Existence of a Limiting Distribution for the Binary GCD Algorithm *

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Abstract

In this article, we prove the existence and uniqueness of a certain distribution function on the unit interval. This distribution appears in Brent's model of the analysis of the binary gcd algorithm. The existence and uniqueness of such a function has been conjectured by Richard Brent in his original paper [1]. Donald Knuth also supposes its existence in [5] where developments of its properties lead to very good estimates in relation with the algorithm. We settle here the question of existence, giving a basis to these results, and study the relationship between this limiting function and the *binary Euclidean operator* B_2, proving rigorously that its derivative is a fixed point of B_2.

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1 Introduction

If u and v are positive integers, their *greatest common divisor* (gcd), written $\gcd(u, v)$ in the sequel, is the largest integer that divides them both. This integer can be computed efficiently using a method discovered more than 2200 years ago: Euclid's algorithm. Quoting D.Knuth [5], this algorithm is the "grand-daddy" of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day. It is however not the best way to find greatest common divisors when dealing with modern computers. In fact, another algorithm, the so-called *binary gcd algorithm*, created by J.Stein [6], requires no division but only subtractions, parity testings and halving of even numbers (which correspond to shifts in binary notations). These procedures are essentially free when compared to the computational cost of divisions.

The idea of the binary gcd algorithm is basically the following: given two positive integers u and v, if halving both numbers is possible at most k time, do it, keeping the values of u and v updated. Then repeat the following procedure until both number are equal, say to l: subtract the smaller from the greater and when the result is even, divide it by the largest power of 2 possible. The gcd of u and v is then $l \cdot 2^k$. This repeated loop will be referred as a "subtract-and-shift cycle" in the sequel.

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The behavior of the binary gcd algorithm is interesting in several ways. On the one hand, it is always important to know the worst case and average case of an algorithm, just from a practical point of view, and this is even more important when the considered algorithm has such a wide application. On the second hand, the machinery elaborated in order to understand the average behavior of the algorithm has led to a deep understanding of it, giving answers as well as rising new questions.

In our case, the worst case the binary gcd algorithm may have to face is a total number of subtractions equal to $1 + \lfloor \log_2 \max(u, v) \rfloor)$, see, e.g., [5].

The exact determination of the average behavior of the binary gcd algorithm is however much more complex than the analysis of its worst case scenario. Two models have been proposed in order to study and analyze the expected behavior of the algorithm. We first describe the model created by R.Brent and gives a short description of the model created by B.Vallée at the end of this introduction.

The first model was created by Richard Brent in 1976 [1]. In his work, R.Brent exhibits a dynamical system describing the binary Euclidean algorithm and provides an heuristic proof of the analysis of the algorithm. This dynamical system is described by the binary Euclidean operator B_2, see (4.1) below, that transforms the density associated to the algorithm, step-by-step. However, the operator B_2 is difficult to analyze, and the question of convergence was left as a conjecture. This approach also suffers from the fact that it lies on an unproven connection between a discrete and a continuous model, see [1] for more details concerning this last point and [2] for a description of the situation 25 years later.

We now describe this model. Suppose that both u and v, with $u > v$, are odd, which is the case after each subtract-and-shift cycle. Every subtract-and-shift cycle forms $u - v$ and shifts this quantity right until obtaining an odd number u' that replace u. Under random conditions, one would expect to have $u' = (u-v)/2^m$ with probability 2^{-m}. This is the heart of Brent's hypothesis. In his model, we suppose that u and v are essentially random, except that they are odd and their ratio v/u has a certain probability distribution. Let g_n be the probability that $\min(u, v)/\max(u, v)$ is greater or equal to x after n subtraction-and-shift cycles have been performed under this assumption. Then the sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation [1, 5]:

$$g_0(x) = 1 - x \ , \ \ g_{n+1}(x) = F(g_n)(x)$$

where, for all $h \in C([0, 1])$,

$$F(h)(x) = \sum_{k\geq 1} 2^{-k} \left(h\left(\frac{x}{x + 2^k}\right) - h\left(\frac{1}{1 + 2^k x}\right)\right), \ x \in [0, 1]. \tag{1.1}$$

In the sequel, we will denote by $F_n(h)$ the partial sums of the above series. Note that the operator F is linear, bounded, since $||F||_\infty \leq 2$, and that the series converges uniformly for any h in $C([0, 1])$. Computational experiments led Richard Brent to conjecture that the functions g_n converge uniformly to a limiting distribution g_∞. Under this conjecture, the function g_∞ satisfies the equality

$$g_\infty(x) = \sum_{k\geq 1} 2^{-k} \left(g_\infty\left(\frac{x}{x + 2^k}\right) - g_\infty\left(\frac{1}{1 + 2^k x}\right)\right)$$

and provides the following estimate. If

$$b = 2 + \int_0^1 \frac{g_\infty(x)}{(1 - x)\ln 2} dt = 2.83297657...,$$

then the expected number of subtract-and-shift cycles in the binary gcd algorithm with starting values u and v is $\ln(uv)/b$.

