Asymptotic behaviour of the Stokes problem in cylinders becoming unbounded in one direction

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Abstract

We study the asymptotics of the Stokes problem in cylinders becoming unbounded in the direction of their axis. First we assume that the applied forces are independent of the axis coordinate, then we assume that they are periodic along the axis of the cylinder. Finally in Section 4, we make an asymptotic analysis under much more general assumptions on the applied forces.

Résumé

Nous faisons une analyse asymptotique du problème de Stokes dans des cylindres qui deviennent infinis dans la direction axiale. Nous considérons tout d’abord le cas où les forces appliquées sont constantes dans la direction de l’axe du cylindre, puis nous traitons le cas de forces périodiques. Enfin, dans la Section 4, nous faisons des hypothèses très générales sur les forces appliquées.

Key words: Stokes equations, cylinders, exponential convergence

1 Introduction and notation

We consider the Stokes problem defined over the cylinder

$$
\Omega_\ell := (-\ell, \ell) \times \omega,
$$

where $\ell > 0$ is a parameter that goes to infinity. The section $\omega$ of the cylinder is a bounded, connected, open subset of $\mathbb{R}^{n-1}$ with Lipschitz boundary $\partial \omega$.

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Preprint submitted to Elsevier 11 May 2008
The unknown \((u_\ell, p_\ell)\) of the Stokes problem, consisting of a velocity field \(u_\ell = (u_\ell^1, \ldots, u_\ell^n) \in (H^1_0(\Omega_\ell))^n\) and a pressure \(p_\ell \in L^2(\Omega_\ell)/\mathbb{R}\) of a fluid, satisfies the equations
\[
-\mu \Delta u_\ell + \nabla p_\ell = f \quad \text{in} \quad (H^{-1}(\Omega_\ell))^n,
\]
\[
\text{div} \, u_\ell = 0 \quad \text{in} \quad \Omega_\ell,
\]
where \(f = (f^1, \ldots, f^n)\) is a given vector field on \(\mathbb{R} \times \omega\) and \(\mu\) is a positive constant describing the viscosity of the fluid.

Our goal is to study the behaviour of the pair \((u_\ell, p_\ell)\) as \(\ell\) goes to infinity, that is, when the domain \(\Omega_\ell\) tends to an infinite cylinder, in the case where \(f\) is independent of \(x_1\) or periodic in the direction \(x_1\).

In the case where \(f\) is independent of the first variable \(x_1\), we show that the velocity fields \(u_\ell\) and the pressures \(p_\ell\) converge in some sense to a solution \((u_\infty, p_\infty)\) of a problem defined over the \((n-1)\)-dimensional set \(\omega\). This problem is in fact a \((n-1)\)-dimensional Stokes problem complemented with an elliptic equation. One cannot expect \((u_\ell, p_\ell)\) to be close to \((u_\infty, p_\infty)\) on the whole cylinder \(\Omega_\ell\), since the solutions \((u_\ell, p_\ell)\) are still influenced by the boundary conditions in the neighbourhood of the ends of the cylinder \([-\ell] \times \omega\) and \([\ell] \times \omega\). We prove instead that \((u_\ell, p_\ell)\) converges to \((u_\infty, p_\infty)\) on every fixed cylinder \(\Omega_{\ell_0}\). However, if the applied forces are orthogonal to the axis of the cylinder, one can give a global approximation (i.e. on the whole cylinder \(\Omega_\ell\)) of the solution by adding a correcting term to the limit solution (see [5]). This kind of problems has been previously considered by Rougirel, Xie, Yeressian and the first author in [6], [20], [9] and [4].

In the case where \(f\) is periodic in the \(x_1\)-direction, the limit will be determined by the solution of a Stokes problem defined on the cell \(Q := (0, 1) \times \omega\). For simplicity, we consider the case of period 1, i.e. when the function \(f\) satisfies
\[
f(x + e_1) = f(x) \quad \text{a.e.} \quad x \in \mathbb{R} \times \omega,
\]
where \(e_1 := (1, 0, \ldots, 0)\), but the result applies for an arbitrary period.

We want to emphasize at this point that the periodic case was also studied in [2] by Baillet, Henrot, Takahashi, for the two dimensional Stokes problem with a different boundary condition (the Navier slip boundary condition), namely \(\text{curl} \, u_\ell = 0\) and \(u_\ell \cdot \nu = 0\) on \(\partial \Omega_\ell\). The case of periodic data for elliptic and parabolic problems was studied in a series of papers by Xie and the first author, see [7], [8], [19] and [20].

