

Improvements in closest point search based on dual HKZ-bases. *

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Abstract

In this paper we review the technique to solve the CVP based on dual HKZ-bases by J. Blömer [4]. The technique is based on the transference theorems given by Banaszczyk [3] which imply some necessary conditions on the coefficients of the closest vectors with respect to a basis whose dual is HKZ reduced. Recursively, starting with the last coefficient, intervals of length i can be derived for the i th coefficient of any closest vector. This leads to $n!$ candidates for closest vectors. In this paper we refine the necessary conditions derived from the transference theorems, giving an exponential reduction of the number of candidates. The improvement is due to the fact that the lengths of the intervals are not independent. In the original algorithm the candidates for a coefficient pair (a_i, a_{i+1}) correspond to the integer points in a rectangle of volume $i \cdot (i + 1)$. In our analysis we show that the candidates for (a_i, a_{i+1}) in fact lie in an ellipse with transverse and conjugate diameter $i + 1$, respectively i . This reduces the overall number of points to be enumerated by an exponential factor of about 0.886^n . We further show how a choice of the coefficients (a_n, \dots, a_{i+1}) influences the interval from which a_i can be chosen. Numerical computations show that these considerations allow to bound the number of points to be enumerated by $n^{0.75n}$ for $10 \leq n \leq 2000$. Under the assumption that the Gaussian heuristic for the length of the shortest nonzero vector in a lattice is tight, this number can even be bounded by $\frac{1}{2^{2n}} n^{n/2}$.

Key Words: CVP, dual lattice, lattice problems, nearest point search**Subject Classification:** 68R05; 94A60

1 Introduction

The closest vector problem (CVP) is the problem of finding a closest lattice point of a given lattice $\mathcal{L} \subset \mathbb{R}^n$ to an arbitrary point t in \mathbb{R}^n . While the problem is proven to be NP-hard (see e.g. [10]), algorithms exist to solve the problem approximately in polynomial time. Babai's nearest plane algorithm [2] is the generic way to get an approximate solution, and the quality of the solution substantially depends on the quality of the basis it is applied to. The algorithm recursively selects the nearest $n - 1, n - 2, \dots, 0$ dimensional plane spanned by the basis vectors. The more orthogonal the basis vectors are, the better the output of the algorithm. E.g. if the basis is LLL-reduced, the point found lies within $2(4/3)^{n/2}$ times the distance of a closest lattice point to t [10]. If the basis vectors are even pairwise orthogonal (note that such a basis does not necessarily exist), it returns a closest vector. Babai's nearest plane algorithm can be modified to output an exact solution by not only considering the nearest, but all planes with distance up to a certain bound in the recursion steps. This is exactly the approach of Kannan [9]. Note that once a plane is fixed, the problem translates to finding a closest lattice point in a lower dimensional lattice, namely the orthogonal projection

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of the lattice onto that plane. Clearly the number of planes to be considered in the lower dimensional lattice is dependent on the choice of the plane in the upper dimension which was realized by Pohst [5]. So instead of looking at all points inside a parallelepiped, the points inside a hyperellipsoid are considered. The running time of Kannan's and Pohst's approach was proven to be $O(n^n)$ in the original considerations [9]. Recently, refined analysis by Hanrot and Stehlé showed that applying Kannan's algorithm to a HKZ-basis the closest vectors can be found by enumerating $2^{O(n)}n^{0.5n}$ points. A more elaborate survey on different methods to solve the problem exactly can be found in e.g. [1]. In [4], a different approach than the one of Kannan [9] is presented. The main difference is that the basis used for closest point search is dual HKZ reduced, e.g. it is a basis whose dual is HKZ reduced. Due to the special form of the basis, the transference theorems proven by Banaszczyk [3] can be used to bound the number of planes to be considered. In each recursion step the number of planes to be considered decreases by 1. Having n planes to consider in the first recursion step, this results in enumeration of $n!$ lattice points. Recently Micciancio gave an algorithm to solve the CVP in time $2^{O(n)}$ based on Voronoi cell computations [11]. The caveat in this approach is the exponential space requirement, and it is not (yet) clear how this can be reduced.

In this paper we give a refined analysis of the approach given in [4]. We show how the overall number of points to be enumerated can be decreased. While in the original algorithm the number of choices of the planes is bounded independently in each step, we examine how the choice of a plane in early recursion steps influences the possible number of choices in following steps. We show how to decrease the number of lattice points to be enumerated by an exponential factor $(\pi/4)^{n/2}$ by deriving how the choices of the planes in two neighboring recursion steps are connected. Further we derive a recursive formula (in the dimension of the lattice) for the number of points to be enumerated when the choices made in early recursion steps are rigorously used to constrain the further choices. A closed form approximation of this formula is still an open problem. However numerical computations show that this number can be bounded by $n^{0.75n}$ for $10 < n \leq 2000$. Given that the shortest vector of the dual lattice satisfies the Gaussian heuristic, we show that this number can even be bounded by $\frac{1}{2^{2\pi}}n^{n/2}$.

The paper is organized as follows. In Section 2 we give some background and introduce notation used throughout the paper. In Section 3 the original algorithm proposed in [4] is described and a motivation for further studies of it is given. In Section 4 we show how the running time can be sped up by a factor $(\pi/4)^{n/2}$. In Section 5, a recursive formula bounding the number of points to be enumerated is derived and its behavior is analyzed. Section 6 shows how the bound on the number of points can even be further reduced under the assumption that the Gaussian Heuristic is tight. Finally in Section 7, Kannan's algorithm and the analysis by Hanrot and Stehlé is quickly reviewed. Some concluding remarks are given in Section 8.

2 Background and Notation

Throughout the paper $\|\cdot\|$ denotes the euclidean norm. Let \mathcal{L} be the discrete subgroup generated by integer linear combinations of k linearly independent vectors b_1, \dots, b_k in \mathbb{R}^n . We call \mathcal{L} a lattice of rank k and dimension n . Given a lattice basis $\{b_1, \dots, b_n\}$ of \mathcal{L} we will usually write it as rows of a matrix B in the following way

$$B = [b_1, \dots, b_n].$$

The lattice points in \mathcal{L} are the integer linear combinations of the basis vectors,

$$\mathcal{L} = \mathcal{L}(B) = \{xB \mid x \in \mathbb{Z}^n\}.$$

By $B^* = [b_1^*, \dots, b_n^*]$ we denote the usual Gram-Schmidt basis corresponding to $B = [b_1, \dots, b_n]$. And by π_i we denote the orthogonal projection

$$\pi_i : \text{span}(b_1, \dots, b_n) \longrightarrow \text{span}(b_1, \dots, b_{i-1})^\perp$$

Further with $\mathcal{L} := \mathcal{L}(b_1, \dots, b_n)$ we have that

$$\mathcal{L}_i := \pi_i(\mathcal{L})$$

is again a lattice of rank $n - i + 1$ with basis $\{\pi_i(b_i), \dots, \pi_n(b_n)\}$.

Definition 2.1 Given a lattice $\mathcal{L} \subset \mathbb{R}^n$ the dual lattice \mathcal{L}^\times is defined by

$$\mathcal{L}^\times := \{v \in \mathbb{R}^n : \langle v, w \rangle \in \mathbb{Z} \forall w \in \mathcal{L}\}$$

There exist a unique dual basis B^\times for every basis B of \mathcal{L} .