In this article, we settle this conjecture by proving that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly towards a function g_∞. In order to do so, we first prove that every element of the sequence is convex and differentiable over $]0,1]$. Then we exhibit a compact set of the Banach space $(C([0,1]), ||.||_\infty)$ which contains g_1 and which is left invariant by the operator F defined by (1.1). This fact assures the existence of accumulation points of the sequence $\{g_n\}_{n\in\mathbb{N}}$, and therefore proves the existence of at least one fixed point of the operator F. We prove the uniqueness via an argument based on the sequence of derivatives $\{g'_n\}_{n\in\mathbb{N}}$. On the way, we study the behavior of this sequence with respect to the binary Euclidean operator B_2, proving that the sequence converges to the unique fixed point of B_2 in the L^1-norm.

The present work provides a proof that the dynamical system studied by Brent possesses indeed a unique limiting distribution. However, it does not shed a new light on the validation of the continuous model. In other words, it makes legitimate the work of R.Brent on the analysis of the binary gcd algorithm, without validating his model. It also answers a 47-points question of D.Knuth [5, p.355, question 32], who grades the problems of [5] on a "logarithmic" scale from 0 to 50.

The second work we were referring to is due to Brigitte Vallée [7] who brings another look to the situation and leads to a successful analysis using rigorous "dynamical" methods. These methods are also the basis for the analysis of several others algorithms [7, 8]. In her work, B.Vallée studies the operator V_2 which describes a slightly different dynamical system. The operator V_2 transforms the density associated to the algorithm where all the subtract-and-shift cycles are gathered together as long as the sign of $u - v$ is constant. As a consequence, the operator V_2 is easier to analyze. B.Vallée shows that the operator V_2 possesses a unique fixed point in some Hardy space and presents a spectral gap. She also proves, based on this spectral gap and with the help of a Tauberian theorem, the connection between the discrete and the continuous model. Quoting D.Knuth, "her methods are sufficiently different that they are not yet known to predict the same behavior as Brent's heuristic model. Thus the problem of analyzing the binary gcd algorithm [...] continues to lead to ever more tantalizing questions of higher mathematics".

Not surprisingly, there is a connection between the two operators B_2 and V_2. We refer the interested reader to [2] for further details regarding this connection.

We will use the notation $||.||_\infty$ and $||.||_1$ for the supremum norm and the L^1-norm of functions defined over $[0,1]$, and $\log_2$ for the logarithm in base 2. Let us recall that a series of function $\sum_{n>0} h_n(x)$ verifies the so-called Weierstrass criterion (see [4, III.4]) over a subset A of $\mathbb{R}$ if we have

$$\sum_{n>0} \sup_{x\in A} |h_n(x)| < \infty. \tag{1.2}$$

Let us mention that the notation for our g_n and g_∞ are different in both [1] and [5]. Brent [1] uses F_n and F_∞ and Knuth [5] uses G_n and G.

2 Convexity and Regularity

We prove in this section that the elements of the sequence $\{g_n\}_{n\in\mathbb{N}}$ are convex and decreasing functions over $[0,1]$ and differentiable over $]0,1]$.

Let m be a 2×2 matrix with real coefficients a, b, c and d . Such a matrix acts naturally on $\mathbb{R}$ via

$$m(x) = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right](x) = \frac{ax+b}{cx+d}$$

and this action satisfies $(m_1 \cdot m_2)(x) = m_1(m_2(x))$ for all pairs m_1, m_2 of 2×2 matrices, where $\cdot$ is the usual matrix product. From now on, we will identify a matrix m with the real function associated to it. Let us define the set M as follows:

$$M = \left\{ m : [0,1] \to \mathbb{R} \mid m(x) = \frac{ax+b}{cx+d} \text{ with } a, b \geq 0,\ c, d > 0\,,\ ad - bc \neq 0 \right\}.$$

Note that for any element m of M, $\operatorname{sgn}(m) := \operatorname{signum}(ad-bc)$ is well-defined, since a common factor at the denominator and the numerator of m does not affect the sign of $ad-bc$. This function satisfies the equality $\operatorname{sgn}(m_1 \circ m_2) = \operatorname{sgn}(m_1) \cdot \operatorname{sgn}(m_2)$ for all pairs m_1, m_2 in M. Let us also define the set S as the set of all series $\sum_{i \in \mathbb{N}} \varepsilon_i m_i$, where $m_i \in M$, satisfying the following three points:

1. $\varepsilon_i = \pm 1$ and $\varepsilon_i \cdot \operatorname{sgn}(m_i) < 0\,, \forall i \in \mathbb{N}$.
2. The series verifies the Weierstrass criterion over $[0,1]$, i.e.,

$$\sum_{i \in \mathbb{N}} \varepsilon_i m_i \in S \implies \sum_{i \in \mathbb{N}} ||m_i||_\infty < +\infty. \tag{2.1}$$