In the last section, we drop any particular constraint on \(f\) and we only assume that the \(L^2\)-norms of \(f\) on \(\Omega_\ell\) have a polynomial growth at infinity. In fact one can prove that the theorems in Section 2 can be obtained as consequences of the general Theorem 13. However, the approach in the proof of Theorem 13 is less natural than the ones in Section 3. At the same time, developing Section 4...
first would require the same techniques that were used in the previous sections.

We now introduce our notation.

A generic point in \( \mathbb{R}^n \), \( n \geq 2 \) is denoted \( x = (x_1, x_2, \ldots, x_n) = (x, x') \), where 
\[
x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}.
\]

We denote by \( \mathcal{L}^k \) the \( k \)-dimensional Lebesgue measure and we say that a property holds \( \mathcal{L}^k \)-a.e. on a set if it holds almost everywhere (on that set) with respect to the measure \( \mathcal{L}^k \). The notation \( \text{int} A \) is used to designate the interior of a set \( A \subset \mathbb{R}^k \) and the notation \( \mathbf{1}_A \) is used for its characteristic function. The outward unit normal to a Lipschitz domain in \( \mathbb{R}^k \) is denoted by \( \nu = (\nu_1, \ldots, \nu_k) \). We use the notation \( B_k(x, r) \) for the \( k \)-dimensional open ball of center \( x \in \mathbb{R}^k \) and radius \( r > 0 \).

For the partial derivatives we use the notation \( \partial_i := \partial/\partial x_i, i = 1, 2, \ldots, n \). The gradient, the laplacian, and the divergence operators defined over \( \mathbb{R}^n \) are respectively denoted 
\[
\nabla = (\partial_1, \ldots, \partial_n), \quad \Delta = \partial_{11} + \cdots + \partial_{nn} \quad \text{and} \quad \text{div } v = \partial_1 v^1 + \cdots + \partial_n v^n,
\]
where \( v = (v^1, v^2, \ldots, v^n) \). We also introduce the operators \( \nabla', \Delta', \text{div}' \) defined by 
\[
\nabla' = (\partial_2, \ldots, \partial_n), \quad \Delta' = \partial_{22} + \cdots + \partial_{nn} \quad \text{and} \quad \text{div}' v = \partial_2 v^2 + \cdots + \partial_n v^n.
\]
If \( v = (v^1, v^2, \ldots, v^n) \) is a \( \mathbb{R}^n \)-valued function, we denote 
\[
v' := (v^2, \ldots, v^n).
\]

For \( O \) a bounded open subset of \( \mathbb{R}^k \), \( k \geq 1 \), \( \mathcal{D}'(O) \) will be the space of distributions over the set \( O \). The space of infinitely differentiable functions with compact support in \( O \) is denoted by \( \mathcal{D}(O) \). We also introduce the quotient space 
\[
\hat{L}^2(O) := L^2(O)/\mathbb{R}
\]
endowed with the norm 
\[
\|v\|_{2,O} := \inf \left\{ \left( \int_O (v + k)^2(x) \, dx \right)^{\frac{1}{2}} ; \ k \in \mathbb{R} \quad \text{is a constant} \right\} \tag{2}
\]
An easy computation shows that 
\[
\|v\|_{2,O} = \|v - \bar{v}\|_{2,O}, \tag{3}
\]

3
where $\overline{v}$ is the average value of $v$, $\overline{v} := \frac{1}{\mathcal{L}^k(O)} \int_O v \, dx$ and $\| \cdot \|_{2,O}$ is the usual norm on $L^2(O)$.

In the right hand side of the above formulae, $v \in L^2(O)$ is an arbitrary representative of $v \in \hat{L}^2(O)$, also denoted $v$ for simplicity. In other words, equality (3) says that the $\hat{L}^2(O)$-norm is given by the $L^2(O)$-norm of the representative with null average value. In particular, the infimum in (2) is attained.

For $m \in \mathbb{N}^*$, we set
$$L^2(O) := (L^2(O))^m$$
and we equip this space with the norm
$$\|v\|_{2,O} := \left\{ \int_O v \cdot v \right\}^{\frac{1}{2}},$$
where “$\cdot$” denote the usual scalar product in $\mathbb{R}^m$.

