(B)| - |\mathrm{Av}_{k-1}(B)|$.

Proof. Consider v_1, v_2 two vertices of $\mathcal{O}v^{\mathrm{Av}(B)}(k)$, and assume that $\mathrm{Av}(B)$ is closed for $\oplus$, for simplicity. Then $\mathrm{lb}(v_1) \oplus \mathrm{lb}(v_2)$ is a permutation in $\mathrm{Av}(B)$, so $W_k(\mathrm{lb}(v_1) \oplus \mathrm{lb}(v_2))$ is a walk in the graph $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ that connects v_1 to v_2. We conclude that $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ is strongly connected.

It follows from Proposition 1.3 that $\dim(P(\mathcal{O}v^{\mathrm{Av}(B)}(k))) = |\mathrm{Av}_k(B)| - |\mathrm{Av}_{k-1}(B)|$ $\square$

Note that Propositions 2.1, 2.2, 2.3, 2.4 imply Theorem 1.7.

3 The feasible region for 312-avoiding permutations

This section is devoted to the proof of Theorem 1.9. The key step in this proof is to show an analogue of Lemma 1.5 for 312-avoiding permutations. More precisely, we have the following.

Lemma 3.1. *Fix $k \in \mathbb{Z}_{\geq 1}$ and $m \geq k$. The map W_k, from the set $\mathrm{Av}_m(312)$ of permutations of size m to the set of walks in $\mathcal{O}v^{\mathrm{Av}(312)}(k)$ of size $m - k + 1$, is surjective.*

To prove the lemma above we have to introduce the following.

Definition 3.2. Given a permutation $\sigma \in \mathcal{S}_n$ and an integer $m \in [n+1]$, we denote by σ^{*m} the permutation obtained from σ by *appending a new final value* equal to m and shifting by $+1$ all the other values larger than or equal to m. Equivalently,

$$\sigma^{*m} := \mathrm{std}(\sigma(1), \dots, \sigma(n), m - 1/2).$$

In order to prove the surjectivity stated in Lemma 3.1, given a walk $w = (e_1, \dots, e_s)$ in $\mathcal{O}v^{\mathrm{Av}(312)}(k)$, we have to exhibit a permutation $\sigma \in \mathrm{Av}(312)$ of size $s+k-1$ such that $W_k(\sigma) = w$. We do that by constructing a sequence of s permutations $(\sigma_i)_{i \leq s} \in (\mathrm{Av}(312))^s$ with size $|\sigma_i| = i+k-1$, in such a way that σ is equal to σ_s. Moreover, we will have that $\mathrm{beg}_{|\sigma_{i+1}|-1}(\sigma_{i+1}) = \sigma_i$.

The first permutation is defined as $\sigma_1 = \mathrm{lb}(e_1)$. To construct σ_{i+1} from σ_i we will show that there exists $\ell \in [|\sigma_i| + 1]$ such that $\mathrm{end}_k(\sigma_i^{*\ell})$ is equal to the pattern $\mathrm{lb}(e_{i+1})$ and $\sigma_i^{*\ell}$ avoids the pattern 312. Then we define $\sigma_{i+1} := \sigma_i^{*\ell}$, determining the sequence $(\sigma_i)_{i \leq s} \in (\mathrm{Av}(312))^s$. Finally, setting $\sigma := \sigma_s$ we have by construction that $W_k(\sigma) = w$ and that $\sigma \in \mathrm{Av}_{s+k-1}(312)$.

Therefore, in order to prove Lemma 3.1 it is enough to prove the following result.

Lemma 3.3. *Let σ be a permutation in $\mathrm{Av}(312)$ such that $\mathrm{end}_k(\sigma) = \pi$ for some $\pi \in \mathrm{Av}_k(312)$. Let $\pi' \in \mathrm{Av}_k(312)$ such that $\pi' \in C_{\mathcal{O}v^{\mathrm{Av}(312)}(k)}(\pi)$. Then there exists $m \in [|\sigma| + 1]$ such that $\sigma^{*m} \in \mathrm{Av}(312)$ and $\mathrm{end}_k(\sigma^{*m}) = \pi'$.*

Proof. We have to distinguish two cases.

Case 1: $\pi'(k) \in \{1, k\}$. We define $h := \mathbb{1}_{\{\pi'(k)=1\}} + (|\sigma|+1)\mathbb{1}_{\{\pi'(k)=k\}}$. In this case one can see that $\sigma^{*h} \in \mathrm{Av}(312)$ - the new final value h cannot create an occurrence of 312 in σ^{*h} - and that $\mathrm{end}_k(\sigma^{*h}) = \pi'$.

Case 2: $\pi'(k) \in [2, k-1]$. Assume that i is the index in the diagram of σ of the dot that corresponds to the dot of height $\pi'(k)$ in the diagram of the pattern $\text{end}_{k-1}(\sigma)$ (for an example see the red dots in Fig. 6). We claim that $\sigma^{*\sigma(i)} \in \text{Av}(312)$ and $\text{end}_k(\sigma^{*\sigma(i)}) = \pi'$. The latter is immediate. It just remains to show that $\sigma' := \sigma^{*\sigma(i)} \in \text{Av}(312)$.

Assume by contradiction that σ' contains an occurrence of 312. Since by assumption $\sigma \in \text{Av}(312)$ then the value 2 of the occurrence 312 must correspond to the final value $\sigma'(|\sigma'|) = \sigma(i)$ of σ'. Moreover, since $\pi' \in \text{Av}(312)$, the 312-occurrence must not occur in the last k elements of σ', that is the 312-occurrence must occur at the values $\sigma'(j), \sigma'(r), \sigma'(|\sigma'|)$ for some indexes $j \leq |\sigma'| - k$ and $j < r < |\sigma'|$. As a consequence, $\sigma'(j) > \sigma'(|\sigma'|)$. Moreover, since $\sigma'(i) = \sigma'(|\sigma'|) + 1$ by construction, it follows that $\sigma'(j) > \sigma'(i)$. We have two cases:

- If $r < i$ then $\sigma'(j), \sigma'(r), \sigma'(i)$ is also an occurrence of 312. A contradiction with the fact that $\sigma \in \text{Av}(312)$.
- If $r > i$ then $\sigma'(i), \sigma'(r), \sigma'(|\sigma'|)$ is also an occurrence of 312. A contradiction with the fact that $\pi' \in \text{Av}(312)$.

This concludes the proof. $\square$

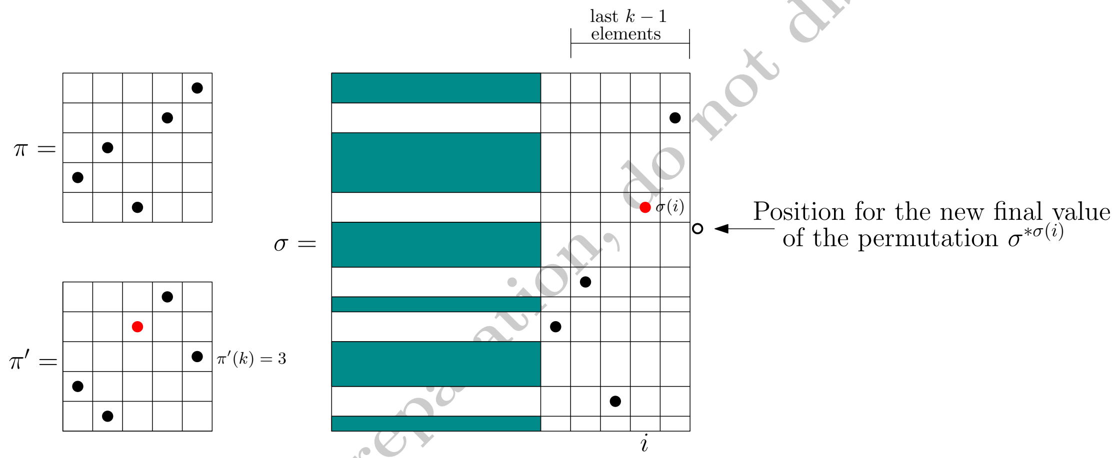


Figure 6: A schema for the proof of Lemma 3.3.