Definition 2.2 Given a basis $B = [b_1, \dots, b_k]$ for a lattice $\mathcal{L} \subset \mathbb{R}^n$ of rank k , then $B^\times = [b_1^\times, \dots, b_k^\times]$ is the reverse dual basis if and only if

$$b_i^\times \in \text{span}(b_1, \dots, b_k) \text{ and } \langle b_i^\times, b_j \rangle = \delta_{i, k-j+1}$$

From now on we will assume that the lattice has full rank, i.e. $k = n$.

Remark 2.3 Given $v = v_1 b_1 + \dots + v_n b_n \in \mathcal{L}$, then

$$v_i = \langle v, b_{n-i+1}^\times \rangle$$

is the i -th coordinate of v with respect to the basis B .

The algorithm of this paper uses lattice bases of special form.

Definition 2.4 A basis $B = [b_1, \dots, b_n]$ of a lattice $\mathcal{L}(b_1, \dots, b_n)$ is called HKZ-basis if and only if it satisfies the following two conditions

1. $\frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle} \leq \frac{1}{2}$ for $j < i$ (size-reduced).
2. The i -th Gram-Schmidt vector satisfies $|b_i^*| = \lambda_1(\pi_i(\mathcal{L}))$.

Definition 2.5 A basis $B = [b_1, \dots, b_n]$ of a lattice $\mathcal{L}(b_1, \dots, b_n)$ is called dual HKZ-basis when its reverse-dual basis $B^\times = [b_1^\times, \dots, b_n^\times]$ is HKZ-reduced.

Lemma 2.6 Let $B = [b_1, \dots, b_n]$ a basis with dual basis $B^\times = [b_1^\times, \dots, b_n^\times]$. Then the dual basis of $[b_1, \dots, b_{n-j}]$ equals $[\pi_{j+1}(b_{j+1}^\times), \dots, \pi_{j+1}(b_n^\times)]$, $n-1 \geq j \geq 0$.

Proof: We start by showing that $\pi_{j+1}(b_i^\times) \in \text{span}(b_1, \dots, b_{n-j})$ for $i \geq j+1$. Clearly $\pi_{j+1}(b_i^\times) \in \text{span}(b_1, \dots, b_n)$. Also $\pi_{j+1}(b_i^\times) \in \text{span}(b_1^\times, \dots, b_j^\times)^\perp$. Hence

$$\pi_{j+1}(b_i^\times) \in \text{span}(b_1, \dots, b_n) \cap \text{span}(b_1^\times, \dots, b_j^\times)^\perp = \text{span}(b_1, \dots, b_{n-j}).$$

It remains to show that $\langle \pi_{j+1}(b_i^\times), b_k \rangle = 1$ if $k = n+1-i$ and $\langle \pi_{j+1}(b_i^\times), b_k \rangle = 0$ if $k \in \{1, \dots, n-j\} \setminus \{n+1-i\}$. This is straightforward as with $k < n-j+1$

$$\langle \pi_{j+1}(b_i^\times), b_k \rangle = \langle b_i^\times, b_k \rangle.$$

□

This proves Lemma 1 in [4]:

Lemma 2.7 If $[b_1, \dots, b_n]$ is a dual HKZ-basis for $\mathcal{L}(b_1, \dots, b_n)$ then $[b_1, \dots, b_k]$ is a dual HKZ-basis for $\mathcal{L}(b_1, \dots, b_k)$, $k = 1, \dots, n$.

Clearly $b_k^* \in \text{span}(b_1, \dots, b_k)$, and as $\langle b_i, b_k^* \rangle = 0$ for $i < k$ it follows that

$$\langle b_k^*, b_k \rangle = \langle b_k^*, b_k^* \rangle = |b_k^*|^2.$$

So we have that $\frac{b_k^*}{|b_k^*|^2}$ is the first basis vector of the basis dual to $[b_1, \dots, b_k]$. Hence we get the following corollary.

Corollary 2.8 $\frac{b_k^*}{\|b_k^*\|^2}$ is a shortest vector in $\mathcal{L}^\times(b_1, \dots, b_k)$ and $\frac{1}{\|b_k^*\|} = |\lambda_1(\mathcal{L}^\times(b_1, \dots, b_k))|$. Further in a dual HKZ-reduced basis $B = [b_1, \dots, b_n]$, $\|b_k^*\|$ is maximal under all possible bases for the sublattice $\mathcal{L}(b_1, \dots, b_k)$.

We will now state a theorem from the geometry of numbers by Banaszczyk [3]. First we need two definitions. Let $\mu(\mathcal{L})$ denote the covering radius of a lattice, i.e.

$$\mu(\mathcal{L}) := \max_{t \in \text{span}(\mathcal{L})} \min_{x \in \mathcal{L}} \|x - t\|.$$

Denote the set of all i -tuples of linearly independent lattice vectors as V_i . Then the i -th minimum $\lambda_i(\mathcal{L})$ of a lattice is defined as

$$\lambda_i(\mathcal{L}) := \min_{(v_1, \dots, v_i) \in V_i} \max_{1 \leq j \leq i} \|v_j\|.$$

Theorem 2.9 (Transference Theorems) *The successive minimas $\lambda_i(\mathcal{L})$ and covering radius $\mu(\mathcal{L})$ of a lattice \mathcal{L} of rank n satisfy the following bounds*

1. $\lambda_i(\mathcal{L}) \cdot \lambda_{n-i+1}(\mathcal{L}^\times) \leq n$, $i = 1, \dots, n$,
2. $\mu(\mathcal{L}) \cdot \lambda_1(\mathcal{L}^\times) \leq \frac{n}{2}$.

With Corollary 2.8 we have that $\mu(\mathcal{L}) \cdot \lambda_1(\mathcal{L}^\times) = \frac{\mu(\mathcal{L})}{\|b_n^*\|}$ so the second inequality in the Transference Theorems implies that

$$\mu(\mathcal{L}) \leq \frac{n}{2} \|b_n^*\|. \quad (2.1)$$

3 Original approach

In this section we review the approach presented in [4]. Given a lattice $\mathcal{L} = \mathcal{L}(b_1, \dots, b_n)$ in \mathbb{R}^n and a vector $t \in \mathbb{R}^n$, we want to find a vector v such that $\|v - t\| \leq \|w - t\|$ for all $w \in \mathcal{L}$. We assume that the basis $B = [b_1, \dots, b_n]$ is dual HKZ reduced.

1. $e = e_1 b_1^* + \dots + e_n b_n^* = t - v$ denotes the error vector,
2. $e^{(i)} := e - \sum_{j=i+1}^n e_j b_j^*$ is the orthogonal projection of the error vector onto $\text{span}(b_1, \dots, b_i)$.
3. $\mu^{(i)}$ denotes the covering radius of $\mathcal{L}(b_1, \dots, b_i)$,
4. $\lambda_1^{\times(i)} := \lambda_1(\mathcal{L}^\times(b_1, \dots, b_i))$.

