Some Inequalities Related to the Seysen Measure of a Lattice

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Abstract

Given a lattice L, a basis B of L together with its dual B^*, the orthogonality measure $S(B) = \sum_i ||b_i||^2||b_i^*||^2$ of B was introduced by M. Seysen [9] in 1993. This measure (the Seysen measure in the sequel, also known as the *Seysen metric* [11]) is at the heart of the Seysen lattice reduction algorithm and is linked with different geometrical properties of the basis [6, 7, 10, 11]. In this paper, we derive different expressions for this measure as well as new inequalities related to the Frobenius norm and the condition number of a matrix.

Key Words: Lattice, orthogonality defect, Seysen measure, HGA inequality
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1 Introduction, Notations and Previous Results

An n-dimensional (real) lattice L is defined as a subset of $\mathbb{R}^m$, $n \leq m$, generated by $B = [b_1|\dots|b_n]^t$, where the b_i are n linearly independent vectors over $\mathbb{R}$ in $\mathbb{R}^m$, as

$$L = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in \mathbb{Z} \right\}.$$

In this paper, the rows of the matrix B span the lattice. Any other matrix $B' = UB$, where $U \in GL_n(\mathbb{Z})$, generates the same lattice. The volume $\operatorname{Vol} L$ of L is the well defined real number $(\det BB^t)^{1/2}$. The dual lattice of L is defined by the basis $B^* = (B^+)^t$, where B^+ is the Moore-Penrose inverse, or pseudo-inverse, of B. If $B^* = [b_1^*|\dots|b_n^*]^t$, then since $BB^+ = I_n$, we have $\langle b_i, b_j^* \rangle = \delta_{i,j}$. Lattice reduction theory deals with the problem of identifying and computing bases of a given lattice whose vectors are *short* and *almost orthogonal*. There are several concepts of reduced bases, such as the concepts of Minkovsky reduced, LLL reduced [5] and Korkin-Zolotarev reduced basis [3]. In 1990, Hastad and Lagarias [1] proved that in all lattices of full rank (i.e., when $n = m$), there exists a basis B such that both B and B^* consist in relatively short vector, i.e., $\max_i ||b_i|| \cdot ||b_i^*|| \leq \exp(O(n^{1/3}))$. In 1993, Seysen [9] improved this upper bound to $\exp(O(\ln^2(n)))$ and suggested to use the expression $S(B) := \sum_i ||b_i||^2||b_i^*||^2$. This definition also allowed him to define a new concept of reduction: a basis B of L is Seysen reduced if $S(B)$ is minimal among all bases of L (see also [4] for a study of this reduction method). A relation between the orthogonality defect [2, 11]

$$\operatorname{od}(B) := 1 - \frac{\det BB^t}{\prod_{i=1}^{n} ||b_i||^2} \in [0, 1]$$

and the Seysen measure $S(B)$ is given in [11] where the following bounds can be found:

$$n \leq S(B) \leq \frac{n}{1-\mathrm{od}(B)}, \tag{1.1}$$

$$0 \leq \mathrm{od}(B) \leq 1-\frac{1}{(S(B)-n+1)^{n-1}}. \tag{1.2}$$

Clearly, the smaller the Seysen measure is, the closer to orthogonal the basis is, showing that the Seysen measure describes the quality of the angle behavior of the vectors in a basis. The length of the different vectors are nevertheless not part of the direct information given by the measure, but Inequality 1.2 gives

$$\prod_{i=1}^{n} ||b_i|| \leq (S(B)-n+1)^{\frac{n-1}{2}} \cdot \mathrm{Vol}\, L$$

which in turn provides the inequality

$$\min_i ||b_i|| \leq (S(B)-n+1)^{(n-1)/2n} (\mathrm{Vol}\, L)^{1/n}. \tag{1.3}$$

Note that such a type of inequality appears in the context of lattice reduction as

$$\min_i ||b_i|| \leq \sqrt{n}\, (\mathrm{Vol}\, L)^{1/n} \quad \text{for Korkin Zolotarev and Minkovsky reduced bases}$$
$$\min_i ||b_i|| \leq (4/3)^{(n-1)/4} (\mathrm{Vol}\, L)^{1/n} \quad \text{for LLL reduced bases.}$$

In this paper, we start by revisiting Seysen's bound $\exp(O(\ln(n)^2))$ by computing the hidden constant in Landau's notation. Then we present new expressions for the Seysen measure, connecting the measure with the condition number and the Frobenius norm of a matrix and allowing us to improve some of the existing bounds. We will from now on suppose that $m = n$, since Equality 3.6 below shows that the Seysen measure is invariant under isometric embeddings.