Building on Proposition 2.2 and Lemma 3.1 we can now prove Theorem 1.9.

Proof of Theorem 1.9. The fact that $P_k^{\text{Av}(312)} = P(\mathcal{O}v^{\text{Av}(312)}(k))$ follows using exactly the same proof of [BP19, Theorem 3.12] replacing Lemma 3.8 and Proposition 3.2 of [BP19] by Lemma 3.1 and Proposition 2.2 of this paper (note that in the proof of [BP19, Theorem 3.12] we also use the fact that the feasible region is closed and this is still true for $P_k^{\text{Av}(312)}$, thanks to Proposition 2.1).

The fact that the dimension of $P_k^{\text{Av}(312)}$ is $C_k - C_{k-1}$ follows from Proposition 2.4 and the well-known fact that the number of permutations of size k avoiding the pattern 312 is equal to the k-th Catalan number. Finally the fact that the vertices of $P_k^{\text{Av}(312)}$ are given by the simple cycles of $\mathcal{O}v^{\text{Av}(312)}(k)$ is a consequence of [BP19, Proposition 2.2]. $\square$

4 The feasible region for monotone avoiding permutations

Fix $\tau = n \cdots 1$, the decreasing pattern of size $n \in \mathbb{Z}_{\geq 1}$. In this section we study $P_k^{\text{Av}(\tau)}$ and we show that this is related to the cycle polytope of the coloured overlap graph $\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k)$, presented in Definition 1.16 - this is Theorem 1.17. We also compute the dimension of $P_k^{\text{Av}(\tau)}$ - this is Theorem 1.18.

4.1 The feasible region is the projection of the cycle polytope of the coloured overlap graph

Recall that we denote $\Pi : \mathbb{R}^{\mathcal{C}_{n-1}(k)} \to \mathbb{R}^{\mathrm{Av}_k(\tau)}$ for the projection mapping that sends the basis element $\vec{e}_{(\pi,\mathfrak{c})}$ to $\vec{e}_\pi$. To prove Theorem 1.17, we start by recalling that $P_k^{\mathrm{Av}(\tau)}$ is a convex set, as established in Proposition 2.2. Thus we only need to describe its extremal points. The proof that these are given by the simple cycles of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ is split into two parts, following the strategy laid out in the first paper of this series [BP19]: we first establish that, for any vertex $\vec{v} \in P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))$, we have that $\Pi(\vec{v})$ is in the feasible region. This is done by way of the walk map $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}$ (see Definition 4.1 below) that transforms a permutation $\sigma \in \mathrm{Av}(\tau)$ into a walk on the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$. Lastly, we see via a factorization theorem that any point in the feasible region results from a sequence of walks in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ that can be asymptotically decomposed into simple cycles; so the feasible region must be in the convex hull of the vectors given by simple cycles.

We recall that we denote $e_1 \bullet e_2 \bullet \dots$ for the walk $(e_1, e_2, \dots)$, to avoid a heavy use of parenthesis, and for two walks w_1, w_2 we denote their concatenation as $w_1 \bullet w_2$.

Definition 4.1 (The coloured walk function)**.** Let σ be a permutation in $\mathrm{Av}_m(\tau)$. To it, it corresponds the following walk $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma)$ on $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ of size $s = m - k + 1$:

$$\mathrm{pat}_{\{1,\dots,k\}}(\mathbb{S}(\sigma)) \bullet \dots \bullet \mathrm{pat}_{\{m-k+1,\dots,m\}}(\mathbb{S}(\sigma)),$$

where we recall that $\mathbb{C}(\sigma)$ is the RITMO colouring of σ, presented in Definition 1.11.

Remark 4.2. Given a permutation σ that avoids τ, each of the restrictions

$$\mathrm{pat}_{\{\ell-k+1,\dots,\ell\}}(\mathbb{S}(\sigma)), \quad \text{for all} \quad \ell \in [k, m],$$

is an inherited $(n-1)$-colouring. The fact that these are $(n-1)$-colourings follows because σ avoids τ, and the fact that these are inherited colouring follows from Observation 1.15 after computations similar to Eq. (3).

Example 4.3. We present the walk $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma)$ corresponding to the permutation $\sigma = 1243756$, for $k = 3$ and $\tau = 321$. The RITMO colouring of σ is 1243756, and the corresponding walk is

$$(123, 132, 213, 132, 312).$$

We can see in Fig. 7 this walk highlighted on the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$.

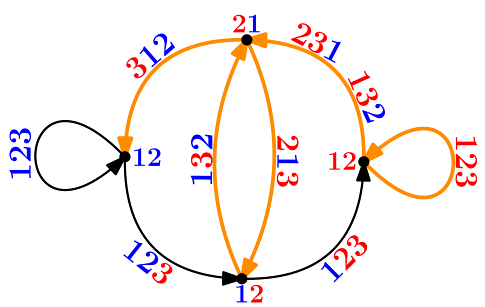


Figure 7: The walk $\mathfrak{C}W_3^{\mathrm{Av}(321)}(1243756)$ in the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$ is highlighted in orange.

With the following preliminary lemma (whose proof is postponed to Section 4.3), we can now present the proof of Theorem 1.17.

Lemma 4.4. *There exists a constant $C = C(k,n)$ such that, for any walk $w = (e_1, \dots, e_j)$ in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ there exists a walk w' in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ of length $|w'| \leq C$ and a permutation σ of size $j+k-1+|w'|$ that satisfies $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma) = w' \bullet w$.*

Proof of Theorem 1.17. Let us first establish a formula for $\widetilde{\text{c-occ}}_k(\sigma)$ with respect to the walk $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma)$. Given a permutation ρ with a colouring $\mathfrak{c}$ we set $\mathrm{per}(\rho, \mathfrak{c}) = \rho$. Given a walk w in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ and a permutation π, define $[\pi : w]$ as the number of edges e in w such that $\mathrm{per}(e) = \pi$. Thus, it easily follows that

$$\widetilde{\text{c-occ}}_k(\sigma) = \frac{1}{|\sigma|} \sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : \mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma)] \vec{e}_\pi \,. \tag{4}$$

On the other hand, from [BP19, Proposition 2.2], the vertices of the cycle polytope $P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))$ are given by the simple cycles of the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$. Specifically, the vertices are given by the vectors $\vec{e}_{\mathcal{C}} \in \mathbb{R}^{\mathcal{C}_k(n-1)}$, for each simple cycle $\mathcal{C}$ of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$, as follows:

$$(\vec{e}_{\mathcal{C}})_{(\pi,\mathfrak{c})} = \frac{\mathbb{1}[(\pi,\mathfrak{c}) \in \mathcal{C}]}{|\mathcal{C}|} \,,$$

for each inherited coloured permutation $(\pi, \mathfrak{c})$. In this way, we have that

$$\Pi(\vec{e}_{\mathcal{C}}) = \frac{1}{|\mathcal{C}|} \sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : \mathcal{C}] \vec{e}_\pi \,. \tag{5}$$

Now let us start by proving the inclusion $\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))) \subseteq P_k^{\mathrm{Av}(\tau)}$. Take $\vec{v}$ a vertex of the polytope $P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))$, that is a vector $\vec{e}_{\mathcal{C}}$ for some simple cycle $\mathcal{C}$ of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$. Because $\mathcal{C}$ is a cycle, we can define the walks $\mathcal{C}^{\bullet \ell}$ obtained by concatenating ℓ times the cycle $\mathcal{C}$. From Lemma 4.4, there exists a walk w'_ℓ with $|w'_\ell| \leq C(k,n)$ and a τ-avoiding permutation σ^ℓ of size $|w'_\ell| + \ell|\mathcal{C}| + k - 1$, such that $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma^\ell) = w'_\ell \bullet \mathcal{C}^{\bullet \ell}$. Now we see that

$$\widetilde{\text{c-occ}}_k(\sigma^\ell) \xrightarrow{\ell \to \infty} \Pi(\vec{v}).$$

