A New Family of Almost Identities

Gérard Maze
gerard.maze@epfl.ch

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Outline of Talk:

1. What is an almost identity?
2. Behaviour of $\sum_{k\geqslant 1} \frac{1}{1+2^k x}$ near 0.
3. The case of $\sum_{k\in\mathbb{Z}} \frac{1}{(2^{-k/2}+2^{k/2})^2}$.
4. The recurrence relation.

1. What is an almost identity?

This talk is about a phenomenon that is absolutely not crucial in mathematics. It is about non-equality between real numbers.

However, there exists non-equalities that are so surprising, so unexpected, that they become simply beautiful.

An almost identity is a non-equality that reveal a strange approximation between two “good arithmetical numbers”.

Examples

Certainly, the most famous instance of such a situation was revealed by S.Ramanujan in the 1914 paper "Modular Equation and Approximation to π". Based on properties of the j-function, he found a family of almost identities, among them

$$
\begin{aligned}
e^{\pi\sqrt{37}} &= 199148648 - 0.219\ldots \cdot 10^{-4}, \\
e^{\pi\sqrt{58}} &= 24591257752 - 0.177\ldots \cdot 10^{-6}, \\
e^{\pi\sqrt{163}} &= 262537412640768744 - 0.749\ldots \cdot 10^{-12}.
\end{aligned}
$$

Another surprising result comes from the average length of a segment in an isosceles right triangle with unit legs. If l is this average length, then

$$
\begin{aligned}
l &= \frac{1}{30}\Big(2 + 4\sqrt{2} + (4+\sqrt{2})\sinh^{-1}(1)\Big) \\
&= 0.4142933026\ldots \\
&= (\sqrt{2}-1) - 0.8\ldots \cdot 10^{-4}.
\end{aligned}
$$

The four examples above are however different in essence: the first three come from a deep property of a complex mathematical object (the j-function) and the last has a good chance to be a genuine arithmetical coincidence.

A natural question that comes to mind in presence of such a non-identity is therefore whether the phenomenon is purely coincidental, or comes from a more subtle process.

For instance, in the equation

$$e^{\pi} - \pi = 19.999099979... ,$$

it is not clear at all whether the almost identity pops up from a deep connection between e and π or just because the expression *happens* to be close to 20.

On the other hand, the fact that the numbers

$$e^{\pi\sqrt{n}}$$

are close to integers for many $n \in \mathbb{N}$ is an indication that the non-equalities can be explained by some mathematical reasonning.

As an example, let us consider the sequence

$$h_n = \frac{n!}{2(\ln(2))^{n+1}},$$

for $1 \leqslant n \leqslant 17$, discovered by D. Hickerson. Here, the phenomenon has a good chance to be no coincidence at all.

Indeed, these numbers are close to integers due to the fact that the quotient is the dominant term in an infinite series for the number of possible outcomes of a race between n people (where ties are allowed).

Recently, J.M. Borwein and P.B. Borwein discovered several families of almost identities, leading to a systematic study of such phenomena. These were based on mathematical concepts that lead to clear explanations. Among the non-identities studied by these authors, let us mention the following striking example:

$$\sum_{k=-\infty}^{\infty} \frac{1}{10^{(k/100)^2}} \cong 100\sqrt{\frac{\pi}{\ln(10)}},$$

correct to at least 18,000 digits.

2. Behaviour of $\sum_{k\geqslant 1} \frac{1}{1+2^k x}$ near 0

The function $f(x) = \sum_{k\geqslant 1} \frac{1}{1+2^k x}$ is important in the analysis of Brent's model of the Binary Euclidian Algorithm. Its behaviour near zero is needed in the estimation of the asymptotic average running time of the algorithm.

Brent's model states the existence of a function $G(x)$, $x \in [0,1]$, satisfying $G(0) = 1$, $G(1) = 0$ and

$$G(x) = \sum_{k\geqslant 1} 2^{-k} \left(G\left(\frac{1}{1+2^k/x}\right) - G\left(\frac{1}{1+2^k x}\right)\right).$$

The existence of the function G has been conjectured by Brent in 1976. If we define the operator F by

$$F(g)(x) := \sum_{k\geqslant 1} 2^{-k}\left(g\left(\frac{1}{1+2^k/x}\right) - g\left(\frac{1}{1+2^k x}\right)\right),$$

then we have

Theorem [GM 04] The operator F has at least one fixed point with $g(0) = 1$ and $g(1) = 0$, i.e., there exists a function that satifies the conditions of Brent's model.

The behaviour of $f(x) = \sum 1/(1+2^k x)$ near 0 is needed in the proof.