So suppose $v = c_1 b_1 + \dots + c_n b_n$, $c_i \in \mathbb{Z}$ is a closest vector to $t = t_1 b_1 + \dots + t_n b_n$, $t_i \in \mathbb{R}$. With (2.1) we get

$$\|v - t\| \leq \mu(\mathcal{L}) \leq \frac{n}{2} \|b_n^*\|,$$

and as $(c_n - t_n)^2 \|b_n^*\|^2 \leq \|v - t\|^2 \leq \left(\frac{n}{2}\right)^2 \|b_n^*\|^2$ we have

$$|c_n - t_n| \leq \frac{n}{2}.$$

Hence we get an interval of length n for the n -th coordinate c_n of v :

$$c_n \in [t_n - n/2, t_n + n/2]. \quad (3.2)$$

As $c_n \in \mathbb{Z}$ we can enumerate n values for c_n . Note that for the orthogonal projection $t^{(n-1)}$ of $t - c_n b_n$ onto $\text{span}(b_1, \dots, b_{n-1})$ we have

$$t^{(n-1)} = t - c_n b_n - \frac{\langle t - c_n b_n, b_n^* \rangle}{\langle b_n^*, b_n^* \rangle} b_n^* = t - c_n b_n - (t_n - c_n) b_n^*,$$

and hence $(t_n - c_n) = e_n$. The following lemma [4] allows to recursively carry the problem to proper sublattices of \mathcal{L} in order to derive corresponding bounds for the other coordinates of v .

Lemma 3.1 *A vector $w \in \mathcal{L}(b_1, \dots, b_i)$ is a closest vector to $t - \sum_{j>i} x_j b_j$, $x_j \in \mathbb{Z}$ if and only if w is a closest vector of the orthogonal projection $t^{(i)}$ of $t - \sum_{j>i} x_j b_j$ onto $\text{span}(b_1, \dots, b_i)$.*

So given c_{i+1}, \dots, c_n and e_{i+1}, \dots, e_n the problem reduces to finding the closest vector to $t^{(i)} = t - \sum_{j=i+1, \dots, n} c_j b_j - \sum_{j=i+1, \dots, n} e_j b_j^*$ in the lattice $\mathcal{L}(b_1, \dots, b_i)$ of rank i . As by Lemma 2.7 $[b_1, \dots, b_i]$ is a dual HKZ basis for $\mathcal{L}(b_1, \dots, b_i)$, we can recursively take the problem to a lower dimension. In dimension $i = 1$, $t^{(1)} \in \text{span}(b_1)$ and we set $c_1 = \lfloor \frac{\langle t^{(1)}, b_1 \rangle}{\langle b_1, b_1 \rangle} \rfloor$ and $e_1 = \frac{\langle t^{(1)}, b_1 \rangle}{\langle b_1, b_1 \rangle} - \lfloor \frac{\langle t^{(1)}, b_1 \rangle}{\langle b_1, b_1 \rangle} \rfloor$ in order to get the closest lattice vector in $\mathcal{L}(b_1)$ to $t^{(1)}$. In fact

$$t^{(1)} - c_1 b_1 - e_1 b_1^* = t - \sum_{j=1}^n c_j b_j - \sum_{j=1}^n e_j b_j^* = 0,$$

assuring that we get a valid pair of vectors $v \in \mathcal{L}$ and error $e \in \mathbb{R}^n$ in the sense that $v + e = t$. Hence we have the following lemma:

Lemma 3.2 *Recursively we can derive $n!$ candidates for a closest vector to t in \mathcal{L} given a dual HKZ-basis for \mathcal{L} .*

We will now give a short motivation for further analysis. The algorithm and the corresponding bound is not optimized at all. Suppose the n -th coordinate e_n of the error vector equals $\frac{n}{2}$. Clearly we have the following inequality $\frac{n}{2} = \frac{\langle e, b_n^* \rangle}{\langle b_n^*, b_n^* \rangle} = \|e\| \frac{1}{\|b_n^*\|} \cos \gamma$. As $\|e\| \leq \mu(\mathcal{L})$, with Equation (2.1) we get $\frac{n}{2} \leq \frac{n}{2} \cos \gamma$. Consequently $\gamma = 0$ which means that the error vector points exactly in the direction of b_n^* . So the error vector can be written as multiple of b_n^* and the coefficients e_1, \dots, e_{n-1} are trivially zero. In the next section we will see how the value of e_i influences the interval length in which e_{i-1} lies.

4 First Improvement

Given the same problem and notation as in Section 3, let us consider the following set

$$T_n := \left\{ (e_1, \dots, e_n) \in \mathbb{R}^n : v = t - \sum_{j=1}^n e_j b_j^* \in \mathcal{L} \text{ and } |e_j| \leq \frac{j}{2} \right\}.$$

In the last section we have seen how all elements of this set can be enumerated recursively and that due to the dual HKZ reducedness of B in fact all closest vectors to t are in the set $\{t - \sum_{j=1}^n e_j b_j^* : (e_1, \dots, e_n) \in T_n\}$. Further in each recursion step the value $e_i + c_i$ is given and as c_i is an integer, the condition $|e_i| \leq \frac{i}{2}$ implies i possible values for e_i . So $|T_n|$ is upper bounded by $n!$.

The goal of this section is to define a subset $T'_n \subset T_n$ still having the property that all closest vectors to t are in the set $\{t - \sum_{j=1}^n e_j b_j^* : (e_1, \dots, e_n) \in T'_n\}$. We will now show how additional constraints on the e_i 's can be derived. Recall that the condition $|e_i| \leq \frac{i}{2}$ comes from the fact that $\|e^{(i)}\| \leq \mu^{(i)} \leq \frac{i}{2} \|b_i^*\|$, where the second inequality is due to the dual HKZ reducedness of the basis. This implies $\|e_i b_i^*\| \leq \|e^{(i)}\| \leq \mu^{(i)} \leq \frac{i}{2} \|b_i^*\|$ and consequently $|e_i| \leq \frac{i}{2}$. However $\|e^{(i)}\| \leq \mu^{(i)}$ is not the only bound on $\|e^{(i)}\|$ we have. Clearly also

$$\|e^{(i)}\|^2 = \|e^{(k)}\|^2 - \sum_{j=i+1}^k e_j^2 \|b_j^*\|^2 \leq \mu^{(k)2} - \sum_{j=i+1}^k e_j^2 \|b_j^*\|^2 \text{ for all } k \geq i.$$

Now if $e_j^2 > \frac{1}{4}$, $j = i+1, \dots, k$ with $\mu^{(k)2} \leq \mu^{(i)2} + \frac{1}{4} \sum_{j=i+1}^k \|b_j^*\|^2$ we have a tighter upper bound

$$\|e^{(i)}\|^2 \leq \mu^{(k)2} - \sum_{j=i+1}^k e_j^2 \|b_j^*\|^2 < \mu^{(i)2}. \quad (4.3)$$

This observation can now be exploited to reduce the size of the intervals in which the e_i 's lie. For all $i = 2, \dots, n$, we derive factors $A_i(e_i) \in \mathbb{R}$ depending on e_i , such that $\|e^{(i-1)}\| \leq A_i(e_i) \mu^{(i-1)}$ and consequently

$|e_{i-1}| \leq A_i(e_i)^{\frac{i-1}{2}}$. Let us define

$$A_i^2(c) := \frac{\frac{i^2}{4} - c^2}{\frac{i^2}{4} - \frac{1}{4}}, \quad i \in \mathbb{N}. \quad (4.4)$$