2 Explicit Constant in Seysen's Bound

We show in this section that the hidden constant in Seysen's bound $\exp(O(\ln(n)^2))$ can be upper bounded by $1+\frac{2}{\ln 2}$. The proof is not new, but revisits some details in the original proof of Seysen [9, Theorem 7] by using explicit bounds given in [5, Proposition 4.2]. Let us define the two main ingredients of the proof. First, if $N(n,\mathbb{R})$ and $N(n,\mathbb{Z})$ are the group of lower triangular unipotent $n \times n$ matrices over $\mathbb{R}$ and $\mathbb{Z}$ respectively (i.e. matrices with 1 in the diagonal), then following [1] and [9], and if $||X||_\infty = \max_{i,j} |X_{ij}|$, we define $S(n)$ for all $n \in \mathbb{N}$ by

$$S(n) = \sup_{A \in N(n,\mathbb{R})} \left(\inf_{T \in N(n,\mathbb{Z})} \max(||TA||_\infty, ||(TA)^{-1}||_\infty) \right).$$

In [9], the author proves that $S(2n) \leq S(n) \cdot \max(1, n/2)$, and concludes that $S(n) = \exp(O((\ln n)^2))$. We would like to point out that the latter is not true in general, unless some other property of the function S is invoked. Indeed, an arbitrary map s defined on the set of odd integers, e.g. $s(2n+1) = \exp(2n+1)$, and extended to $\mathbb{N}$ with the rule $s(2n) = n/2 \cdot s(n)$ satisfies the condition $s(2n) \leq s(n) \cdot \max(1, n/2)$ but we have $s(n) \neq \exp(O((\ln n)^2))$ in general. This point seems to have been overlooked in [9]. However, in our case, we have the following in addition.

Lemma 2.1 $\forall n \leq m \in \mathbb{N}, S(n) \leq S(m)$

Proof: It is not difficult to see that for all $A \in N(n,\mathbb{R})$, there exists a matrix $T_A \in N(n,\mathbb{Z})$ such that

$$\inf_{T \in N(n,\mathbb{Z})} \max(||TA||_\infty, ||(TA)^{-1}||_\infty) = \max(||T_A A||_\infty, ||(T_A A)^{-1}||_\infty).$$

See the Remark following Definition 4 of [9] for the details. As a consequence, in order to prove the lemma, it is sufficient to show that

$$\sup_{A \in N(n,\mathbb{R})} \max(||T_A A||_\infty, ||(T_A A)^{-1}||_\infty) \leq \sup_{A' \in N(n+1,\mathbb{R})} \max(||T_{A'} A'||_\infty, ||(T_{A'} A')^{-1}||_\infty). \tag{2.4}$$

Let us consider the map i from $N(n,\mathbb{R})$ to $N(n+1,\mathbb{R})$ defined by mapping a matrix A to the block matrix $\mathrm{diag}\,(1,A)$. The map i is a group homomorphism and thus $i(A)^{-1} = i(A^{-1}) = \mathrm{diag}\,(1,A^{-1})$. We claim that for all $A \in N(n,\mathbb{R})$ and all $T \in N(n,\mathbb{Z})$, we have

$$\max(||i(TA)||_\infty, ||i(TA)^{-1}||_\infty) = \max(||TA||_\infty, ||(TA)^{-1}||_\infty). \tag{2.5}$$