3. The following series converges:

$$\sum_{i \in \mathbb{N}} \frac{|a_i d_i - b_i c_i|}{c_i d_i} < +\infty\,, \text{ where } m_i(x) = \frac{a_i x + b_i}{c_i x + d_i}. \tag{2.2}$$

Note that the series (2.2) is well-defined since a common factor at the denominator and the numerator of m_i does not affect the terms of the series. A typical element g of S can be written as

$$g(x) = \sum_{i \in \mathbb{N}} \frac{a_i x + b_i}{c_i x + d_i}. \tag{2.3}$$

For sake of clarity, let us recall two facts about series of functions: first, if a series of functions satisfies the Weierstrass criterion (1.2) on a set, then it converges uniformly and absolutely on it, and the limit does not depend on any permutation of the sum. This result applies also for double sums. Second, if the derivatives of the partial sums of a convergent series of function converges uniformly then the series is differentiable and its derivative is the limit of the derivatives of the partial sums, see, e.g., Thm. 2.13, Thm. 2.9, Thm. 4.3 and Thm. 6.18 of [4].

The definition of the set S takes its roots in the following two lemmas, which are the keystones of the article.

Lemma 2.1 *Every function g in S is a convex, decreasing, continuous function over $[0,1]$ and continuously differentiable over any compact set of $]0,1]$.*

Proof: A function m in M is convex and decreasing if and only if $\operatorname{sgn}(m) < 0$. Indeed, we have

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2} < 0 \iff ad - bc < 0,$$

and

$$\left(\frac{ax+b}{cx+d}\right)'' = -2c\cdot\frac{ad-bc}{(cx+d)^3} > 0 \iff ad-bc<0.$$

Using the first two points of the above definition, any element g in S is a uniform limit of convex, decreasing and continuous functions, and is therefore convex, decreasing and continuous. Let us prove now that any element of S is continuously differentiable over any compact interval of $]0,1]$. Let $0<\varepsilon<1$. For g as in (2.3), the partial sums of g' satisfy

$$\left(\sum_{i=0}^{N}\frac{a_ix+b_i}{c_ix+d_i}\right)' = \sum_{i=0}^{N}\frac{a_id_i-b_ic_i}{(c_ix+d_i)^2}$$

and the definition of S shows that this series satisfies the Weierstrass criterion over $[\varepsilon,1]$ since for all $x\in[\varepsilon,1]$,

$$\frac{|a_id_i-b_ic_i|}{(c_ix+d_i)^2} \le \frac{|a_id_i-b_ic_i|}{(c_i\varepsilon+d_i)^2} \le \varepsilon^{-2}\frac{|a_id_i-b_ic_i|}{(c_i+d_i)^2}$$

and

$$\frac{1}{(c+d)^2} \le \frac{1}{4cd} \quad \forall c,d>0,$$

yields

$$\varepsilon^{-2}\sum_{i\ge 0}\frac{|a_id_i-b_ic_i|}{(c_i+d_i)^2} \le \frac{\varepsilon^{-2}}{4}\cdot\sum_{i\ge 0}\frac{|a_id_i-b_ic_i|}{c_id_i} < +\infty.$$

Thus, the partial sums of derivative converge uniformly over $[\varepsilon,1]$ to a limiting function which is the derivative of g. This finishes the proof. □

Lemma 2.2 *Let F_n be the partial sums of the series (1.1). If $g:[0,1]\longrightarrow\mathbb{R}$ is a function in S, then $F_n(g)\in S$ for all $n\in\mathbb{N}$ and $F(g)\in S$.*

Proof: Let us define the following particular elements of M:

$$\mu_k(x) = \begin{bmatrix} 1 & 0 \\ 1 & 2^k \end{bmatrix}(x) = \frac{x}{x+2^k} \quad \text{and} \quad \nu_k(x) = \begin{bmatrix} 0 & 1 \\ 2^k & 1 \end{bmatrix}(x) = \frac{1}{2^kx+1}.$$

Note that these functions map the interval $[0,1]$ in itself and satisfy

$$\operatorname{sgn}(\mu_k)>0 \text{ and } \operatorname{sgn}(\nu_k)<0.$$

Let us prove that if $g(x)=\sum_{i\in\mathbb{N}}\varepsilon_i m_i(x)$ is in S, then $F(g)$ lies inside S. The proof for the partial sums $F_n(g)$ is similar, although infinite sums might become finite. We have

$$\begin{aligned} F(g)(x) &= \sum_{k\ge 1}2^{-k}\left(g(\mu_k(x))-g(\nu_k(x))\right) \\ &= \sum_{k\ge 1}2^{-k}\left(\sum_{i\in\mathbb{N}}\varepsilon_i m_i(\mu_k(x)) - \sum_{i\in\mathbb{N}}\varepsilon_i m_i(\nu_k(x))\right). \end{aligned} \tag{2.4}$$