In fact, we have that

$$\begin{aligned}
\widetilde{\text{c-occ}}_k\left(\sigma^\ell\right) &\overset{(4)}{=} \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : \mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma^\ell)] \vec{e}_\pi \\
&= \frac{\ell}{|\sigma^\ell|} \left(\sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : \mathcal{C}] \vec{e}_\pi \right) + \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : w'_\ell] \vec{e}_\pi \\
&\overset{(5)}{=} \frac{\ell|\mathcal{C}|}{|\sigma^\ell|} \Pi(\vec{e}_{\mathcal{C}}) + \frac{1}{|\sigma^\ell|} \vec{z}_\ell \\
&= \left(1 - \frac{k-1+|w'_\ell|}{|\sigma^\ell|}\right) \Pi(\vec{e}_{\mathcal{C}}) + \frac{1}{|\sigma^\ell|} \vec{z}_\ell \,,
\end{aligned}$$

where $\vec{z}_\ell = \sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : w'_\ell] \vec{e}_\pi$. However, because $|w'_\ell| \leq C(k,n)$, we have that

$$\frac{k-1+|w'_\ell|}{|\sigma^\ell|} \xrightarrow{\ell \to \infty} 0 \,,$$

$$\frac{1}{|\sigma^\ell|} ||\vec{z}_\ell|| \leq \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\tau)} [\pi : w'_\ell] = \frac{|w'_\ell|}{|\sigma^\ell|} \xrightarrow{\ell \to \infty} 0 \,.$$

Therefore $\widetilde{\text{c-occ}}_k(\sigma^\ell) \to \Pi(\vec{e}_{\mathcal{C}})$. This, together with Proposition 2.2, shows the desired inclusion.

For the other inclusion, consider a vector $\vec{v} \in P_k^{\text{Av}(\tau)}$, so that there is a sequence of τ-avoiding permutations σ^ℓ such that $\widetilde{\text{c-occ}}_k(\sigma^\ell) \xrightarrow{\ell\to\infty} \vec{v}$ and that $|\sigma^\ell| \xrightarrow{\ell\to\infty} +\infty$. Fix $\varepsilon > 0$, and let M be an integer such that $\ell \geq M$ implies $||\widetilde{\text{c-occ}}_k(\sigma^\ell) - \vec{v}|| < \frac{1}{2}\varepsilon$ and $|\sigma^\ell| > \frac{6k!}{\varepsilon}$. The set of edges of the walk $\mathfrak{C}W_k^{\text{Av}(\tau)}(\sigma^\ell)$ can be split into $\mathcal{C}_1^{(\ell)} \uplus \cdots \uplus \mathcal{C}_j^{(\ell)} \uplus \mathcal{T}^{(\ell)}$, where each $\mathcal{C}_i^{(\ell)}$ is a simple cycle of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ and $\mathcal{T}^{(\ell)}$ is a path that does not repeat vertices, so $|\mathcal{T}^{(\ell)}| < V(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)) \leq (k-1)!$ (for a precise explanation of this fact see [BP19, Lemma 3.13]). Thus, we get

$$
\begin{aligned}
\widetilde{\text{c-occ}}_k(\sigma^\ell) &\overset{(4)}{=} \frac{1}{|\sigma^\ell|} \sum_{\pi \in \text{Av}_k(\tau)} [\pi : \mathfrak{C}W_k^{\text{Av}(\tau)}(\sigma^\ell)]\vec{e}_\pi \\
&= \frac{1}{|\sigma^\ell|} \sum_{i=1}^{j} \sum_{\pi \in \text{Av}_k(\tau)} [\pi : \mathcal{C}_i^{(\ell)}]\vec{e}_\pi + \frac{1}{|\sigma^\ell|} \sum_{\pi \in \text{Av}_k(\tau)} [\pi : \mathcal{T}^{(\ell)}]\vec{e}_\pi \\
&\overset{(5)}{=} \frac{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1}{|\sigma^\ell|} \sum_{i=1}^{j} \frac{|\mathcal{C}_i^{(\ell)}|}{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1} \Pi(\vec{e}_{\mathcal{C}}) + \frac{1}{|\sigma^\ell|} \sum_{\pi \in \text{Av}_k(\tau)} [\pi : \mathcal{T}^{(\ell)}]\vec{e}_\pi \,.
\end{aligned}
$$

Now we set $\vec{x} := \sum_{i=1}^{j} \frac{|\mathcal{C}_i^{(\ell)}|}{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1} \Pi(\vec{e}_{\mathcal{C}_i^{(\ell)}})$ and $\vec{y} := \frac{1}{|\sigma^\ell|} \sum_{\pi \in \text{Av}_k(\tau)} [\pi : \mathcal{T}^{(\ell)}]\vec{e}_\pi$, in such a way that $\vec{x} \in \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))$ arises as a convex combination of vectors corresponding to simple cycles - note that $\sum_{i=1}^{j} |\mathcal{C}_i^{(\ell)}| = |\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1$. We simply get that

$$
\widetilde{\text{c-occ}}_k(\sigma^\ell) = \frac{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1}{|\sigma^\ell|} \vec{x} + \vec{y} \,.
$$

Thus,

$$
\text{dist}\left(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi\left(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))\right)\right) \leq ||\widetilde{\text{c-occ}}_k(\sigma^\ell) - \vec{x}|| \leq \frac{|\mathcal{T}^{(\ell)}| + k - 1}{|\sigma^\ell|} ||\vec{x}|| + ||\vec{y}|| \,. \tag{6}
$$

Observe that $||\vec{y}|| \leq \frac{1}{|\sigma^\ell|} \sum_{\pi \in \text{Av}_k(\tau)} [\pi : \mathcal{T}^{(\ell)}] = \frac{|\mathcal{T}^{(\ell)}|}{|\sigma^\ell|} \leq \frac{(k-1)!}{|\sigma^\ell|}$. Also, because the sum of the non-negative coordinates of $\vec{x}$ is one, we have that $||\vec{x}|| \leq 1$ and so that $\frac{|\mathcal{T}^{(\ell)}| + k - 1}{|\sigma^\ell|} ||\vec{x}|| \leq \frac{(k-1)! + k - 1}{|\sigma^\ell|}$. Then, we can simplify Eq. (6) to

$$
\text{dist}(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))) \leq \frac{(k-1)! + k - 1 + (k-1)!}{|\sigma^\ell|} \leq \frac{3k!}{|\sigma^\ell|} \,,
$$

so for $\ell \geq M$ we have that $\text{dist}(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))) < \frac{1}{2}\varepsilon$. As a consequence, for $\ell \geq M$,

$$
\text{dist}(\vec{v}, \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))) \leq ||\vec{v} - \widetilde{\text{c-occ}}_k(\sigma^\ell)|| + \text{dist}(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))) < \varepsilon \,.
$$

Noting that $\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))$ is a closed set, since ε is generic, we obtain that $\vec{v}$ is in the polytope $\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)))$, concluding the proof of the theorem. $\square$

It just remains to prove Lemma 4.4. This is the goal of the next two sections.

4.2 Preliminary results: basic properties of RITMO colourings and their relations with active sites

We begin by stating (without proof) some basic properties of the RITMO colouring. We suggest to compare the following lemma with Fig. 8.

Lemma 4.5. *Let σ be a permutation, and consider $\mathbb{C}(\sigma)$ its RITMO colouring.*

1. *If $i < j \in [|\sigma|]$ such that $\sigma(i) > \sigma(j)$, then $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$.*
2. *If $i < j \in [|\sigma|]$ such that $\sigma(i) < \sigma(j)$ and $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$, then there exists $k > i$ such that $\sigma(k) > \sigma(j)$ and $\mathbb{C}(\sigma)(k) = \mathbb{C}(\sigma)(i)$.*
3. *If $i < j \in [|\sigma|]$ such that $\sigma(i) < \sigma(j)$ and $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$, then there exists $h > i$ such that $\sigma(h) > \sigma(j)$ and $\mathbb{C}(\sigma)(h) = \mathbb{C}(\sigma)(j) - 1$.*

We remark that Properties 2 and 3 arise as particular cases of the same general result: if $i < j \in [|\sigma|]$ with $\sigma(i) < \sigma(j)$ and $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$, then there are indices $j > k_1 > k_2 > \cdots > k_l > i$, with $l = \mathbb{C}(\sigma)(j) - \mathbb{C}(\sigma)(i)$, such that $\mathbb{C}(\sigma)(k_s) = \mathbb{C}(\sigma)(j) - s$ for all $s \in [\ell]$, and $\sigma(j) < \sigma(k_1) < \cdots < \sigma(k_l)$. We opt to single out Properties 2 and 3 because these will be enough for our applications.