We obtain the following lemma

Lemma 4.1 *If $c^2 \geq \frac{1}{4}$, then we have*

$$\mu^{(i)2} - c^2 \|b_i^*\|^2 \leq A_i^2(c) \cdot \mu^{(i-1)2}.$$

Proof: We have to show that

$$\left(\frac{i^2}{4} - \frac{1}{4}\right) \left(\mu^{(i)2} - c^2 \|b_i^*\|^2\right) \leq \left(\frac{i^2}{4} - c^2\right) \mu^{(i-1)2}.$$

Since $\mu^{(i)2} - \frac{1}{4} \|b_i^*\|^2 \leq \mu^{(i-1)2}$, it is sufficient to show that

$$\left(\frac{i^2}{4} - \frac{1}{4}\right) \left(\mu^{(i)2} - c^2 \|b_i^*\|^2\right) \leq \left(\frac{i^2}{4} - c^2\right) \left(\mu^{(i)2} - \frac{1}{4} \|b_i^*\|^2\right).$$

This is true since

$$\left(c^2 - \frac{1}{4}\right) \mu^{(i)2} \leq \left(c^2 - \frac{1}{4}\right) \frac{i^2}{4} \|b_i^*\|^2.$$

□

We can now prove the core lemma, which gives the factor by which the error vector is smaller than the covering radius.

Lemma 4.2 *Under the previous assumptions and notations:*

$$\|e^{(i-1)}\|^2 \leq A_i^2(e_i) \cdot \mu^{(i-1)2}. \quad (4.5)$$

Proof: We separate the two cases where $|e_i| < \frac{1}{2}$, $|e_i| \geq \frac{1}{2}$ respectively. If $|e_i| < \frac{1}{2}$, then $A_i^2(e_i) > 1$ and the proposition follows by $\|e^{(i-1)}\|^2 \leq \mu^{(i-1)2}$. If $|e_i| \geq \frac{1}{2}$, the claim follows from

$$\|e^{(i-1)}\|^2 = \|e^{(i)}\|^2 - e_i^2 \|b_i^*\|^2 \leq \mu^{(i)2} - e_i^2 \|b_i^*\|^2,$$

and Lemma 4.1. □

So with $e_{i-1}^2 \|b_{i-1}^*\|^2 \leq \|e^{(i-1)}\|^2 \leq A_i^2(e_i) \cdot \mu^{(i-1)2}$ and $\frac{\mu^{(i-1)2}}{\|b_{i-1}^*\|^2} \leq \frac{(i-1)^2}{2}$ we immediately obtain the following bound.

Corollary 4.3 *Using the notation from before,*

$$e_{i-1}^2 \left(\frac{i^2}{4} - \frac{1}{4}\right) + e_i^2 \frac{(i-1)^2}{4} \leq \frac{i^2 (i-1)^2}{4}. \quad (4.6)$$

So we define

$$T'_n := \{(e_1, \dots, e_n) \in T_n : \text{Equation (4.6) holds for all } i = 2, \dots, n\}.$$

We are interested on an upper bound on the volume of T'_n giving us an upper bound on the number of points we have to enumerate to get the closest vectors. Let us first assume that n is even. Clearly

$$T'_n \subset T''_n := \bigotimes_{i=1}^{n/2} \{(e_{2i-1}, e_{2i}) \in \mathbb{R}^n : \text{Equation (4.6) holds}\}.$$

The volume of T_n'' can be computed as the product of the volumes of the 2-dimensional ellipses.

$$\text{vol}(T_n'') = \left(\frac{\pi}{4}\right)^{n/2} \prod_{i=1}^{n/2} \left(2i(2i-1) \frac{i}{\sqrt{i^2-1/4}}\right) = \left(\frac{\pi}{4}\right)^{n/2} n! \prod_{i=1}^{n/2} \frac{i}{\sqrt{i^2-1/4}}.$$

As

$$\prod_{i=1}^{n/2} \frac{i}{\sqrt{i^2-1/4}} = \sqrt{\frac{4}{3}} \prod_{i=2}^{n/2} \frac{i}{\sqrt{i^2-1/4}} < \sqrt{\frac{4}{3}} \prod_{i=2}^{n/2} \frac{i}{\sqrt{i^2-1}} = \sqrt{\frac{4}{3}} \underbrace{\sqrt{\frac{n}{n/2+1}}}_{< \sqrt{2}} < 2.$$

We obtain

$$\text{vol}(T_n'') < 2 \left(\frac{\pi}{4}\right)^{n/2} n!.$$

In the case where n is odd consider

$$T_n' \subset T_n''' := \{|e_1| \leq \frac{1}{2}\} \otimes \bigotimes_{i=1}^{(n-1)/2} \{(e_{2i}, e_{2i+1}) \in \mathbb{R}^n : \text{Equation (4.6) holds}\}.$$

The volume of T_n''' then is

$$\text{vol}(T_n''') = \left(\frac{\pi}{4}\right)^{(n-1)/2} \prod_{i=1}^{(n-1)/2} 2i(2i+1) \frac{2i+1}{\sqrt{(2i+1)^2-1}} = \left(\frac{\pi}{4}\right)^{(n-1)/2} n! \prod_{i=1}^{(n-1)/2} \frac{i+1/2}{\sqrt{(i+1/2)^2-1/4}}.$$

As $\prod_{i=1}^{(n-1)/2} \frac{i+1/2}{\sqrt{(i+1/2)^2-1/4}} < \prod_{i=1}^{(n-1)/2} \frac{i}{\sqrt{i^2-1/4}}$ we get the same bound

$$\text{vol}(T_n''') < 2 \left(\frac{\pi}{4}\right)^{n/2} n!.$$

Theorem 4.4 *Given a dual HKZ basis B of a full rank lattice $\mathcal{L} \subset \mathbb{R}^n$ all closest vectors to a given point $t \in \mathbb{R}^n$ can be found by recursively enumerating at most $2 \left(\frac{\pi}{4}\right)^{n/2} n!$ lattice points.*

So with $\sqrt{\pi/4} \approx 0.886$ we get an exponential gain of roughly 0.886^n compared to the original considerations.

5 Further improvement

Recall the starting point of the considerations of the previous section. We have an upper bound on $\|e^{(i)}\|^2$:

$$\|e^{(i)}\|^2 \leq \mu^{(k)2} - \sum_{j=i+1}^k e_j^2 \|b_j^*\|^2. \quad (5.7)$$

Note that the bound (5.7) is decreasing with increasing e_j 's and in fact if they satisfy $|e_j| > \frac{1}{2}$ then as in Equation (4.3),

$$\mu^{(k)2} - \sum_{j=i+1}^k e_j^2 \|b_j^*\|^2 < \mu^{(i)2}.$$

In the original approach (see Section 3), only the case $k = i$ was considered. In Section 4 we considered the case where $k = i + 1$ and we got that

$$\|e^{(i)}\|^2 \leq A_{i+1}^2(e_{i+1}) \cdot \mu^{(i)2},$$

where $A_{i+1}^2(e_{i+1}) := \left(\frac{(i+1)^2}{4} - e_{i+1}^2\right) \left(\frac{(i+1)^2}{4} - \frac{1}{4}\right)^{-1}$. From that we derived that pairs of coefficients (e_i, e_{i+1}) lie inside a 2-dimensional ellipsoid of volume $\frac{\pi}{4}(i+1)^2\sqrt{\frac{i}{i+2}}$. The goal of this section is to generalize this method to more than just 2-tuples of coefficients. Consider

$$\begin{aligned} \|e^{(i-1)}\|^2 = \|e^{(i)}\|^2 - e_i^2\|b_i^*\|^2 &\leq A_{i+1}^2(e_{i+1}) \cdot \mu^{(i)2} - e_i^2\|b_i^*\|^2 \\ &= A_{i+1}^2(e_{i+1}) \left(\mu^{(i)2} - \frac{e_i^2}{A_{i+1}^2(e_{i+1})} \|b_i^*\|^2 \right). \end{aligned} \quad (5.8)$$