The function $f(x) = \sum_{k \geqslant 1} \frac{1}{1+2^k x}$ is divergent near 0. Let us try to figue out its rate of divergence. First, if $x = 2^{-m}$, $m \in \mathbb{N}$, we have

$$f(2^{-m}) = \underbrace{\sum_{k=1}^{m} \frac{1}{1+2^{k-m}}}_{\cong m} + \underbrace{\sum_{k \geqslant 1} \frac{1}{1+2^k}}_{\text{constant}}$$

which leads to $f(x) = -\lg(x) - 1/2 + r(x)$, with r bounded and $r(2^{-m}) \xrightarrow[m\to\infty]{} 0$.

What is more precisely the behaviour of $r(x)$ near zero?

Suppose that $f(x) = -\lg(x) + 1/2 + ax + O(x^2)$. Then

$$\begin{aligned} 0 &= \lim_{m\to\infty} \left(f(2^{-s}) - s)\right)' \big|_{s=m} \\ &= \lim_{m\to\infty} \ln(2) \cdot \sum_{k=1}^{\infty} \frac{\left(2^{(k-m)}\right)}{\left(1 + 2^{(k-m)}\right)^2} - 1 \\ &= \ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{-k/2} + 2^{k/2}\right)^2} - 1. \end{aligned}$$

This leads to the equality:

$$\ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{-k/2} + 2^{k/2}\right)^2} = 1.$$

However, this is not true. In fact, we have

$$\ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{-k/2} + 2^{k/2}\right)^2} = 1 + 0.48\ldots \cdot 10^{-10}.$$

Is this a coincidence, or is a more subtle process hidden?

Let us define the real numbers u_n as follows:

$$u_n := \ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{k/2} + 2^{-k/2}\right)^n}, \quad n \in \mathbb{N} - \{0\}.$$

It turns out that

$$
\begin{aligned}
u_1 &= \pi + 0.53\ldots \cdot 10^{-11} \\
u_2 &= 1 + 0.48\ldots \cdot 10^{-10} \\
u_3 &= \frac{\pi}{2^3} + 0.22\ldots \cdot 10^{-9} \\
u_4 &= \frac{1}{6} + 0.67\ldots \cdot 10^{-9} \\
u_5 &= \frac{3\pi}{2^7} + 0.15\ldots \cdot 10^{-8} \\
u_6 &= \frac{1}{30} + 0.29\ldots \cdot 10^{-8} \qquad \text{etc} \ldots
\end{aligned}
$$

At this point it is difficult to believe that the almost identities generated by the u_n are genuine coincidences. There must be an explanation, and we might expect a closed formula for the almost identities

$$u_1 \cong \pi \quad u_3 \cong \frac{\pi}{8} \quad u_5 \cong \frac{3\pi}{128} \quad u_7 \cong \frac{5\pi}{1024} \quad u_9 \cong \frac{35\pi}{32768}$$

$$u_2 \cong 1 \quad u_4 \cong \frac{1}{6} \quad u_6 \cong \frac{1}{30} \quad u_8 \cong \frac{1}{140} \quad u_{10} \cong \frac{1}{630}$$

3. The case of $\sum_{k\in\mathbb{Z}} \frac{1}{(2^{-k/2}+2^{k/2})^2}$.

Consider again our first almost identity, namely

$$\ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{-k/2} + 2^{k/2}\right)^2} = 1 + 0.48... \cdot 10^{-10}$$

based on the function

$$f(x) = \sum_{k\geqslant 1} \frac{1}{1 + 2^k x}.$$

The game plan is to express the functions $f(x)$ in a completely different manner in order to compute the series. The keystone of this process is the Mellin transform. Recall that the Mellin transform of a locally Lebesgue integrable function $f(x)$ over $]0,\infty[$ is the function

$$f^*(s) = \int_0^\infty f(x)x^{s-1}ds.$$

The conditions $f(x) \sim_0 O(x^u)$ and $f(x) \sim_\infty O(x^v)$, with $u > v$ guarantee that $f^*(s)$ exists in the strip $-u < \Re s < -v$.

Mellin's inversion formula states that if f is continuous and $c \in]-u, -v[$, then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds,$$

and in a neighbourhood of 0, we have

$$f(x) = \sum_{\Re s_i < c} \operatorname{Res}(f^*(s) x^{-s}, s_i).$$

The Mellin transform has the following property: let $h(x)$ be a locally Lebesgue integrable function over $]0,\infty[$, $f(x) = \sum_{k=1}^{\infty} h(2^k x)$, and suppose that the convergence is uniform in $]0,\infty[$. Then

$$\begin{aligned} f^*(s) &= \int_0^\infty \sum_{k=1}^{\infty} h(2^k x) x^{s-1} dx \\ &= \sum_{k=1}^{\infty} \int_0^\infty h(y) y^{s-1} 2^{-ks} dy \\ &= \frac{h^*(s)}{2^s - 1}. \end{aligned}$$

In our case $h(x) = 1/(1+x)$.