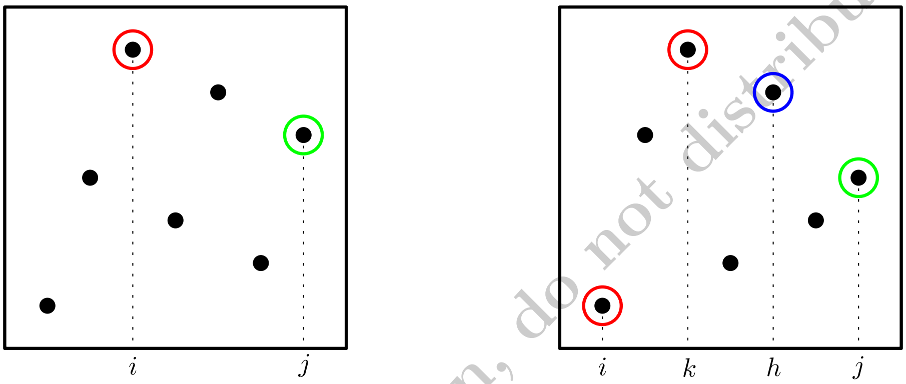


Figure 8: A schema for Lemma 4.5. The left-hand side is usefull for Property 1 and the right-hand side for Properties 2 and 3.

We now introduce a key definition.

Definition 4.6. Given a coloured permutation $(\pi, \mathfrak{c})$, and a pair (y, f) with $y \in [|\pi|+1]$, $f \geq 1$, we define the coloured permutation $(\pi, \mathfrak{c})^{*(y,f)}$ to be the permutation π^{*y} together with the colouring $\mathfrak{c}^{*f} : [|\pi| + 1] \to \mathbb{Z}_{\geq 1}$ that has $\mathrm{pat}_{[|\pi|]}(\mathfrak{c}^{*f}) = \mathfrak{c}$ and $\mathfrak{c}^{*f}(|\pi| + 1) = f$.

Let $(\pi, \mathfrak{c})$ be an inherited $(n - 1)$-coloured permutation. An *active site* is a pair (y, f) with $y \in [|\pi| + 1]$ and $f \in [n - 1]$, such that $(\pi, \mathfrak{c})^{*(y,f)}$ is an inherited $(n - 1)$-coloured permutation.

We present the following analogue of Lemma 4.5.

Lemma 4.7. *Let (y, f) be an active site of an inherited coloured permutation $(\pi, \mathfrak{c})$, and consider some index $i \in [|\pi|]$. Then*

1. *if $\mathfrak{c}(i) \geq f$, then $y > \pi(i)$;*
2. *if $\pi(i) < y$ and $\mathfrak{c}(i) < f$, then there exists $k > i$ such that $\pi(k) \geq y$ and $\mathfrak{c}(k) = \mathfrak{c}(i)$.*
3. *if $\pi(i) < y$ and $\mathfrak{c}(i) < f$, then there exists $k > i$ such that $\pi(k) \geq y$ and $\mathfrak{c}(k) = f - 1$.*

Proof. Let σ be a permutation such that $\mathrm{end}_{|\pi|+1}(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})^{*(y,f)}$, which exists because (y, f) is an active site of $(\pi, \mathfrak{c})$. The lemma is an immediate consequence of Lemma 4.5, applied to the RITMO colouring $\mathbb{C}(\sigma)$, and for $j = |\sigma|$. □

We now observe a correspondence between edges of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ and active sites of some coloured permutations.

Observation 4.8. Fix an inherited coloured permutations $(\pi_1, \mathfrak{c}_1)$ of size $k-1$. Then there exists a bijection between the set of edge $e \in \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ from $(\pi_1, \mathfrak{c}_1)$ and the set of active sites (y, f) of $(\pi_1, \mathfrak{c}_1)$. Specifically, this correspondence between edges and active sites is given as follows:

$$e = (\pi, \mathfrak{c}) \mapsto (\pi(k), \mathfrak{c}(k)), \quad (y, f) \mapsto (\pi_1, \mathfrak{c}_1)^{*(y,f)}.$$

Fix now an inherited coloured permutation $(\pi, \mathfrak{c})$. By definition, there exists some σ_0 that satisfies $\mathrm{end}_{|\pi|}(\mathbb{S}(\sigma_0)) = (\pi, \mathfrak{c})$. The goal of the next section is to show that regardless of the permutation σ_0 chosen, if (y, f) is an active site of $(\pi, \mathfrak{c})$ then there exists an index $i \in [|\sigma_0| + 1]$ such that

$$\mathrm{end}_{|\pi|+1}(\mathbb{S}(\sigma_0^{*i})) = (\pi, \mathfrak{c})^{*(y,f)}.$$

We already know that there exists a permutation σ_1 such that $\mathrm{end}_{|\pi|+1}(\mathbb{S}(\sigma_1)) = (\pi, \mathfrak{c})^{*(y,f)}$; here we are interested in finding out if σ_1 can arise as an extension of σ_0. In this way it is easy to see that the main hurdle in establishing the proof of Lemma 4.4 - that for a walk we can iteratively construct the corresponding permutation - can be surpassed.

The following two definitions refer to important values of permutations and their patterns.

Definition 4.9. Let π and σ be two permutations such that $\pi = \mathrm{end}_{k-1}(\sigma)$. For a dot at height $\ell \in [|\pi|]$ in the diagram of π, we define $\widetilde{\ell}$ to be the height of the corresponding dot in the diagram of σ. Algebraically we have that $\widetilde{\ell} = \sigma(|\sigma| - |\pi_1| + \pi_1^{-1}(\ell))$. We use the convention that $\widetilde{|\pi_1| + 1} = |\sigma| + 1$ and $\widetilde{0} = 0$.

See Fig. 9 for an example. We have the following simple result.

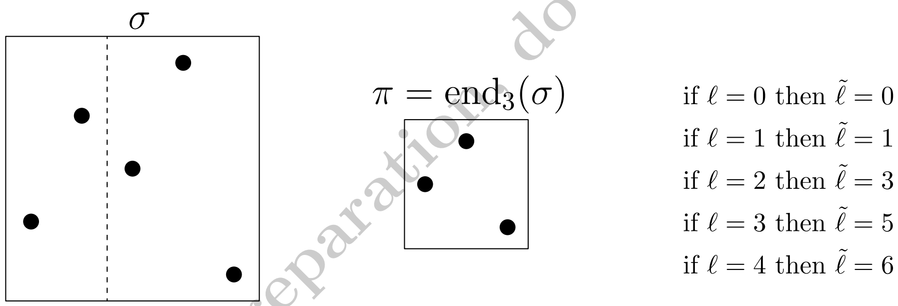


Figure 9: A schema for the Definition 4.9. On the left-hand side the permutation $\sigma = 24351$, in the middle the pattern $\pi = 231$ induced by the last three indices of σ, and on the right-hand side the quantities $\tilde{\ell}$.

Lemma 4.10. *Let σ, π be permutations such that $\pi = \mathrm{end}_{k-1}(\sigma)$. Now let $y \in [|\pi| + 1]$ and $\iota \in [|\sigma| + 1]$. Then we have that*

$$\mathrm{end}_k(\sigma^{*\iota}) = \pi^{*y} \iff \widetilde{y-1} < \iota \leq \widetilde{y}.$$

Definition 4.11. Given a permutation σ and a colour $c \in \{1, 2, \dots\}$, we define $z_\sigma(c)$ to be the smallest number $\iota \in [|\sigma| + 1]$ such that $\mathbb{C}(\sigma^{*\iota})(|\sigma| + 1) \leq c$. We use the convention that $z_\sigma(0) = |\sigma| + 2$.