First, if $\max(||i(TA)||_\infty, ||i(TA)^{-1}||_\infty) = 1$, then the above equality is straightforward, due to the definition of $||.||_\infty$. Let us then consider the case where the maximum is not 1. Notice that since $||X||_\infty \geq 1$ is true for all matrix X in $N(m,\mathbb{R})$, we have that $\max(||X||_\infty, ||X^{-1}||_\infty) \geq 1$ and so $\max(||i(TA)||_\infty, ||i(TA)^{-1}||_\infty) > 1$. As a consequence the maximum in $\max(||i(TA)||_\infty, ||i(TA)^{-1}||_\infty)$ is achieved by one of the entries of $i(TA)$ or $i(TA)^{-1}$, and this entry cannot be the one in the upper left corner. The maximum is then the same for both sides of (2.5). This proves the above claim. Now, since

$$\sup_{A' \in N(n+1,\mathbb{R})} \max(||T_{A'}A'||_\infty, ||(T_{A'}A')^{-1}||_\infty) \geq \max(||i(TA)||_\infty, ||i(TA)^{-1}||_\infty) = \max(||TA||_\infty, ||(TA)^{-1}||_\infty),$$

is true for all $A \in N(n,\mathbb{R})$, taking the supremum on the left hand side, we see that Inequality 2.4 is correct. □

This lemma makes the following inequalities valid:

$$S(n) = S(2^{\log_2 n}) \leq S(2^{\lceil \log_2 n \rceil}) \leq 2^{\lceil \log_2 n \rceil - 2} \cdot 2^{\lceil \log_2 n \rceil - 3} \cdot \ldots \cdot 2 \cdot 1 \leq \exp\left(\frac{(\ln n)^2}{2\ln 2}\right).$$

The second ingredient we need is related to the Korkin-Zolotarev reduced bases of a lattice L. Such bases are well known, see e.g. [5], and one of their properties is the following: if B is a Korkin-Zolotarev reduced basis of L, and if $B = HK$, where $H = (h_{ij})$ is a lower triangular matrix and K is an orthogonal matrix, then for all $1 \leq i \leq j \leq n$, we have

$$h_{jj}^2 > h_{ii}^2 (j-i+1)^{-1-\ln(j-i+1)}.$$

This is a direct consequence of [5, Proposition 4.2] and the fact that the concept of Korkin-Zolotarev reduction is recursive. See [9] for the details. In [9], the author concludes that $\frac{h_{ii}^2}{h_{jj}^2} = \exp(O((\ln n)^2))$ but we have the more precise statement that

$$\frac{h_{ii}^2}{h_{jj}^2} \leq \exp((\ln(j-i+1))^2 + \ln(j-i+1)) \leq \exp((\ln n)^2 + \ln n).$$

Let us now revisit the proof of [9, Theorem 7] by making use of the previous inequalities. This theorem states that for every lattice L there is a basis $\tilde{B} = [\tilde{b_1}|\ldots|\tilde{b_n}]^t$ with reciprocal basis $\tilde{B}^* = [\tilde{b_1}^*|\ldots|\tilde{b_n}^*]^t$ which satisfies

$$||\tilde{b_i}|| \cdot ||\tilde{b_i}^*|| \leq \exp(c_2(\ln n)^2)$$

for all i and for a fixed c_2, independent of n. We explicit now an upper bound for the constant c_2. Given a lattice L and a Korkin-Zolotarev reduced basis $B = HK$ as above, the proof of [9, Theorem 7] shows that there exists a basis $\tilde{B}$, constructed from B, such that

$$||\tilde{b_i}||^2 \cdot ||\tilde{b_i}^*||^2 \leq n^2 \cdot \max_{k \geq j} \left\{ \frac{h_{jj}^2}{h_{kk}^2} \right\} \cdot S(n)^4$$

Making use of the previous inequalities, we can write

$$||\tilde{b_i}||^2 \cdot ||\tilde{b_i}^*||^2 \leq n^2 \cdot \exp((\ln n)^2 + \ln n) \cdot \exp\left(\frac{4(\ln n)^2}{2\ln 2}\right) = \exp\left(\left(\frac{2}{\ln 2} + 1\right)(\ln n)^2 + 3\ln n\right).$$

which shows that $c_2 < \frac{1}{\ln 2} + \frac{1}{2} + \frac{3}{2\ln n} < \frac{1}{\ln 2} + \frac{1}{2} + \frac{3}{2\ln 2} = \frac{5}{2\ln 2} + \frac{1}{2}$ and gives the following proposition:

Proposition 2.2 *For every lattice L there is a basis B which satisfies*

$$S(B) \leq \exp\left(\left(\frac{2}{\ln 2} + 1\right)(\ln n)^2 + 4\ln n\right).$$