So under the condition that $\frac{e_i^2}{A_{i+1}^2(e_{i+1})} \geq \frac{1}{4}$, by Lemma 4.1 we have

$$\|e^{(i-1)}\|^2 \leq A_{i+1}^2(e_{i+1}) A_i^2 \left(\frac{e_i^2}{A_{i+1}^2(e_{i+1})} \right) \mu^{(i-1)2}.$$

Note that if $|e_{i+1}|, |e_i| > \frac{1}{2}$, $A_{i+1}^2(e_{i+1}) < 1$ and $A_i^2 \left(\frac{e_i^2}{A_{i+1}^2(e_{i+1})} \right) < 1$. Clearly the bigger $|e_{i+1}|, |e_i|$ the smaller the bound on $\|e^{(i-1)}\|$ becomes.

Definition 5.1 For e_n, \dots, e_1 recursively define C_{n+1}^2, \dots, C_1^2 by

$$C_{n+1}^2 := 1 \quad \text{and} \quad C_{i-1}^2 := \begin{cases} 1 & \text{if } |e_{i-1}| < \frac{1}{2}, \\ C_i^2 A_{i-1}^2 \left(\frac{e_{i-1}}{C_i} \right) & \text{else.} \end{cases}$$

Note that $C_i^2 \leq 1$ for all i .

Proposition 5.2 For $i = n, \dots, 2$ we have

$$\|e^{(i-1)}\|^2 \leq C_i^2 \mu^{(i-1)2}.$$

Proof: The proof goes by reverse induction on i . For $i = n - 1$ the result follows by Proposition 4.2. Assume the results holds for i . If $|e_i| < \frac{1}{2}$, $C_i^2 = 1$ and the proposition follows trivially. For the case $|e_i| \geq \frac{1}{2}$, note that

$$\|e^{(i-1)}\|^2 = \|e^{(i)}\|^2 - e_i^2\|b_i^*\|^2 \leq C_{i+1}^2 \mu^{(i)2} - e_i^2\|b_i^*\|^2 = C_{i+1}^2 \mu^{(i)2} - e_i^2\|b_i^*\|^2.$$

We also have that $\frac{e_i^2}{C_{i+1}^2} > \frac{1}{4}$ and with Lemma 4.1,

$$C_{i+1}^2 \left(\mu^{(i)2} - \frac{e_i^2}{C_{i+1}^2} \|b_i^*\|^2 \right) \leq C_{i+1}^2 A_i^2 \left(\frac{e_i}{C_{i+1}} \right) \mu^{(i-1)2} = C_i^2 \mu^{(i-1)2}.$$

□

With the Transference Theorems the following corollary follows immediately:

Corollary 5.3 For $i = n, \dots, 2$ we have

$$e_{i-1}^2 \leq C_i^2 \left(\frac{i-1}{2} \right)^2.$$

Under the assumption that a few consecutive e_j 's are at least one half in absolute value, e.g. $|e_k|, \dots, |e_i| \geq \frac{1}{2}$, the next lemma will give a closed form expression for C_i depending on e_k, \dots, e_i . As a corollary of the next lemma and Proposition 5.2, we will see how e_k, \dots, e_i satisfy a $(k - i + 1)$ -dimensional ellipsoid equation.

Lemma 5.4 Let $n \geq k \geq i \geq 1$. Under the assumption that $|e_k|, \dots, |e_i| \geq \frac{1}{2}$ and either $k = n$ or $|e_{k+1}| < \frac{1}{2}$ we have

$$C_i^2 = \frac{\prod_{j=i}^k \frac{j^2}{4}}{\prod_{j=i}^k \left(\frac{j^2}{4} - \frac{1}{4} \right)} - \sum_{j=i+1}^k \left(e_j^2 \frac{\prod_{l=i}^{j-1} \frac{l^2}{4}}{\prod_{l=i}^j \left(\frac{l^2}{4} - \frac{1}{4} \right)} \right) - e_i^2 \frac{1}{\frac{i^2}{4} - \frac{1}{4}}.$$

Proof: We go by reverse induction on i . The case $i = k$ follows by definition. Assume the result holds for $i + 1$. Then

$$C_i^2 = A_i^2 \left(\frac{e_i}{C_{i+1}} \right) C_{i+1}^2 = C_{i+1}^2 \frac{\frac{i^2}{4}}{\frac{i^2}{4} - \frac{1}{4}} - e_i^2 \frac{1}{\frac{i^2}{4} - \frac{1}{4}}.$$

Plugging in C_{i+1}^2 immediately gives the result. \square

Note that from $e^{(i)2} \leq C_{i+1}^2 \mu^{(i)2}$ and the Transference Theorems we obtain $e_i^2 \leq \frac{i^2}{4} C_{i+1}^2$. So under the condition that $|e_k|, \dots, |e_{i+1}| > \frac{1}{2}$ we have that

$$e_i^2 \leq \frac{i^2}{4} \left(\frac{\prod_{j=i+1}^k \frac{j^2}{4}}{\prod_{j=i+1}^k \left(\frac{j^2}{4} - \frac{1}{4} \right)} - \sum_{j=i+2}^k \left(e_j^2 \frac{\prod_{l=i+1}^{j-1} \frac{l^2}{4}}{\prod_{l=i+1}^j \left(\frac{l^2}{4} - \frac{1}{4} \right)} \right) - e_{i+1}^2 \frac{1}{\frac{(i+1)^2}{4} - \frac{1}{4}} \right). \quad (5.9)$$

The following corollary follows immediately.

Corollary 5.5 *If $|e_k|, \dots, |e_{i+1}| > \frac{1}{2}$ for $1 \leq i < k \leq n$, then*

$$e_i^2 + \sum_{j=i+1}^k \left(e_j^2 \frac{\prod_{l=i}^{j-1} \frac{l^2}{4}}{\prod_{l=i+1}^j \left(\frac{l^2}{4} - \frac{1}{4} \right)} \right) \leq \frac{\prod_{j=i}^k \frac{j^2}{4}}{\prod_{j=i+1}^k \left(\frac{j^2}{4} - \frac{1}{4} \right)}. \quad (5.10)$$

As in Section 4 we now define a set S_n such that all closest vectors to t are in $\{t - \sum_{j=1}^n e_j b_j^* : (e_1, \dots, e_n) \in S_n\}$:

$$S_n := \left\{ (e_1, \dots, e_n) \in T_n : |e_i| \leq \frac{i}{2} \cdot C_{i+1} \right\}.$$

Note that by Corollary 5.3, S_n has the desired property.