See Fig. 10 for an example.

Remark 4.12. Fix a permutation σ and a colour f. Assume that there exists a maximal index p of σ such that $\mathbb{C}(\sigma)(p) = f$. Then it is immediate to observe that $z_\sigma(f) = \sigma(p) + 1$. Otherwise, if such a p does not exist, then $z_\sigma(f) = 1$.

We have the following simple result.

Lemma 4.13. *Let σ be a permutation, $f \in \mathbb{Z}_{\geq 1}$ a colour and $\iota \in [|\sigma| + 1]$. Then we have that*

$$\mathbb{C}(\sigma^{*\iota})(|\sigma| + 1) = f \iff z_\sigma(f) \leq \iota < z_\sigma(f - 1).$$

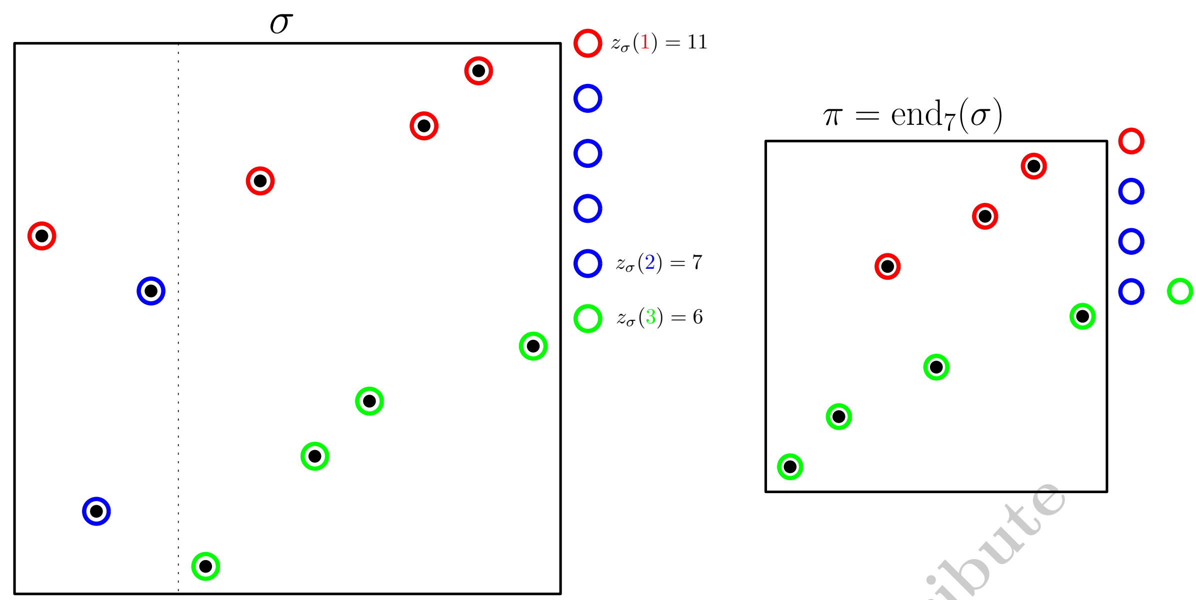


Figure 10: **Left:** A permutation $\sigma \in \mathrm{Av}(4321)$ colored with its RITMO colouring. **Right:** The inherited permutation $\pi = \mathrm{end}_7(\sigma)$. We draw with some coloured circles on the right of each diagram the active sites of the two coloured permutations. Finally the quantities $z_\sigma(1), z_\sigma(2), z_\sigma(3)$, defined in Definition 4.11, are highlighted on the right of the diagram of the permutation σ.

4.3 The proof of main lemma

In the following we will prove Lemma 4.4, where we find for any walk w in the overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$, a permutation σ such that $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma) = w' \bullet w$, where w' is an additional walk in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ of length $|w'| \leq C(k,n)$.

This is done by induction. In the inductive step, given ρ such that $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\rho) = w' \bullet w$, we observe that by taking $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\rho^{*\iota})$ we extend the walk w by precisely one edge, and by choosing the correct ι we can force the last entry of $\rho^{*\iota}$ to have the correct height and the correct colour in order to obtain the envisaged extension of the walk. To see that, we will use Lemma 4.10 and Lemma 4.13 in order to argue that we can always choose such a ι in the intersection of two intervals. The crucial part of the proof is to establish that these intervals have always non-trivial intersection.

Proof of Lemma 4.4. We start by defining the desired constant $C = C(k,n)$. Recall that the edges of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ are inherited permutations. Therefore, for each edge $e = (\pi, \mathfrak{c}) \in E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))$ we can chose σ_e, one among the smallest permutations such that $(\pi, \mathfrak{c}) = \mathrm{end}_k(\mathbb{S}(\sigma_e))$. Define $C(k,n) := \max_{e \in E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))} |\sigma_e| + n - k - 1$. We claim that this is the desired constant.

We will prove a stronger version of the lemma, by constructing a permutation σ such that $\mathbb{C}(\sigma)$ is a surjective $(n-1)$-colouring and $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma) = w' \bullet w$. This will be proven by induction on the length of the walk $j = |w|$.

We first consider the case $j = 1$. In this case, the walk $w = (e_1)$ has a unique edge, and we can select $\sigma = (n-1) \cdots 1 \oplus \sigma_{e_1}$. In this way, it is clear that $\mathbb{C}(\sigma)$ is a surjective $(n-1)$-colouring, because σ has a monotone decreasing subsequence of size $n-1$, while it is clearly τ-avoiding. Furthermore, because $\mathrm{end}_k(\mathbb{S}(\sigma)) = \mathrm{end}_k(\mathbb{S}(\sigma_{e_1})) = e_1$, we have that $\mathfrak{C}W_k^{\mathrm{Av}(\tau)}(\sigma) = w' \bullet e_1$ for some path w' such that $|w' \bullet e_1| = |w'| + 1 = |\sigma| - k + 1 = |\sigma_{e_1}| + n - k$. Therefore we have that $|w'| = |\sigma_{e_1}| + n - 1 - k \leq C$, concluding the base case.

We now consider the case $j \geq 2$. Take a walk $w = (e_1, \dots, e_j)$ in $\mathfrak{COv}^{\mathcal{Av}(1\dots n)}(k)$, and consider (by inductive hypothesis) the permutation σ such that $\mathfrak{CW}_k^{\mathrm{Av}(\tau)}(\sigma) = w' \bullet (e_1, \dots, e_{j-1})$ for some walk w' of size at most C and such that $\mathbb{C}(\sigma)$ is a surjective $(n-1)$-colouring. We are going to show that there exists a suitable $\iota \in [|\sigma|+1]$ such that $\mathfrak{CW}_k^{\mathrm{Av}(\tau)}(\sigma^{*\iota}) = w' \bullet (e_1, \dots, e_j)$ and such that $\mathbb{C}(\sigma^{*\iota})$ is a surjective $(n-1)$-colouring.

It is enough to show that we can find an index $\iota \in [|\sigma|+1]$ such that $\mathrm{end}_k(\mathbb{C}(\sigma^{*\iota})) = e_j$ and that $\sigma^{*\iota}$ is τ-avoiding. Indeed, in this case the colouring $\mathbb{C}(\sigma^{*\iota})$ is clearly a surjective $(n-1)$-colouring. Furthermore, we have that $\mathfrak{CW}_k^{\mathrm{Av}(\tau)}(\sigma^{*\iota}) = w' \bullet (e_1, \dots, e_{j-1}, e_j)$, concluding the induction step, as $|w'| \leq C$ by hypothesis.