Based on the Weierstrass criterion (2.1), we have

$$\begin{aligned} ||F(g)||_\infty &\le \sum_{k\ge 1}2^{-k}\left(\sum_{i\in\mathbb{N}}||m_i\circ\mu_k||_\infty + \sum_{i\in\mathbb{N}}||m_i\circ\nu_k||_\infty\right) \\ &\le \sum_{k\ge 1}2^{-k}\left(\sum_{i\in\mathbb{N}}||m_i||_\infty + \sum_{i\in\mathbb{N}}||m_i||_\infty\right) < +\infty \end{aligned} \tag{2.5}$$

and therefore the double sums in (2.4) can be rearranged in any simple sum

$$F(g)(x) = \sum_{i\in\mathbb{N}} \varepsilon_i M_i(x)$$

where $\varepsilon_i M_i(x)$ is either of the type $\varepsilon_j \cdot 2^{-k} m_j(\mu_k(x))$ or of the type $-\varepsilon_j \cdot 2^{-k} m_j(\nu_k(x))$. Clearly, $F(g)$ has the correct structure to be an element of S. We must now prove that this function fulfills the three points of the definition of the set S. Inequality (2.5) shows that the latter series fulfills the Weierstrass criterion, directly proving the second point. Since

$$\varepsilon_j \cdot \mathrm{sgn}\,(2^{-k} \cdot (m_j \circ \mu_k)) = \varepsilon_j \cdot \mathrm{sgn}\,(m_j) \cdot \mathrm{sgn}\,(\mu_k) = \varepsilon_j \cdot \mathrm{sgn}\,(m_j) < 0,$$

and

$$\varepsilon_j \cdot \mathrm{sgn}\,(-2^{-k} \cdot (m_j \circ \nu_k)) = -\varepsilon_j \cdot \mathrm{sgn}\,(m_j) \cdot \mathrm{sgn}\,(\nu_k) = \varepsilon_j \cdot \mathrm{sgn}\,(m_j) < 0,$$

the first point is verified. Let us check the validity of the third point. A straightforward computation shows that if g is as in (2.3) then the analogue series as (2.2) for $F(g)$ is the following double series:

$$\sum_{k\geq 1}\sum_{i\in\mathbb{N}} \frac{2|a_i d_i - b_i c_i|}{(c_i + d_i) d_i \cdot 2^k}.$$

This series is convergent since the inequality $(c_i + d_i)d_i > c_i d_i$ yields the following estimate:

$$\begin{aligned} \sum_{k\geq 1}\sum_{i\in\mathbb{N}} \frac{2|a_i d_i - b_i c_i|}{(c_i + d_i) d_i \cdot 2^k} &< 2\sum_{k\geq 1}\frac{1}{2^k}\cdot\sum_{i\in\mathbb{N}}\frac{|a_i d_i - b_i c_i|}{c_i d_i} \\ &= 2\sum_{i\in\mathbb{N}}\frac{|a_i d_i - b_i c_i|}{c_i d_i} < +\infty. \end{aligned}$$

This shows that the third point is fulfilled and the lemma is then proven. □

Proposition 2.3 *Every element of the sequence* $\{g_n\}_{n\in\mathbb{N}\setminus\{0\}}$ *is in* S. *Thus every element of the sequence* $\{g_n\}_{n\in\mathbb{N}}$ *is a convex, continuous and decreasing function over* $[0,1]$, *continuously differentiable over* $]0,1]$.

Proof: Since $g_0(x) = 1 - x$, g_0 fulfills the conditions of the claim. The function g_1 is as follows

$$g_1(x) = \sum_{k\geq 1} 2^{-k}\left(\frac{1}{1+2^k x} - \frac{x}{x+2^k}\right) = \sum_{k\geq 1}\left(\frac{1}{2^k + 2^{2k}x} - \frac{x}{2^k x + 2^{2k}}\right),$$

and a straightforward computation shows that the function g_1 above is an element of S. Lemma 2.2 shows by induction that g_n is an element of S and is therefore convex, decreasing, continuous over $[0,1]$ and continuously differentiable over any compact subset of $]0,1]$ by Lemma 2.1. □

3 Existence of an accumulation point

In the current section, we prove that the sequence $\{g_n\}_{n\in\mathbb{N}}$ possesses at least one accumulation point in the Banach space of continuous function defined over $[0,1]$, with the supremum norm. Let us define the following two subsets of this Banach space:

$$K_1 = \overline{S} \cap \{g|\, g(0) = 1, g(1) = 0\}, \quad (3.1)$$

$$K_2 = \{g \in C([0,1])\,|\, 1 + 3/2 \cdot x\log_2 x - 5x \leq g(x) \leq 1 - x\}, \quad (3.2)$$

where $\overline{S}$ is the closure of S in the supremum norm and $\log_2$ is the logarithm in base 2. Note that any element of $\overline{S}$ is a decreasing, convex and continuous function, being a uniform limit of such functions. The definition of K_2 seems odd at first sight. The key point is that a function in K_2 cannot come close to 1 with a too steep slope when x goes to 0. We start with the following proposition:

Proposition 3.1 *The operator F verifies the following properties:*

1. $F(K_1) \subset K_1$,
2. $F(K_1 \cap K_2) \subset K_2$,

and therefore $F(K_1 \cap K_2) \subset K_1 \cap K_2$.