Since

$$h^*(s) = \int_0^\infty \frac{x^{s-1}}{1+x} dx = \frac{\pi}{\sin \pi s},$$

we have, in a neighbourhood of 0,

$$\begin{aligned} f(x) &= \sum_{\Re s_i < 1/2} \operatorname{Res}\left(\left(\frac{\pi}{\sin \pi s}\right) \frac{x^{-s}}{2^s - 1}, s_i\right) \\ &= -\lg(x) - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{(-2)^k}{2^k - 1} x^k \\ &\quad - \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{\sin(2k\pi \lg(x))}{\sinh\left(2k\pi^2 / \ln(2)\right)}. \end{aligned}$$

We see that $f(x)$ is the sum of $-\lg(x) - 1/2$ (as expected), a power series and the term

$$-\frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{\sin(2k\pi \lg(x))}{\sinh\left(2k\pi^2/\ln(2)\right)}.$$

The latter is a lg-periodic function whose absolute value never exceeds $8 \cdot 10^{-12}$.

Based on the new expression of $f(x)$, we can now compute the exact value of u_2.

We have

$$u_2 = \sum_{k\in\mathbb{Z}} \frac{1}{(2^{-k/2}+2^{k/2})^2} = \lim_{m\to\infty} \left(f(2^{-s})\right)'\Big|_{s=m}$$

$$= 1+\varepsilon,$$

where

$$\varepsilon = \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{2k\pi}{\sinh\left(2k\pi^2/\ln(2)\right)} = 0.4885109... \cdot 10^{-10}.$$

Using the same tools, we obtain the real value of u_1:

$$u_1 = \ln(2) \cdot \sum_{k \in \mathbb{Z}} \frac{1}{2^{-k/2} + 2^{k/2}} \quad = \quad \pi + \varepsilon,$$

where

$$\varepsilon = \sum_{k=1}^{\infty} \frac{2\pi}{\cosh\left(2k\pi^2 / \ln(2)\right)} = 0.5389144... \cdot 10^{-11}.$$

In the process, the function $1/(1+x)$ has been replaced by $-2 \cdot (\arctan(\sqrt{x}) - \pi/2)$.

4. The recurrence relation

Having found the roots of the mystery related to the non-equalities $u_1 \neq \pi$ and $u_2 \neq 1$, we would now like to extend the above results for $u_3, u_4, u_5, \ldots$. The situation is simply stated with the equation

$$u_n = \left(\frac{1}{4} \cdot \frac{n-2}{n-1}\right) u_{n-2} + r_n,$$

where

$$0 < r_n \leqslant r_{11} = 0.88351435449... \cdot 10^{-8}.$$

In fact, if

$$c_k = \frac{\prod_{j=0}^{l-2}(j^2+4\pi^2k^2/\ln(2)^2)}{(2l-2)!} \quad \text{when } n = 2l,$$

$$b_k = \frac{2\pi k\prod_{j=0}^{l-2}((j+1/2)^2+4\pi^2k^2/\ln(2)^2)}{\ln(2)(2l-2)!} \quad \text{when } n = 2l+1.$$

then

$$r_n = \begin{cases} \dfrac{2\pi}{\ln(2)(n-1)} \cdot \displaystyle\sum_{k=1}^{\infty} c_k \frac{2k\pi}{\sinh(2k\pi^2/\ln 2)} & n \text{ even,} \\ \dfrac{2\pi}{\ln(2)(n-1)} \cdot \displaystyle\sum_{k=1}^{\infty} b_k \frac{2k\pi}{\cosh(2k\pi^2/\ln 2)} & n \text{ odd.} \end{cases}$$

The entire theory used here to explain why the numbers u_n are so close to elements in $\mathbb{Q} \cup \pi\mathbb{Q}$ has nothing to do with the presence of 2 in the denominator of

$$\frac{1}{\left(2^{-k/2} + 2^{k/2}\right)^n}.$$

One can argue that any sum of the type

$$\ln(m) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(m^{-k/2} + m^{k/2}\right)^n}$$

has the potential to lie close to $\mathbb{Q}$ or $\pi\mathbb{Q}$ depending on the parity of n.

As a matter of fact, we have, for example,

$$\ln(4)\cdot\sum_{k=-\infty}^{\infty}\frac{1}{2^{-k}+2^{k}} = \pi + 0.82\ldots\cdot 10^{-5},$$

$$\ln(9)\cdot\sum_{k=-\infty}^{\infty}\frac{1}{3^{-k}+3^{k}} = \pi + 0.15\ldots\cdot 10^{-2},$$

$$\ln(4)\cdot\sum_{k=-\infty}^{\infty}\frac{1}{(2^{-k}+2^{k})^{2}} = 1 + 0.37\ldots\cdot 10^{-4}.$$

For further discussions and details, please have a look at

http://algo.epfl.ch/~gerard/

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