We now show that we can find an index $\iota \in [|\sigma|+1]$ such that $\mathrm{end}_k(\mathbb{C}(\sigma^{*\iota})) = e_j$. If so, the RITMO colouring of $\sigma^{*\iota}$ has precisely $n-1$ colours, so it follows that $\sigma^{*\iota}$ is τ-avoiding. First, let $(\pi, \mathfrak{c}) = \mathrm{beg}(e_j)$. For an entry of height $l \in [|\pi|]$ in the diagram of π, we recall that $\widetilde{l} \in [|\sigma|]$ denotes the height of the corresponding entry in the diagram of σ, as in Definition 4.9. Let (y, f) be the active site of $(\pi, \mathfrak{c})$ corresponding to the edge e_j, so that $f \in [n-1]$ and $y \in [|\pi|+1]$.

From Lemma 4.10, we have that $\mathrm{end}_k(\sigma^{*\iota}) = \mathrm{per}(e_j)$ if and only if

$$\widetilde{y-1} < \iota \leq \widetilde{y}\,. \tag{7}$$

From Lemma 4.13, we have that $\mathbb{C}(\sigma^{*\iota})(|\sigma|+1) = f$ if and only if

$$z_\sigma(f) \leq \iota < z_\sigma(f-1)\,. \tag{8}$$

This gives us two intervals, and our goal is to show that these intervals have a non-trivial intersection, concluding that the desired number ι exists.

Claim. $z_\sigma(f) \leq \widetilde{y}$.

Assume by sake of contradiction that $z_\sigma(f) > \widetilde{y}$. If $y = |\pi|+1$, then $\widetilde{y} = |\sigma|+1$ by convention. This gives a contradiction because $f \geq 1$ and so $z_\sigma(f) \leq |\sigma|+1$. Thus $y < |\pi|+1$. Let $p \in [|\sigma|]$ be the maximal index such that $\mathbb{C}(\sigma)(p) = f$. We know that such a p exists, because $\mathbb{C}(\sigma)$ is a surjective $(n-1)$-colouring. By maximality of p, it follows that $\sigma(p) + 1 = z_\sigma(f) > \widetilde{y}$ (see Remark 4.12). We now split the proof into two cases; when p is included in the last $|\pi|$ indexes of σ and when it is not:

- **Assume that** $p > |\sigma| - |\pi|$. Let $q = p - (|\sigma| - |\pi|) > 0$. Because by assumption $\mathrm{end}_{k-1}(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$, we have that $f = \mathbb{C}(\sigma)(p) = \mathfrak{c}(q)$. Since we know that $\sigma(p)+1 = z_\sigma(f) > \widetilde{y}$, we have that $\pi(q) + 1 > y$. This contradicts the Property 1 of Lemma 4.7, as the active site (y, f) satisfies both $\mathfrak{c}(q) \geq f$ and $\pi(q) \geq y$.
- **Assume that** $p \leq |\sigma| - |\pi|$. Then $\sigma(p) \neq \widetilde{y}$, so $\sigma(p) > \widetilde{y}$. Using the Property 1 of Lemma 4.5 with $i = p$ and $j = \sigma^{-1}(\widetilde{y})$, we have that $f = \mathbb{C}(\sigma)(p) < \mathbb{C}(\sigma)(\sigma^{-1}(\widetilde{y}))$. So $\mathfrak{c}(\pi^{-1}(y)) = \mathbb{C}(\sigma)(\sigma^{-1}(\widetilde{y})) > f$. But this contradicts again the Property 1 of Lemma 4.7 for $i = \pi^{-1}(y)$, as the active site (y, f) satisfies both $\mathfrak{c}(\pi^{-1}(y)) > f$ and $y \leq \pi(\pi^{-1}(y))$.

Therefore, in both cases we have a contradiction.

Claim. $z_\sigma(f-1) > \widetilde{y-1} + 1$.

Assume by contradiction that $z_\sigma(f-1) \leq \widetilde{y-1} + 1$. If $f = 1$, then recall that we convention $z_\sigma(0) = |\sigma|+2$, so we have $\widetilde{y-1} \geq |\sigma|+1$. But $y \leq |\pi|+1$ so $\widetilde{y-1} \leq |\sigma|$, a contradiction. Thus $f > 1$. Let p be the maximal index in $[|\sigma|]$ such that $\mathbb{C}(\sigma)(p) = f-1$. We know that such a p exists, because $\mathbb{C}(\sigma)$ is a surjective $(n-1)$-colouring. By construction, $\sigma(p)+1 = z_\sigma(f-1) \leq \widetilde{y-1} + 1$ (see Remark 4.12), so $\sigma(p) \leq \widetilde{y-1}$. As above, we now split the proof into two cases; when p is included in the last $|\pi|$ indexes of σ and when it is not:

- **Assume that** $p > |\sigma| - |\pi|$. Let $q = p - (|\sigma| - |\pi|) > 0$. Because by assumption $\text{end}_{k-1}(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$, we have that $f - 1 = \mathbb{C}(\sigma)(p) = \mathfrak{c}(q)$. Since we know that $\sigma(p) \leq \widetilde{y-1}$, we have that $\pi(q) \leq y - 1$. Thus, by Property 2 of Lemma 4.7, there exists some $k > q$ such that $\mathfrak{c}(k) = \mathfrak{c}(q) = f - 1$. The existence of such k contradicts the maximality of p, as we get that $k + (|\sigma| - |\pi|) > p$ has $\mathbb{C}(\sigma)(k + (|\sigma| - |\pi|)) = \mathfrak{c}(k) = f - 1$.

- **Assume that** $p \leq |\sigma| - |\pi|$. Let $r = \sigma^{-1}(\widetilde{y-1})$. Then $r > |\sigma| - |\pi| \geq p$ and so $p \neq r$. It follows that $\sigma(p) = z_\sigma(f-1) < \widetilde{y-1}$.

  We now claim that $\mathbb{C}(\sigma)(r) < f - 1$. Indeed, if $\mathbb{C}(\sigma)(r) = f - 1$, because $p < r$ we have immediately a contradiction with the maximality of p. Moreover, if $\mathbb{C}(\sigma)(r) > f - 1$, Property 2 of Lemma 4.5 guarantees that there is some $k > p$ such that $\sigma(k) > \sigma(r)$ and $\mathbb{C}(\sigma)(k) = f - 1$. Again, we have a contradiction with the maximality of p.

  Now let $q = r - (|\sigma| - |\pi|)$, and observe that $\mathfrak{c}(i) = \mathbb{C}(\sigma)(r) < f - 1$. On the other hand, because $r = \sigma^{-1}(\widetilde{y-1})$, we have $\pi(q) = y - 1$. Because (y, f) is an active site of $(\pi, \mathfrak{c})$, Property 3 of Lemma 4.7 guarantees that there is some index $k > q$ of π such that $\mathfrak{c}(k) = f - 1$. But this contradicts again the maximality of p, as we would have that $\mathbb{C}(\sigma)(k + |\sigma| - |\pi|) = f - 1$ all the while $k + |\sigma| - |\pi| > q + |\sigma| - |\pi| = r > p$.

Therefore, in both cases we have a contradiction.

Using the two claims above, we can conclude that the intervals in Eqs. (7) and (8) have a non-trivial intersection, and therefore the desired index ι exists. Consequently, we can construct the desired permutation $\sigma^{*\iota}$. $\square$

4.4 Dimension of the feasible region

The computation of the dimension of $P_k^{\text{Av}(\tau)}$ is based on the description given in Theorem 1.17. This allows us to compute a lower bound by carefully studying the kernel of the map Π.

Proof of Theorem 1.18. From Theorem 1.7 we have that $\dim(P_k^{\text{Av}(\tau)}) \leq |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|$. Therefore, we just have to establish that $\dim(P_k^{\text{Av}(\tau)}) \geq |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|$.