Proof: The map F being continuous, we have $F(\overline{S}) \subset \overline{F(S)}$. Lemma 2.2 implies that $\overline{F(S)} \subset \overline{S}$, therefore $F(\overline{S}) \subset \overline{S}$. The fact that $F(g)(0) = 1$ and $F(g)(1) = 0$ when $g(0) = 1$ and $g(1) = 0$ is straightforward. This proves the first point.

Suppose g is a function in $K_1 \cap K_2$. The inequality $F(g)(x) \leq 1 - x$ is obvious since, $F(g)$ being an element of $\overline{S}$, is convex and lies below the secant joining $(0,1)$ to $(1,0)$. It remains to show that

$$F(g)(x) \geq 1 + 3/2 \cdot x \log_2 x - 5x.$$

Based on the definitions of F and K_2, we have

$$\begin{aligned} F(g)(x) &= \sum_{k\geq 1} 2^{-k} \left(g(x/(x+2^k)) - g(1/(1+2^k x)) \right) \\ &\geq \sum_{k\geq 1} 2^{-k} \Bigg(1 + \frac{3}{2} \cdot \left(\frac{x}{x+2^k} \cdot \left(\log_2 x - \log_2(x+2^k) \right) \right) \\ &\quad -5 \cdot \frac{x}{x+2^k} - 1 + \frac{1}{1+2^k x} \Bigg) \\ &= \frac{3}{2} \cdot x \log_2 x \cdot \left(\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{1}{x+2^k} \right) - \frac{3}{2} \cdot x \cdot \left(\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{\log_2(x+2^k)}{x+2^k} \right) \\ &\quad -5x \cdot \left(\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{1}{x+2^k} \right) + \left(\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{1}{1+2^k x} \right). \end{aligned}$$

Note that, for $x \in [0,1]$, we have

$$\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{1}{x+2^k} \leq \sum_{k\geq 1} \frac{1}{4^k} = \frac{1}{3},$$

and

$$\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{\log_2(x+2^k)}{x+2^k} \leq \sum_{k\geq 1} \frac{1}{2^k} = 1.$$

Based on Mellin's transform, the equality

$$\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{1}{1+2^k x} = 1 + x \log_2 x + x \cdot P(\log_2 x) + \frac{x}{2} - \sum_{k\geq 2} (-1)^k \frac{2^{k-1}}{2^{k-1}-1} x^k \tag{3.3}$$

where

$$P(y) = \frac{2\pi}{\ln 2} \cdot \sum_{k \geq 1} \frac{\sin 2\pi k y}{\sinh(2k\pi^2/\ln 2)}$$

can be proven. A proof can also be found in [5, p. 644], where it appears as the main step in the computation of the function g_1. As a matter of fact, the function $P(y)$ is small, and can be bounded in absolute value by $8 \cdot 10^{-12}$, c.f. [5]. We will however only need a far less accurate bound. Since $\sinh(t) > e^t/4$ for $t > \ln 2/2$, we have

$$|P(y)| < \frac{2\pi}{\ln 2} \cdot \sum_{k \geq 1} \left(4e^{-2\pi^2/\ln 2}\right)^k = \frac{2\pi}{\ln 2} \cdot \frac{4}{e^{2\pi^2/\ln 2} - 4} = 1.5549.. \cdot 10^{-11} < 1/4.$$

For $x \in [0, 1]$, the terms of the alternating sums on the right-hand-side of (3.3) decrease in absolute value. This sum can therefore be bounded above by its first term, $2x^2$. Since $x \log_2 x \leq 0$ over $[0, 1]$, the previous estimation of $F(g)$ becomes

$$\begin{aligned}
F(g)(x) \quad &\geq \quad \frac{3}{2} \cdot x \log_2 x \cdot \frac{1}{3} - \frac{3}{2} \cdot x \cdot 1 \\
&\quad\; -5 \cdot x \cdot \frac{1}{3} + 1 + x \log_2 x - x \cdot \frac{1}{4} + \frac{x}{2} - 2x^2 \\
&= \quad 1 + \frac{3}{2} \cdot x \log_2 x - \frac{35}{12} x - 2x^2 \\
&= \quad 1 + \frac{3}{2} \cdot x \log_2 x - 5x + \underbrace{\left(\frac{25}{12} x - 2x^2\right)}_{\geq 0} \\
&\geq \quad 1 + \frac{3}{2} \cdot x \log_2 x - 5x.
\end{aligned}$$

This last estimate finishes the proof of the proposition. □

Note that the proof of the previous proposition also shows that if a function g is convex and in K_2, then $F(g)$ is in K_2 as well. Indeed, the only property needed from K_1 in the proof that $F(K_1 \cap K_2) \subset K_2$ is the convexity of elements in K_1. Let us state this result as a corollary:

Corollary 3.2 *If a function $g : [0, 1] \to \mathbb{R}$ is convex and in K_2, then $F(g)$ is in K_2.*

We turn now to a result of compactness.