3 Explicit Expression for the Seysen Measure

In this section, we present different expressions for the Seysen measure. First, let us recall the following known expression for the measure. Given a basis B of L, by definition of B^*, for all $0 \leq j \leq n$, the vector b_j^* is orthogonal to L_j, where L_j is the sublattice of L generated by all the vectors of B except b_j. If β_j is the angle between b_j and b_j^* and α_j is the angle between b_j and L_j, we have $\cos^2 \beta_i = \sin^2 \alpha_i$ and

$$S(B) = \sum_i ||b_i||^2 ||b_i^*||^2 = \sum_i \frac{\langle b_i, b_i^* \rangle^2}{\cos^2 \beta_i} = \sum_i \frac{1}{\sin^2 \alpha_i}. \tag{3.6}$$

This has already been used in [4, 11]. We introduce now the following new representation, which can be used to define the Seysen measure without any references to the dual basis:

Proposition 3.1 *For every lattice L, if $B = [b_1| \dots |b_n]^t$ is a basis of L with $B = D \cdot V$ where $D = \operatorname{diag}(||b_1||, \dots, ||b_n||)$, then*

$$S(B) = ||V^{-1}||^2$$

where $||.||$ is the Frobenius norm, i.e., $||X|| = \sqrt{\sum_{i,j} |x_{ij}|^2}$.

Proof: Let $M = BB^t$. Using $||X||^2 = \operatorname{tr}(XX^t)$ and $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$, we have

$$||V^{-1}||^2 = \operatorname{tr}(V^{-1}(V^{-1})^t) = \operatorname{tr}(D^2 M^{-1}) = \sum_i ||b_i||^2 \cdot \left(M^{-1}\right)_{i,i}.$$

Since $M^{-1} = \frac{1}{\det M} \operatorname{comat}(M)$, where $\operatorname{comat}(M)$ is the comatrix of M, we have

$$\left(M^{-1}\right)_{i,i} = \frac{1}{\det M} \operatorname{comat}(M)_{i,i} = \frac{\det M^{i,i}}{\det M}$$

where $M^{i,i}$ is the square matrix obtained from M by deleting the i-th row and the i-th column of M. So if B^i is the matrix obtained by deleting the i-th row of B, we have

$$\det M^{i,i} = \det B^i (B^i)^t = (\operatorname{Vol} L_i)^2$$

which gives

$$\frac{\det M^{i,i}}{\det M} = \frac{(\operatorname{Vol} L_i)^2}{(\operatorname{Vol} L)^2} = \frac{(\operatorname{Vol} L_i)^2}{(||b_i|| \cdot \operatorname{Vol} L_i \cdot \sin \alpha_i)^2} = \frac{1}{||b_i||^2 \sin^2 \alpha_i}.$$

Finally,

$$||V^{-1}||^2 = \sum_i ||b_i||^2 \cdot \left(M^{-1}\right)_{i,i} = \sum_i ||b_i||^2 \cdot \frac{1}{||b_i||^2 \sin^2 \alpha_i} = S(B).$$

□

Another way of looking at the previous result is with the help of the (Frobenius) condition number of an invertible matrix X which is defined as $\kappa(X) = ||X|| \cdot ||X^{-1}||$.

Corollary 3.2 *With the above notation, we have $S(B) = \frac{\kappa(V)^2}{n}$.*

By defining the matrix U as $U = VV^t$, then $BB^t = DUD$, where D is as above, and if θ_{ij} is the angle between b_i and b_j, then $U = (\cos \theta_{ij})_{ij}$. The matrix U is a symmetric positive definite matrix, and the eigenvalues $\lambda_1, \dots, \lambda_n$ of U are real positive.

Corollary 3.3 *With the above notation, we have $S(B) = \operatorname{tr}(U^{-1}) = \sum_i \frac{1}{\lambda_i}$.*

From the equality $BB^t = DUD$, we have $(\mathrm{Vol}\, L)^2 = \det U \cdot \prod_i ||b_i||^2$ which in turn leads to

$$\prod_i ||b_i|| = (\det U)^{-1/2} \cdot \mathrm{Vol}\, L = \left(\prod_i \frac{1}{\lambda_i}\right)^{1/2} \cdot \mathrm{Vol}\, L. \tag{3.7}$$