First, recall that the projection $\Pi : \mathbb{R}^{\mathcal{C}_{n-1}(k)} \to \mathbb{R}^{\operatorname{Av}_k(\tau)}$ maps $\vec{e}_{(\pi,\mathfrak{c})} \mapsto \vec{e}_\pi$. Let $V = \operatorname{span}\{P(\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k))\}$. From the rank nullity theorem applied to the restriction $\Pi|_V$ we have that

$$\dim V = \dim \operatorname{im} \Pi|_V + \dim \ker \Pi|_V \,. \tag{9}$$

Note that the graph $\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k)$ is strongly connected (this can be proved with the argument used in Proposition 2.4). Therefore, from Proposition 1.3 and the fact that $\vec{0}$ is not in the affine span of $P(\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k))$, we have that

$$\dim V = 1 + |E(\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k))| - |V(\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k))| = 1 + |\mathcal{C}_{n-1}(k)| - |\mathcal{C}_{n-1}(k-1)| \,. \tag{10}$$

In addition, from Theorem 1.17 we have $\operatorname{im}\Pi|_V = \operatorname{span}\{\Pi(P(\mathfrak{CO}v^{\mathcal{A}v(\tau)}(k)))\} = \operatorname{span}\{P_k^{\text{Av}(\tau)}\}$, and so

$$\dim \operatorname{im} \Pi|_V = \dim \operatorname{span}\{P_k^{\text{Av}(\tau)}\} = 1 + \dim P_k^{\text{Av}(\tau)} \,, \tag{11}$$

because $\vec{0}$ is not in the affine span of $P_k^{\text{Av}(\tau)}$. Eqs. (9) to (11) together give us that

$$1 + |\mathcal{C}_{n-1}(k)| - |\mathcal{C}_{n-1}(k-1)| = 1 + \dim P_k^{\text{Av}(\tau)} + \dim \ker \Pi_V \,.$$

We now claim that $\dim \ker \Pi|_V \leq |\mathcal{C}_{n-1}(k)| - |\operatorname{Av}_k(\tau)| - |\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_{k-1}(\tau)|$. This is enough to conclude, as we get that

$$\dim P_k^{\text{Av}(\tau)} = |\mathcal{C}_{n-1}(k)| - |\mathcal{C}_{n-1}(k-1)| - \dim \ker \Pi|_V \geq |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)| \,.$$

To compute $\dim \ker \Pi|_V$, notice that $\ker \Pi|_V$ is a vector space given by two types of equations: the ones defining $\ker \Pi$ and the ones defining V. It arises then as the kernel of an $|\operatorname{Av}_k(\tau)| + |\mathcal{C}_{n-1}(k-1)|$ by $|\mathcal{C}_{n-1}(k)|$ matrix A.

We now describe this matrix A. It can be split as $A = \begin{bmatrix} A_{\text{ker}} \\ A_V \end{bmatrix}$, where A_{ker} is an $|\operatorname{Av}_k(\tau)|$ by $|\mathcal{C}_{n-1}(k)|$ matrix defined for a permutation $\pi \in \operatorname{Av}_k(\tau)$ and a coloured permutation $(\pi', \mathfrak{c}) \in \mathcal{C}_{n-1}(k)$ as $(A_{\text{ker}})_{\pi,(\pi',\mathfrak{c})} = \mathbb{1}_{\{\pi=\pi'\}}$, and A_V is the $|\mathcal{C}_{n-1}(k-1)|$ by $|\mathcal{C}_{n-1}(k)|$ incidence matrix of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$. In Example 4.14 one can see an example of such matrix for $k=3$, $n=3$ and at page 25 for $k=3$, $n=4$. We have that $\dim \ker \Pi|_V = \operatorname{nul} A = |\mathcal{C}_{n-1}(k)| - \operatorname{rk} A$, so our goal is to establish that

$$\operatorname{rk} A \geq |\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|.$$

This will be done by finding a suitable non-singular minor of A with size $|\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|$.

Construction of the minor. We are going to select a subsets $\mathfrak{C}\mathcal{E}$ of columns and a subset $\mathcal{V}$ of rows of the matrix A, both of cardinality $|\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|$.

We start by determining the set $\mathfrak{C}\mathcal{E}$. For each vertex v of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$, consider the active site $(k,1)$, which is always an active site, and the corresponding edge e (which we write from now on as $\operatorname{comp}(v)$), according to Observation 4.8. We call this the *completion process* of v. Notice that in this case we have $\operatorname{beg}(e) = v$. As a result, we can define the set of edges $\mathfrak{C}\mathcal{E}^k(k)$ obtained by the completion process of all $v \in \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ - note that the notation $\mathfrak{C}\mathcal{E}^k(k)$ recalls that the permutations in $\mathfrak{C}\mathcal{E}^k(k)$ are colored, end with the value k and have size k. Because for each vertex $v \in \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ there is exactly one distinct edge $e \in \mathfrak{C}\mathcal{E}^k(k)$ such that $\operatorname{beg}(e) = v$, we have that $|\mathfrak{C}\mathcal{E}^k(k)| = |V(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k))| = |\mathcal{C}_{n-1}(k-1)|$.

Let $\mathcal{N}\mathcal{E}^k(k)$ be the set of permutations $\sigma \in \operatorname{Av}_k(\tau)$ that satisfy $\sigma(k) \neq k$ - note that the notation $\mathcal{N}\mathcal{E}^k(k)$ recalls that the permutations in $\mathcal{N}\mathcal{E}^k(k)$ are not colored (indeed there is no $\mathfrak{C}$), do not end with the value k and have size k. Let $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ be a set of edges of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$ such that for each permutation $\sigma \in \mathcal{N}\mathcal{E}^k(k)$, there is exactly one distinct edge $e \in \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ such that $e = (\sigma, \mathfrak{c})$ for some colouring $\mathfrak{c}$. It is clear that $|\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)| = |\mathcal{N}\mathcal{E}^k(k)| = |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|$. It is also clear that the sets $\mathfrak{C}\mathcal{E}^k(k)$ and $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ are disjoint. Define $\mathfrak{C}\mathcal{E} := \mathfrak{C}\mathcal{E}^k(k) \uplus \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$, so that $|\mathfrak{C}\mathcal{E}| = |\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)|$.

Consider $\gamma = 1 \cdots k$ to be the increasing permutation of size k. We prove (for later use) that $\mathfrak{C}\mathcal{E}^k(k)$ has a unique cycle, which is the loop $\mathbb{S}(\gamma) = 1 \cdots k$ at the vertex $\operatorname{beg}_{k-1}(\mathbb{S}(\gamma)) = 1 \cdots k-1$. For a coloured permutation, we define the number of *trailing reds* to be the number of consecutive elements that are coloured red in the end of the permutation. In this way, for any edge $e \in \mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$, the number of trailing reds of $\operatorname{beg}_{k-1}(e)$ is strictly smaller than the number of trailing reds of $\operatorname{end}_{k-1}(e)$ (by definition of completion process), so a cycle cannot be formed.

We now determine the set $\mathcal{V}$. On $\operatorname{Av}_k(\tau) \uplus \mathcal{C}_{n-1}(k-1)$, consider the set

$$\mathcal{V} = \mathcal{N}\mathcal{E}^k(k) \uplus \{\gamma\} \uplus \left(\mathcal{C}_{n-1}(k-1) \setminus \{\operatorname{beg}_{k-1}(\mathbb{S}(\gamma))\}\right),$$

where we note that $\operatorname{beg}_{k-1}(\mathbb{S}(\gamma))$ is an an inherited permutation. Observe that $|\mathcal{V}| = |\operatorname{Av}_k(\tau)| - |\operatorname{Av}_{k-1}(\tau)| + |\mathcal{C}_{n-1}(k-1)|$.

Proof that the minor is non-singular. We establish now that the minor of A determined by $\mathfrak{C}\mathcal{E}$ and $\mathcal{V}$ is non-singular, by presenting two orders on these sets so that the corresponding minor becomes upper-triangular with non-zero entries in the diagonal. Recall that

$$\mathfrak{C}\mathcal{E} = \{\mathbb{S}(\gamma)\} \uplus (\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}) \uplus \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$$

and that

$$\mathcal{V} = \{\gamma\} \uplus \left(\mathcal{C}_{n-1}(k-1) \setminus \{\operatorname{beg}_{k-1}(\mathbb{S}(\gamma))\}\right) \uplus \mathcal{N}\mathcal{E}^k(k).$$

We will define two total orders in these sets that preserves the order described by the decompositions above.