Proposition 3.3 *The set $K_1 \cap K_2$ is compact in the Banach space $(C([0, 1]), ||.||_\infty)$.*

Proof: The set K_1 is closed in $(C([0, 1]), ||.||_\infty)$ being the intersection of two closed sets. The set K_2 is clearly closed as well. Consider the set of Hölder functions over $[0, 1]$ (with parameter 1/2). These are the functions f for which

$$N_{1/2}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1/2}} < \infty.$$

Based on an argument of equicontinuity, it can be verified that the set

$$K_{A,B} = \{f \in C([0, 1]) \,|\, ||f||_\infty \leq A,\, N_{1/2}(f) \leq B\}$$

is compact in $C([0,1])$ for any $A, B > 0$, see e.g., Chap.4, Sect.6 of [3]. Let us show that $K_1 \cap K_2 \subset K_{1,5}$. The only non-trivial point to be checked is the fact that if $g \in K_1 \cap K_2$, then $N_{1/2}(g) \leq 5$. The function g being decreasing and convex, we have, for $0 < y < x < 1$,

$$\frac{|g(x) - g(y)|}{|x - y|^{1/2}} \leq \frac{g(0) - g(x - y)}{\sqrt{x - y}} = \frac{1 - g(h)}{\sqrt{h}}, \quad \text{with } h = x - y > 0.$$

Using a property of the elements of K_2, we also have

$$\frac{1 - g(h)}{\sqrt{h}} \leq \frac{-3/2h \log_2 h + 5h}{\sqrt{h}} = 5\sqrt{h} - 3/2\sqrt{h} \log_2 h.$$

The maximum value of the latter function, defined over $[0, 1]$, is reached for $h = 1$ and therefore

$$\frac{|g(x) - g(y)|}{|x - y|^{1/2}} \leq \left[5\sqrt{h} - 3/2\sqrt{h} \log_2 h\right]_{h=1} = 5.$$

Taking the supremum, we obtain the expected result. The set $K_1 \cap K_2$ being a closed subset of a compact metric space, it is itself compact. This proves the proposition. □

The previous two propositions give directly the next corollary, since any compact set in a metric space satisfies the Bolzano-Weierstrass condition:

Corollary 3.4 *The sequence $\{g_n\}_{n \in \mathbb{N}}$ possesses at least one accumulation point in $K_1 \cap K_2$.*

Proof: The function g_0 is convex and in K_2. By Corollary 3.2, $F(g_0) = g_1$ is therefore in K_2. This function is also in K_1 (see the proof of Proposition 2.3) and thus any element of the sequence $\{g_n\}_{n \in \mathbb{N}}$ but g_0 is in the compact $K_1 \cap K_2$. The conclusion follows by the Bolzano-Weierstrass property. □

4 Behavior of the derivatives and uniqueness of the accumulation point

In the current section, we prove that the sequence $\{g_n\}_{n \in \mathbb{N}}$ in fact possesses only one accumulation point g_∞. This proves that the sequence converges to this well-defined function in $K_1 \cap K_2$ since a sequence in a compact metric space with only one accumulation point converges to this point. In order to achieve this goal, we study the sequence of derivatives $\{g'_n\}_{n \in \mathbb{N}}$ in the topology of the L^1-norm over $]0, 1]$. Then, based on a property of the binary Euclidean operator B_2, defined below by (4.1), we show the uniqueness of the accumulation points of both the sequences $\{g_n\}_{n \in \mathbb{N}}$ and $\{g'_n\}_{n \in \mathbb{N}}$.

The sequence of derivatives $\{g'_n\}_{n \in \mathbb{N}}$ does not converges uniformly, or at least we do not know it, and therefore nothing tells us that an accumulation point g_∞ possesses a derivative which is the uniform limit of the derivatives of the subsequence. However, g_n being convex for all n, we will see in the sequel that the sequence of derivatives does converge but in a weaker topology, the topology of $L^1([0, 1])$. Recall that any convex function h over $[0, 1]$ is differentiable almost everywhere in $[0, 1]$ and that it is absolutely continuous [3], i.e.,

$$h(x) = h(0) + \int_0^x h'(t)dt, \quad \forall x \in [0, 1].$$

The following lemma sheds light on the convergence of the derivatives of convex functions:

Lemma 4.1 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of convex functions defined over $[a,b]$, differentiable over $]a,b[$, and converging uniformly to a function f. Then:*