5.1 Bounding the set S_n

We will now bound the number of elements in S_n . Clearly

$$S_n \subset S'_n := \left\{ (e_1, \dots, e_n) \in \mathbb{R}^n : |e_i| \leq \frac{i}{2} \cdot C_{i+1}, \quad i = 1, \dots, n \right\}.$$

For $k = 1, \dots, n$, when $C_{k+1} = 1$, define $a_k = \text{vol}(S'_k)$ and set $a_0 = 1$. Clearly $a_1 = 1$ and $a_n = \text{vol}(S'_n)$. Further define

$$V_{j,k} := \begin{cases} 1 & \text{for } j = k, \\ \text{vol} \left\{ (e_{j+1}, \dots, e_k) \in \mathbb{R}^{k-j} : \text{Equation (5.10) holds} \right\} & \text{else.} \end{cases}$$

For a given element $(e_1, \dots, e_k) \in \mathbb{R}^k$ we can define

$$\tau := \max_{1 \leq i \leq k} \{i : C_i = 1\} = \max_{1 \leq i \leq k} \left\{ i : |e_i| \leq \frac{1}{2} \right\},$$

allowing to write

$$a_k = a_{\tau-1} V_{\tau,k}.$$

We can now partition S'_n into disjoint sets, depending on the possible values of τ , as

$$S'_k = \bigcup_{1 \leq \tau \leq k} \left\{ (e_1, \dots, e_k) \in \mathbb{R}^k : |e_i| \leq \frac{i}{2} \cdot C_{i+1} \text{ and } \max_{1 \leq i \leq k} \{i : C_i = 1\} = \tau \right\}$$

So in the case where $C_{k+1} = 1$, we have

$$a_k = \sum_{1 \leq j \leq k} a_{j-1} V_{j,k}. \quad (5.11)$$

In particular

$$\text{vol}(S'_n) = a_n = \sum_{1 \leq j \leq n} a_{j-1} V_{j,n}.$$

Using the well known formula for the volume of an ellipsoid, $V_{j,k}$ can be computed (see Appendix A) as

$$V_{j,k} = \frac{\pi^{(k-j)/2}}{\Gamma\left(\frac{k-j}{2} + 1\right)} \frac{k!}{j! 2^{k-j}} \left(\frac{k+1}{j+1}\right)^{1/2} \left(\frac{k}{k+1}\right)^{(k-j)/2}.$$

Clearly

$$V_{j,k} \leq \left(\frac{\pi}{4}\right)^{(k-j)/2} \frac{1}{\Gamma\left(\frac{k-j}{2} + 1\right)} \frac{k!}{j!} \left(\frac{k+1}{j+1}\right)^{1/2}.$$

Plugging this into Equation (5.11), for $k = 1, \dots, n$ we get

$$a_k \leq \sum_{1 \leq j \leq k} a_{j-1} \left(\frac{\pi}{4}\right)^{(k-j)/2} \frac{1}{\Gamma\left(\frac{k-j}{2} + 1\right)} \frac{k!}{j!} \left(\frac{k+1}{j+1}\right)^{1/2},$$

which leads to

$$\begin{aligned} \frac{a_k}{\sqrt{k+1}(k+1)!} \left(\frac{4}{\pi}\right)^{k/2} &\leq \frac{1}{k+1} \sum_{1 \leq j \leq k} \frac{a_{j-1}}{\sqrt{j+1}j!} \left(\frac{4}{\pi}\right)^{j/2} \frac{1}{\Gamma\left(\frac{k-j}{2} + 1\right)} \\ &\leq \frac{1}{k+1} \sum_{1 \leq j \leq k} \frac{a_{j-1}}{\sqrt{j}j!} \left(\frac{4}{\pi}\right)^{j/2} \frac{1}{\Gamma\left(\frac{k-j}{2} + 1\right)}. \end{aligned}$$

So we have a recursively defined upper bound for a_k . We will now derive a nicer recursion, the goal to upper bound a_k remains the same however. Define

$$\sigma_k := \frac{a_k}{\sqrt{k+1}(k+1)!} \left(\frac{4}{\pi}\right)^{k/2} \text{ for } k = 0, \dots, n.$$

As $a_0 = 1$, we get the following recursive relation

$$\sigma_k \leq \begin{cases} 1 & \text{for } k = 0, \\ \frac{1}{k+1} \sum_{j=1}^k \frac{\sigma_{j-1}}{\Gamma\left(\frac{k-j}{2} + 1\right)} & \text{for } k \geq 1. \end{cases}$$

So setting $s_0 := 1$ and

$$s_k := \frac{1}{k+1} \sum_{j=1}^k \frac{s_{j-1}}{\Gamma\left(\frac{k-j}{2} + 1\right)},$$

then $\sigma_k \leq s_k$ and it is enough to derive an upper bound on s_k . We can define the following sequence for $n \geq 2$:

$$\begin{aligned} c_n &:= \frac{\log s_n}{n \log n} + \frac{1}{2n} + \frac{(n+2) \log(n+2)}{n \log n} - \frac{1}{\log n} + \frac{\log(\pi/4)}{2 \log n} \\ &\geq \frac{\log\left(s_n \sqrt{n}(n+1)! (\pi/4)^{n/2}\right)}{n \log n} \\ &\geq \frac{\log a_n}{n \log n}, \end{aligned} \tag{5.12}$$

where Eq. (5.12) is valid because $(n+1)! \leq e \left(\frac{n+2}{e}\right)^{n+2}$. Then

$$\text{vol}(S'_n) = a_n \leq n^{c_n n}.$$

Deriving any useful and provable explicit bound on s_n , and therefore on c_n , seems to be a nontrivial task. However, numerical computations of c_n suggest that $c_n < 0.75$ for $10 < n \leq 2000$ (compare Figure 1).

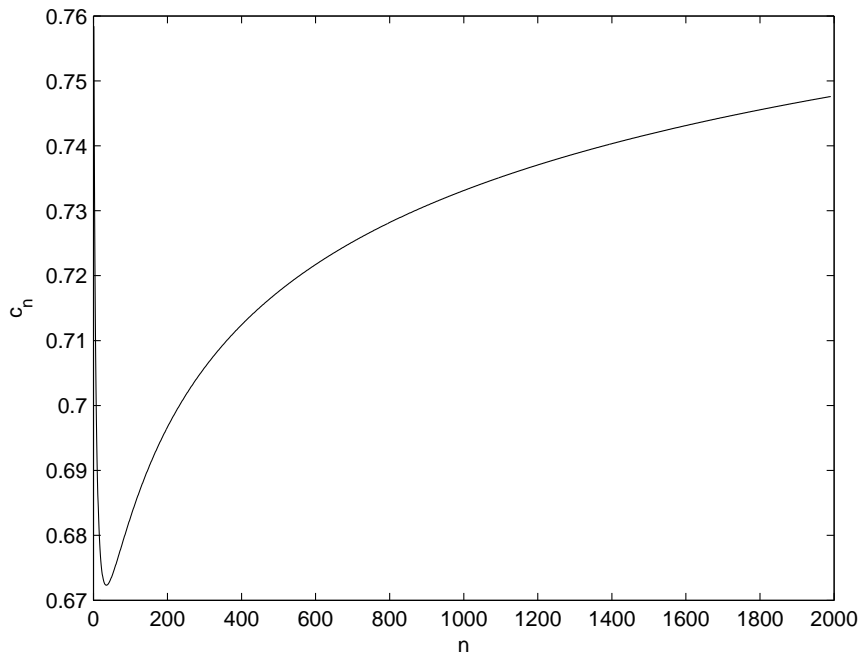


Figure 1: The behaviour of c_n for $10 \leq n \leq 2000$.