Let us denote by $\leq_{\mathbb{S}(\gamma)}$ and $\leq_\gamma$ the trivial orders in $\{\mathbb{S}(\gamma)\}$ and $\{\gamma\}$. Consider a total order $\leq_{\mathcal{NE}^k(k)}$ in the set $\mathcal{NE}^k(k)$, and construct the corresponding total order $\leq_{\mathfrak{C}\mathcal{NE}^k(k)}$ in $\mathfrak{C}\mathcal{NE}^k(k)$ according to the bijection described above between $\mathcal{NE}^k(k)$ and $\mathfrak{C}\mathcal{NE}^k(k)$.

Additionally, in $\mathcal{C}_{n-1}(k-1)\setminus\{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ define the partial order $\leq_\mathcal{C}$ by setting $v_1 \leq_\mathcal{C} v_2$ if there is an edge $e \in \mathfrak{C}\mathcal{E}^k(k)$ such that $e = v_2 \to v_1$. Equivalently, $v_1 \leq_\mathcal{C} v_2$ if $\mathrm{end}_{k-1}(\mathrm{comp}(v_2)) = v_1$, that is, if the completion process of v_2 gives an edge pointing to v_1. We extend transitively $\leq_\mathcal{C}$ to become a partial order. This is a partial order because the edges in $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$ do not form any cycle, as explained above. We fix an extension of the partial order $\leq_\mathcal{C}$ to a total order on $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ and we still denote it by $\leq_\mathcal{C}$.

Finally, by identifying the edges $e \in \mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$ with the edges in the set $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ via the mapping $e \mapsto \mathrm{beg}_{k-1}(e)$, the total order $\leq_\mathcal{C}$ on $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ induces a total order also on the set $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$ that we denote $\leq_{\tilde{\mathcal{C}}}$.

Now define the following two total orders:

$$\begin{aligned} \leq_{\mathfrak{C}\mathcal{E}} &:= \leq_{\mathbb{S}(\gamma)} \bullet \leq_{\tilde{\mathcal{C}}} \bullet \leq_{\mathfrak{C}\mathcal{NE}^k(k)}, \quad \text{on } \mathfrak{C}\mathcal{E}, \\ \leq_\mathcal{V} &:= \leq_\gamma \bullet \leq_\mathcal{C} \bullet \leq_{\mathcal{NE}^k(k)}, \quad \text{on } \mathcal{V}. \end{aligned}$$

Under these total orders, one can see that the minor $\mathcal{V} \times \mathfrak{C}\mathcal{E}$ of the matrix A becomes

$$A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}} = \begin{array}{c} \begin{array}{ccc} \mathbb{S}(\gamma) & \mathfrak{C}\mathcal{E}^k(k)\setminus\{\mathbb{S}(\gamma)\} & \mathfrak{C}\mathcal{NE}^k(k) \end{array} \\ \begin{bmatrix} 1 & A_1 & A_2 \\ Z_1 & B & A_3 \\ Z_2 & Z_3 & C \end{bmatrix} \end{array} \begin{array}{c} \gamma \\ \mathcal{C}_{n-1}(k-1)\setminus\{\mathrm{beg}_{k-1}\mathbb{S}(\gamma)\} \\ \mathcal{NE}^k(k) \end{array}.$$

It is immediate to argue that Z_1 and Z_2 are zero matrices. That Z_3 is a zero matrix follows from the observation that for any edge $(\sigma, \mathfrak{c}) \in \mathfrak{C}\mathcal{E}^k(k)$ we have that $\sigma(k) = k$, so $\sigma \notin \mathcal{NE}^k(k)$. The matrix C is the identity matrix by definition of the two orders $\leq_{\mathcal{NE}^k(k)}$ on $\mathcal{NE}^k(k)$ and $\leq_{\mathfrak{C}\mathcal{NE}^k(k)}$ on $\mathfrak{C}\mathcal{NE}^k(k)$.

We finally claim that the matrix B is upper triangular. Recall that the matrix B is a minor of the incidence matrix $A_\mathcal{V}$ of the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\tau)}(k)$. Consider a non-zero off-diagonal entry $B_{e,v}$. Since it is off-diagonal then $\mathrm{beg}(e) \neq v$ by definition of the orders $\leq_\mathcal{C}, \leq_{\tilde{\mathcal{C}}}$. Moreover, since it is non-zero, we must have $\mathrm{end}_{k-1}(e) = v$, and so $v \leq_\mathcal{C} \mathrm{beg}_{k-1}(e)$ by definition of $\leq_\mathcal{C}$. We conclude, by definition of $\leq_{\tilde{\mathcal{C}}}$, that the entry $B_{e,v}$ is above the diagonal. Conversely, if $B_{e,v}$ is a diagonal entry, then $\mathrm{beg}(e) = v$ and so $B_{e,v} = 1$ is non-zero.

We conclude that $A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}}$ is an upper triangular matrix with non-zero entries on the diagonal. This concludes the proof that $\mathrm{rk}\, A \geq |\operatorname{Av}_k(\tau)| + |\mathcal{C}_{n-1}(k-1)| - |\operatorname{Av}_{k-1}(\tau)|$. $\square$

Example 4.14 (The case $n = 3$ and $k = 3$)**.** As alluded to above, we present the matrix A, introduced in the proof of Theorem 1.18 for the case $n = 3$ and $k = 3$:

$$A := \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ \color{red}{12} \\ \color{blue}{12} \\ \color{blue}{12} \\ \color{red}{21} \end{array} \begin{array}{c} \begin{array}{ccccccccc} \color{red}{123} & \color{blue}{123} & \color{blue}{123} & \color{blue}{123} & \color{red}{132} & \color{blue}{132} & \color{red}{213} & \color{red}{231} & \color{red}{312} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \end{array}.$$

We also present the corresponding upper triangular minor of dimension $|\operatorname{Av}_3(321)| + |\mathcal{C}_2(2)| - |\operatorname{Av}_2(321)|$. Some choices were made to obtain this matrix that we clarify here. The set $\mathfrak{C}\mathcal{NE}^3(3)$ of edges in bijection with $\mathcal{NE}^3(3) = \{231, 132, 312\}$ via the mapping per was chosen to be $\mathfrak{C}\mathcal{NE}^3(3) = \{231, 132, 312\}$, but could have been, for instance, $\{231, 132, 312\}$. The ordering on these sets must be coherent with the mapping per, thus we fix

$$\{132 < 231 < 312\}, \quad \{132 < 231 < 312\}\,.$$

For the order on the set $\mathcal{C}_{n-1}(k-1) \setminus \{\operatorname{beg}_{k-1} \mathbb{S}(\gamma)\} = \{12, 12, 21\}$, we have to choose a linear order such that $12 \leq 12$ and $12 \leq 21$, thus the following works:

$$\{12 < 12 < 21\}\,.$$

Finally, the corresponding order in $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\} = \{123, 123, 213\}$ is

$$\{\operatorname{comp}(12) < \operatorname{comp}(12) < \operatorname{comp}(21)\} = \{123 < 123 < 213\}\,.$$

In this way, the matrix $A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}}$ is upper triangular:

$$A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}} \coloneqq \begin{array}{c} \\ 123 \\ 12 \\ 12 \\ 21 \\ 132 \\ 231 \\ 312 \end{array} \begin{array}{c} \begin{array}{ccccccc} 123 & 123 & 123 & 213 & 132 & 231 & 312 \end{array} \\ \left[\begin{array}{c|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

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$$A := \begin{array}{cc} & \begin{array}{ccccccccccccccccccccccccccccc} 123 & 123 & 123 & 123 & 123 & 123 & 123 & 123 & 123 & 123 & 132 & 132 & 132 & 132 & 132 & 132 & 213 & 213 & 213 & 213 & 231 & 231 & 231 & 231 & 312 & 312 & 312 & 312 & 321 \end{array} \\ \begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 21 \\ 21 \\ 21 \end{array} & \left[\begin{array}{ccccccccccccccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \end{array}.$$

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