1. *If E is the subset of point of $]a,b[$ where f is differentiable, then for all x_0 in E, the sequence $\{f_n'(x_0)\}_{n\in\mathbb{N}}$ converges to $f'(x_0)$.*
2. *If the functions f_n are monotone, then the sequence $\{f_n'\}_{n\in\mathbb{N}}$ converges to f' in the L^1-norm.*

Proof: Let $x_0 \in E$. There exists $h_0 > 0$ such that $[x_0 - h_0, x_0 + h_0] \subset]a,b[$. The functions f_n being convex, we have $\forall n \in \mathbb{N}$, $\forall\, 0 < h < h_0$,

$$\frac{f_n(x_0 - h) - f_n(x_0)}{h} \leq f_n'(x_0) \leq \frac{f_n(x_0 + h) - f_n(x_0)}{h}.$$

When n goes to infinity, this leads to the following inequalities $\forall\, 0 < h < h_0$,

$$\frac{f(x_0 - h) - f(x_0)}{h} \leq \liminf_n f_n'(x_0) \leq \limsup_n f_n'(x_0) \leq \frac{f(x_0 + h) - f(x_0)}{h}.$$

Taking the limit when h goes to 0, we finally have

$$f'(x_0) \leq \liminf_n f_n'(x_0) \leq \limsup_n f_n'(x_0) \leq f'(x_0), \text{ i.e., } \lim_{n\to\infty} f_n'(x_0) = f'(x_0).$$

In order to prove the second point, note that since f is convex, the set $[a,b] \setminus E$ has measure 0, and thus f_n' converges to f' almost everywhere. Without loss of generality, suppose the functions f_n are increasing, i.e., $f_n' \geq 0$ almost everywhere. The functions f_n and f are absolutely continuous, and thus

$$\lim_{n\to\infty} \int_a^b |f_n'| = \lim_{n\to\infty} \int_a^b f_n' = \lim_{n\to\infty} (f_n(a) - f_n(b)) = f(a) - f(b) = \int_a^b f' = \int_a^b |f'|.$$

A direct application of the dominated convergence theorem shows that f_n' converges to f' in L^1, see also Ex.21, p.57 of [3]. □

Consider the following linear operator, obtained by taking the formal derivative of the series (1.1):

$$B_2(h)(x) = \sum_{k\geq 1} \left(\frac{1}{x+2^k}\right)^2 h\left(\frac{x}{x+2^k}\right) + \left(\frac{1}{1+2^k x}\right)^2 h\left(\frac{1}{1+2^k x}\right). \tag{4.1}$$

This operator is referred as the “binary Euclidean operator” in the literature. It was first studied by Richard Brent [1]. It is not clear at first sight for what class of function the operator B_2 should be defined. If we consider its action on $L^1([0,1])$, then the operator B_2 is a contraction with respect to the L_1-norm:

$$\begin{aligned}
\int_0^1 |B_2(h)(t)|dt \quad &\leq \quad \sum_{k\geq 1} \int_0^1 \left(\frac{1}{t+2^k}\right)^2 \left|h\left(\frac{t}{t+2^k}\right)\right| dt \\
&\qquad + \int_0^1 \left(\frac{1}{1+2^k t}\right)^2 \left|h\left(\frac{1}{1+2^k t}\right)\right| dt \\
&= \quad \sum_{k\geq 1} 2^{-k} \left(\int_0^{1/(1+2^k)} |h(y)|dy + \int_{1/(1+2^k)}^1 |h(y)|dy\right) \\
&= \quad \int_0^1 |h(y)|dy. \qquad (4.2)
\end{aligned}$$

The first equality comes from the changes of variables $y = t/(t + 2^k)$ in the first integral and $y = 1/(1 + 2^k t)$ in the second integral. As a consequence, the operator B_2 can be defined over the entire Banach space $L^1([0, 1])$, and this operator is continuous with respect to the topology generated by its norm:

$$B_2 : L^1([0,1]) \longrightarrow L^1([0,1]) \text{ and } B_2 \in \mathcal{L}(L^1([0,1]), L^1([0,1])).$$

This property was already noticed by Brent in his original article [1]. Here is a first application of Lemma 4.1:

Proposition 4.2 *If h is a function in K_1, c.f. (3.1), then $F(h)' = B_2(h')$ in $L^1([0, 1])$.*

Proof: Let $h \in S \cap \{g|\, g(0) = 1, g(1) = 0\}$. Let us define

$$f_n(x) = F_n(h)(x) = \sum_{k=1}^{n} 2^{-k} \left(h\left(\frac{x}{x+2^k}\right) - h\left(\frac{1}{1+2^k x}\right)\right).$$

By Lemma 2.2 and 2.1, these functions are convex, decreasing and continuously differentiable over $]0, 1]$. Therefore, over $]0, 1]$, we have

$$f_n'(x) = \sum_{k=1}^{n} \left(\frac{1}{x+2^k}\right)^2 h'\left(\frac{x}{x+2^k}\right) + \left(\frac{1}{1+2^k x}\right)^2 h'\left(\frac{1}{1+2^k x}\right).$$

The sequence $\{f_n'\}_{n\in\mathbb{N}}$, being the partials sum of the series that defines B_2, converges to $B_2(h')$ in $L^1([0, 1])$. The condition of Lemma 4.1 are fulfilled and since the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly towards $F(h)$, we have $F(h)' = B_2(h')$ in $L^1([0, 1])$.