6 Hermite factor

In this section we will point out the influence of the hermite factor of the dual lattice on the running time of the algorithm. While the considerations in the previous section give a reduction in the running time for all lattices, this section will only give an improvement in the case where the length λ_1^\times of a shortest vector in the dual lattice satisfies $\lambda_1^\times \geq (\text{vol}(\mathcal{L}^\times))^{1/n}$. Let

$$\alpha := \sqrt{\gamma(\mathcal{L}^\times)} = \frac{\lambda_1(\mathcal{L}^\times)}{(\text{vol}(\mathcal{L}^\times))^{1/n}},$$

denote the hermite factor of the dual lattice \mathcal{L} . Now consider the following bound on the length of the error vector e

$$\|e\|^2 = e_1^2 \|b_1^*\|^2 + \dots + e_n^2 \|b_n^*\|^2 \leq \left(\frac{n}{2}\right)^2 \|b_n^*\|. \quad (6.13)$$

Again the number of coefficients satisfying this inequality can be approximated by the volume of the ellipsoid:

$$\begin{aligned} V &:= \left| \left\{ (e_1, \dots, e_n) \in \mathbb{R}^n : v = t - \sum_{j=1}^n e_j b_j^* \in \mathcal{L} \text{ and } \left\| \sum_{j=1}^n e_j b_j^* \right\| \leq \frac{n}{2} \|b_n^*\| \right\} \right| \\ &= \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \left(\frac{n}{2}\right)^n \frac{\|b_n^*\|^n}{\prod_{j=1}^n \|b_j^*\|}. \end{aligned} \quad (6.14)$$

Note that if $B = [b_1, \dots, b_n]$ is dual HKZ reduced, then $\|b_n^*\| = \frac{1}{\lambda_1(\mathcal{L}^\times)}$ and $\prod_{j=1}^n \|b_j^*\| = \text{vol}(\mathcal{L}) = \frac{1}{\text{vol}(\mathcal{L}^\times)}$. Consequently

$$\frac{\|b_n^*\|^n}{\prod_{j=1}^n \|b_j^*\|} = \left(\frac{\text{vol}(\mathcal{L}^\times)}{\lambda_1(\mathcal{L}^\times)}\right)^n = \left(\frac{1}{\alpha}\right)^n. \quad (6.15)$$

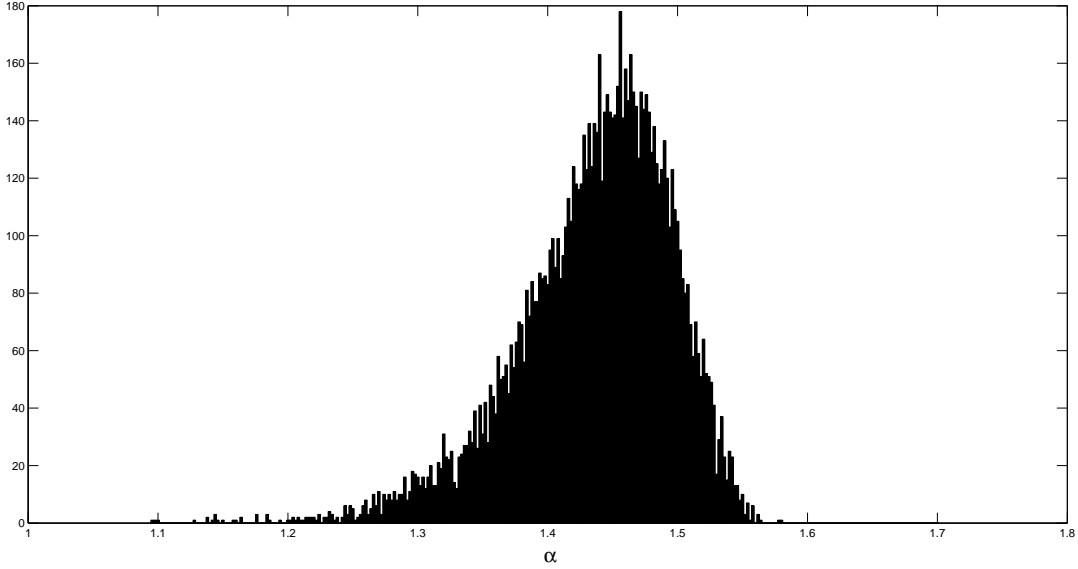


Figure 2: Histogram of hermite factors of random lattices with 200-bit prime determinant and dimension 30.

So

$$\begin{aligned}
 V &= \left| \left\{ (e_1, \dots, e_n) \in \mathbb{R}^n : v = t - \sum_{j=1}^n e_j b_j^* \in \mathcal{L} \text{ and } \left\| \sum_{j=1}^n e_j b_j^* \right\| \leq \frac{n}{2} \|b_n^*\| \right\} \right| \\
 &= \left(\frac{\pi}{4}\right)^{n/2} \frac{1}{\Gamma(\frac{n}{2} + 1)} \left(\frac{n}{2}\right)^n \left(\frac{1}{\alpha}\right)^n.
 \end{aligned} \tag{6.16}$$

While α can be smaller than 1, the Gaussian heuristic [12] suggest that it is bigger than one:

$$\alpha = \frac{\lambda_1(\mathcal{L}^\times)}{(\text{vol } \mathcal{L}^\times)^{1/n}} \approx \frac{\Gamma(\frac{n}{2} + 1)^{1/n}}{\sqrt{\pi}}. \tag{6.17}$$

In fact tests with random integer lattices in the sense of Goldstein and Meier [6] suggest that the heuristic is quite tight for higher dimensions (> 30). E.g. for dimension $n = 30$, the Gaussian heuristic suggest that $\alpha \approx 1.43$, which is supported by the histogram in Figure 6. Assuming that this is in fact the case and plugging in the Gaussian heuristic into formula (6.16), we obtain

$$V = \left(\frac{\pi}{4}\right)^{n/2} \frac{1}{\Gamma(\frac{n}{2} + 1)} \left(\frac{n}{2}\right)^n \left(\frac{1}{\alpha}\right)^n \approx \left(\frac{\sqrt{n}}{4}\right)^n.$$

So in this case, the average number of points to be enumerated would be

$$\left(\frac{\sqrt{n}}{4}\right)^n = \frac{1}{2^{2n}} n^{n/2}. \tag{6.18}$$

7 Kannan's algorithm

In this section we quickly review Kannan's algorithm and the complexity analysis done by Hanrot and Stehlé [7, 8]. In contrast to Blömers approach, Kannan's algorithm takes as input a HKZ reduced basis B . Let

$e = e_1 b_1^* + \dots + e_n b_n^*$ again denote the error vector $v - t$. Hanrot and Stehlé in their analysis use the fact that

$$e_1^2 \|b_1^*\| + \dots + e_k^2 \|b_k^*\| \leq \frac{1}{4} \sum_{i=1}^k \|b_i^*\|^2 \leq \frac{k}{4} \max_{1 \leq j \leq k} \|b_j^*\|^2. \quad (7.19)$$

Clearly the volume of the ellipsoid defined by Equation (7.19) depends on the lengths of the Gram-Schmidt vectors. Let us define $C(0) := 1$ and for $k \geq 1$

$$C(k) := |\{(e_1, \dots, e_k) \in \mathbb{R}^k : \text{Equation (7.19) holds}\}|.$$