The arithmetic-geometric mean inequality applied to the λ_i's, $(\prod_i 1/\lambda_i)^{1/n} \leq \frac{1}{n}\sum_i 1/\lambda_i$, immediately gives the inequality

$$\prod_i ||b_i|| \leq \left(\frac{1}{n}\sum_i \frac{1}{\lambda_i}\right)^{\frac{n}{2}} \cdot \mathrm{Vol}\, L = \left(\frac{S(B)}{n}\right)^{\frac{n}{2}} \cdot \mathrm{Vol}\, L.$$

However, we also have the equality $\sum_i \lambda_i = \mathrm{tr}\, U = n$, which affords a slightly better upper bound for the geometric mean. Indeed, the harmonic-geometric-arithmetic mean inequalities applied to the $1/\lambda_i$'s imply that if $g = (\prod_i 1/\lambda_i)^{1/n}$, $h = \left(\frac{1}{n}\sum_i \lambda_i\right)^{-1} = 1$ and $a = \frac{1}{n}\sum_i \frac{1}{\lambda_i} = \frac{S(B)}{n}$, then we have $h \leq g \leq a$, but we also have the following result, which is [8, Corollary 3.1].

Lemma 3.4 *With the above notations, if $\alpha = 1/n$, we have*

$$g \leq \left(\frac{a - h(1-2\alpha) - \sqrt{(a-h)(a-h(1-2\alpha)^2)}}{2\alpha}\right)^{\alpha} \left(\frac{a + h(1-2\alpha) + \sqrt{(a-h)(a-h(1-2\alpha)^2)}}{2(1-\alpha)}\right)^{1-\alpha}.$$

This leads to the following inequality:

Proposition 3.5 *With the above notation, we have*

$$\prod_i ||b_i|| \leq e^{1/2} \cdot \left(\frac{S(B)+1}{n}\right)^{\frac{n-1}{2}} \cdot \mathrm{Vol}\, L. \tag{3.8}$$

Proof: Since $(1-2/n)^2 \leq 1$, we have

$$(a-h)^2 \leq (a-h)(a-h(1-2/n)^2) \leq (a-h(1-2/n)^2)^2$$

and thus the upper bound of the previous Lemma gives

$$g \quad \leq \quad \left(\frac{a-h(1-2/n)-(a-h)}{2/n}\right)^{1/n} \left(\frac{a+h(1-2/n)+(a-h(1-2/n)^2)}{2(1-1/n)}\right)^{1-1/n}.$$

After suitable simplification, we obtain

$$g \quad \leq \quad a \cdot \left(\frac{h}{a}\right)^{1/n} \cdot \left(1 + \frac{h}{a}\cdot\left(1-\frac{2}{n}\right)\cdot\frac{1}{n}\right)^{1-1/n} \cdot \left(1+\frac{1}{n-1}\right)^{1-1/n}.$$

Since $(1+\frac{1}{n-1})^{n-1} < e$, taking the n-th power of both sides of the previous inequality gives

$$\prod_i 1/\lambda_i < e \cdot \left(\frac{S(B)+1-\frac{2}{n}}{n}\right)^{n-1} < e \cdot \left(\frac{S(B)+1}{n}\right)^{n-1}.$$

The result follows by applying the previous inequality to Equation (3.7). □

This is an improvement by a factor of roughly $n^{n/2}$ of the bound given by (1.3), and can be used to strengthen the bound of the orthogonality defect (1.1):

Corollary 3.6 *With the above notations, we have*

$$\mathrm{od}(B) \leq 1 - \frac{1}{e}\left(\frac{n}{S(B)+1}\right)^{n-1}$$

Combining the previous proposition with the explicit bound of Proposition 2.2, we have the following proposition:

Proposition 3.7 *For every lattice L, if $B = [b_1|\ldots|b_n]^t$ is a Seysen reduced basis, then*

$$\min_i ||b_i|| \leq \exp\left(\left(\frac{1}{\ln 2} + \frac{1}{2}\right)(\ln n)^2 + O(\ln n)\right) \cdot (\mathrm{Vol}\, L)^{1/n}.$$

4 Conclusion

In this article, we gave an explicit upper bound for the constant hidden inside Landau's notation of the original bound of the Seysen measure [9]. We also developed the connection between the Seysen measure and standard linear algebra concepts such as the Frobenius norm and the condition number of a matrix. This allowed us to improve known upper bounds for the Seysen measure and the orthogonality defect.

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