In general, if $h \in \overline{S} \cap \{g|\, g(0) = 1, g(1) = 0\}$, there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ of function in $S \cap \{g|\, g(0) = 1, g(1) = 0\}$ that converges uniformly to h. We use again Lemma 4.1 to have that $h_n' \longrightarrow h'$ in L^1. Since every h_n belongs to $S \cap \{g|\, g(0) = 1, g(1) = 0\}$ the partial result above applies and thus $F(h_n)' = B_2(h_n')$ for all $n \in \mathbb{N}$. Since $\{F(h_n)\}_{n\in\mathbb{N}}$ is a sequence of decreasing convex functions that converges uniformly to $F(h)$, thanks to the continuity of F with respect to $||.||_\infty$, we can once again apply Lemma 4.1 to this sequence. Taking the limit leads to the result since $F(h_n)' \longrightarrow F(h)'$ in L^1 and $B(h_n') \longrightarrow B(h')$ in L^1 as well, because of the continuity of B_2 with respect to $||.||_1$. □

Based on these properties, we can prove the following expected theorem, using another time Lemma 4.1:

Theorem 4.3 *The sequence $\{g_n\}_{n\in\mathbb{N}}$ possesses a unique accumulation point, and therefore converges uniformly to a limiting function g_∞ which is a fixed point of the linear operator F. The sequence of derivatives $\{g_n'\}_{n\in\mathbb{N}}$ converges almost everywhere and in the L^1-norm to g_∞', which is a fixed point of the linear operator B_2.*

Proof: Let us consider g_∞, an accumulation point of the sequence $\{g_n\}_{n\in\mathbb{N}}$. Based on Proposition 4.2, we have

$$g_\infty' = F(g_\infty)' = B_2(g_\infty') \text{ in } L^1([0,1]).$$

Consider the following sequence of non-negative real numbers:

$$u_n = \int_0^1 |g_\infty' - g_n'|dt = ||g_\infty' - g_n'||_1 \ , \quad n \in \mathbb{N}.$$

Then, using the previous equality, Proposition 4.2 and the fact that B_2 is a contraction, we see that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is decreasing:

$$\begin{aligned} u_{n+1} &= ||g'_\infty - g'_{n+1}||_1 \\ &= ||g'_\infty - F(g_n)'||_1 \\ &= ||B_2(g'_\infty) - B_2(g'_n)||_1 \\ &= ||B_2(g'_\infty - g'_n)||_1 \\ &\leq ||g'_\infty - g'_n||_1 \\ &= u_n. \end{aligned}$$

Let $\{g_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{g_n\}_{n\in\mathbb{N}}$ that converges to g_∞. Note that the conditions of Lemma 4.1 are fulfilled and therefore the sequence $\{g'_{n_k}\}_{k\in\mathbb{N}}$ converges in $L^1([0,1])$ and almost everywhere to g'_∞. This implies that the decreasing sequence $\{u_n\}_{n\in\mathbb{N}}$ possesses a subsequence that converges to 0 and therefore

$$\lim_{n\to\infty} u_n = 0.$$

In other words, the sequence $\{g'_n\}_{n\in\mathbb{N}}$ converges to g'_∞ in $L^1([0,1])$. Thus, since

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \left(g_n(0) + \int_0^x g'_n(t)dt \right) = 1 + \int_0^x g'_\infty(t)dt = g_\infty(x),$$

we see that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges point-wise to g_∞. This makes impossible the existence of another accumulation point. As explained at the beginning of the section, this shows that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges to g_∞. □

5 Conclusion

Theorem 4.3 shows that the operator B_2 has a unique eigenfunction with eigenvalue 1. Computational experiments [2] show that the next eigenvalues seem to be conjugate complex numbers λ and $\overline{\lambda}$ close to $0.1735 \pm 0.00884i$, with $|\lambda_1| = |\lambda_2| = 0.1948$. Therefore, B_2 seems to present the spectral gap Vallée's operator V_2 possesses. The method described in this article does not seem to extend in such a way that this spectral gap can be proved. This would imply the exponential speed of convergence already suspected in [1].

We did not prove that the function g'_∞ is continuous over $]0,1]$, which is strongly suspected. If proven, this continuity would directly imply the uniform convergence of the sequence of continuous and increasing functions $\{g'_n\}_{n\in\mathbb{N}}$ over any compact set of $]0,1]$ because of a theorem of Dini.

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