We derive a recursive bound for $C(n)$: Let $\tau := \operatorname{argmax}_{1 \leq j \leq n} \|b_j^*\|$. Then we have the following inequality

$$e_\tau^2 \|b_\tau^*\| + \dots + e_n^2 \|b_n^*\| \leq e_1^2 \|b_1^*\| + \dots + e_n^2 \|b_n^*\| \leq \frac{n}{4} \|b_\tau^*\|^2. \quad (7.20)$$

As B is HKZ reduced, also the $n - \tau + 1$ dimensional lattice $\pi_\tau(B)$ is HKZ reduced. Consequently by Hermite's [12] bound we have that

$$\|b_\tau^*\| \leq \sqrt{\frac{n - \tau + 5}{4}} \cdot \operatorname{vol}(\pi_\tau(\mathcal{L}))^{1/(n - \tau + 1)}.$$

Let us consider

$$C(\tau, n) := |\{(e_\tau, \dots, e_n) \in \mathbb{R}^{n - \tau + 1} : \text{Equation (7.20) holds}\}|.$$

We can compute the volume of $C(\tau, n)$ using the Ellipsoid formula

$$\begin{aligned} C(\tau, n) &= \frac{\pi^{(n - \tau + 1)/2}}{\Gamma(\frac{n - \tau + 1}{2} + 1)} \left(\frac{n}{4}\right)^{(n - \tau + 1)/2} \frac{\|b_\tau^*\|^{n - \tau + 1}}{\prod_{j=\tau}^n \|b_j^*\|} \\ &= \frac{\pi^{(n - \tau + 1)/2}}{\Gamma(\frac{n - \tau + 1}{2} + 1)} \left(\frac{n}{4}\right)^{(n - \tau + 1)/2} \frac{\|b_\tau^*\|^{n - \tau + 1}}{\operatorname{vol}(\pi_\tau(\mathcal{L}))} \\ &\leq \left(\frac{\pi}{4}\right)^{(n - \tau + 1)/2} \frac{1}{\Gamma(\frac{n - \tau + 1}{2} + 1)} \left(\frac{n - \tau + 5}{4}\right)^{(n - \tau + 1)/2} n^{(n - \tau + 1)/2}. \end{aligned}$$

Consequently we get

$$C(n) \leq C(\tau - 1)C(\tau, n) = C(\tau - 1)2^{c(n - \tau + 1)} n^{(n - \tau + 1)/2},$$

for some constant c . This gives

$$C(n) \leq 2^{c'n} n^{n/2},$$

for some constant c' . As in the previous section we can state the result in the case where the Gaussian heuristic is reached, i.e. $\|b_\tau^*\| = \frac{\Gamma(\frac{n - \tau + 1}{2} + 1)^{1/(n - \tau + 1)}}{\sqrt{\pi}} \cdot \operatorname{vol}(\pi_\tau(\mathcal{L}))^{1/(n - \tau + 1)}$. This gives

$$C(\tau, n) = \left(\frac{n}{4}\right)^{(n - \tau + 1)/2},$$

and

$$C(n) = \left(\frac{\sqrt{n}}{2}\right)^n = \frac{1}{2^n} n^{n/2}.$$

8 Conclusion

We have seen that given a dual HKZ-basis, we can solve the closest vector problem using the approach by Blömer [4] by enumerating $n^{c_n n}$ lattice points, with $c_n < 0.75$ for $10 < n \leq 2000$. Kannan's algorithm runs faster, as refined analysis thereof implies [8]. Using Kannan's algorithm, which as input takes a HKZ-basis, it is enough to enumerate $n^{n/2+o(n)}$ lattice points. On the other hand we have seen that if the shortest vector of the dual lattice satisfies the Gaussian heuristic, the transference theorems imply that is enough to enumerate all lattice points inside a ellipsoid of volume $\left(\frac{\sqrt{n}}{4}\right)^n$ in order to find the closest vectors. If the same assumption is made for all gram-schmidt vectors of the HKZ-basis used in Kannan's algorithm, the closest lattice points lie inside an ellipsoid of volume $\left(\frac{\sqrt{n}}{2}\right)^n$. Referring to the case where the Gaussian heuristic as tight as average case, Table 1 gives an overview on the complexities.

Approach	original	refined (worst case)	refined (average)
Kannan	$n^{n+o(n)}$	$2^{O(n)}n^{n/2}$	$2^{-2n}n^{n/2}$
Blömer	$n!$	$n^{c_n n}$	$2^{-n}n^{n/2}$

Table 1: Overview on the number of points to enumerate.

A Computation of $V_{\tau,k}$ in Section 5

$$\begin{aligned}
V_{\tau,k} &= \frac{\pi^{(k-\tau)/2}}{\Gamma\left(\frac{k-\tau}{2} + 1\right)} \left(\frac{\prod_{j=\tau+1}^k \frac{j}{2}}{\prod_{j=\tau+2}^k \left(\frac{j^2}{4} - \frac{1}{4}\right)^{1/2}} \right)^{k-\tau+1} \prod_{j=\tau+2}^k \frac{\prod_{i=\tau+2}^j \left(\frac{i^2}{4} - \frac{1}{4}\right)^{1/2}}{\prod_{i=\tau+1}^{j-1} \frac{i}{2}} \\
&= \frac{\pi^{(k-\tau)/2}}{\Gamma\left(\frac{k-\tau}{2} + 1\right)} \left(\frac{\tau+1}{2}\right)^{k-\tau} \left(\prod_{j=\tau+2}^k \frac{\frac{j}{2}}{\left(\frac{j^2}{4} - \frac{1}{4}\right)^{1/2}} \right)^{k-\tau} \\
&\quad \frac{k!}{(\tau+1)^{k-\tau-1}(\tau+1)!} \prod_{j=\tau+2}^k \prod_{i=\tau+2}^j \frac{\left(\frac{i^2}{4} - \frac{1}{4}\right)^{1/2}}{\frac{i}{2}} \\
&= \frac{\pi^{(k-\tau)/2}}{\Gamma\left(\frac{k-\tau}{2} + 1\right)} \frac{k!}{\tau!2^{k-\tau}} \left(\frac{(\tau+2)k}{(\tau+1)(k+1)}\right)^{(k-\tau)/2} \left(\frac{\tau+1}{\tau+2}\right)^{(k-\tau-1)/2} \left(\frac{k+1}{\tau+2}\right)^{1/2} \\
&= \frac{\pi^{(k-\tau)/2}}{\Gamma\left(\frac{k-\tau}{2} + 1\right)} \frac{k!}{\tau!2^{k-\tau}} \left(\frac{k+1}{\tau+1}\right)^{1/2} \left(\frac{k}{k+1}\right)^{(k-\tau)/2} \\
&\leq \frac{\pi^{(k-\tau)/2}}{\Gamma\left(\frac{k-\tau}{2} + 1\right)} \frac{k!}{\tau!2^{k-\tau}} \left(\frac{k}{\tau+1}\right)^{1/2} \\
&= \left(\frac{\pi}{4}\right)^{(k-\tau)/2} \frac{1}{\Gamma\left(\frac{k-\tau}{2} + 1\right)} \frac{k!}{\tau!} \left(\frac{k}{\tau+1}\right)^{1/2}
\end{aligned}$$

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