If $U \subset \mathbb{R}^k$ is an unbounded open set, then we define
$$L^2_{\text{loc}}(U) := \{ v \in L^2_{\text{loc}}(U) ; v \in L^2(O) \text{ for all bounded open set } O \subset U \}.$$ and
$$\hat{L}^2_{\text{loc}}(U) := L^2_{\text{loc}}(U)/\mathbb{R}, \quad L^2_{\text{loc}}(U) := (L^2_{\text{loc}}(U))^m.$$ Similarly we set
$$H^1(O) := (H^1(O))^m, \quad H^1_0(O) := (H^1_0(O))^m, \quad H^{-1}(O) := (H^{-1}(O))^m,$$
where $H^1(O), H^1_0(O)$ and $H^{-1}(O)$ are the usual Sobolev spaces constructed on $L^2(O)$ - see [10], [12], [4]. In this paper $m$ will be equal to $n$ or $n-1$, the choice being obvious from the context. If $k = m$, then we also define
$$\hat{H}^1_0(O) := \{ v \in H^1_0(O) ; \text{div } v = 0 \}.$$ Next, we define the spaces
$$\tilde{H}^1_0(O) := \{ v \in H^1_0(O) ; \int_O v \, dx = 0 \}, \quad \tilde{H}^1(O) := H^1_0(O) \cap \tilde{H}^1(O).$$ On the spaces $H^1_0(O)$ and $\hat{H}^1_0(O)$ we will use the norm
$$\|\nabla v\|_{2,O} := \left\{ \int_O \nabla v \cdot \nabla v \right\}^{\frac{1}{2}}$$
where for \( u, v \in H^1(\Omega) \),

\[
\nabla u \cdot \nabla v := \sum_{i=1}^{m} \nabla u^i \cdot \nabla v^i,
\]

the product between \( \nabla u^i \) and \( \nabla v^i \) being the usual scalar product in \( \mathbb{R}^k \).

We now define some functional spaces well suited for the study of the Stokes problem \((1)\) in cylinders \( \Omega_\ell \) becoming unbounded.

The most relevant situation is in dimension 3 and is described by the figure below

![Diagram](image)

We are mainly interested in two situations: first when the applied forces are constant along the axis of the cylinder and then when they are periodic in the direction of this axis.

In the case of forces independent of the coordinate along the axis, the following space will be used:

\[
\mathbb{V}(\omega) := \bar{H}_0^1(\omega) \times \hat{H}_0^1(\omega) = \{ v = (v^1, v') ; v^1 \in \bar{H}_0^1(\omega), v' \in \hat{H}_0^1(\omega) \}.
\]

For the periodic case, we will need:

\[
\mathbb{H}_{\text{per}}^1(Q) := \{ v \in \mathbb{H}^1(Q) ; v = 0 \text{ on } (0, 1) \times \partial \omega \text{ and } v(0, \cdot) = v(1, \cdot) \},
\]

\[
\hat{\mathbb{H}}_{\text{per}}^1(Q) := \{ v \in \mathbb{H}^1_{\text{per}}(Q) ; \text{div } v = 0 \text{ in } Q \}
\]
and

\[
\mathbb{V}_{\text{per}}(Q) := \{ v \in \hat{\mathbb{H}}_{\text{per}}^1(Q) ; v^1 \in H^1(Q) \},
\]
where

\[
Q := (0, 1) \times \omega.
\]
2 Statements of the main results

The setting we have chosen is not the most general for which the methods described in Section 3 work, but it is more intuitive. More general cases are treated in the remarks following the proofs.

Let \( \omega \) be a bounded Lipschitz domain in \( \mathbb{R}^{n-1} \), \( n \geq 2 \) and \( f \in L^2_{\text{loc}}(\mathbb{R} \times \omega) \). Throughout the paper, \( \mu \) is a positive constant.

Then for all \( \ell > 0 \), there exists a unique solution \((u_\ell, p_\ell)\) to the problem

\[
\begin{cases}
(u_\ell, p_\ell) \in \mathbb{H}^1_0(\Omega_\ell) \times \hat{L}^2(\Omega_\ell), \\
\mu \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx - \int_{\Omega_\ell} p_\ell \text{div} v \, dx = \int_{\Omega_\ell} f \cdot v \, dx \quad \text{for all} \quad v \in \mathbb{H}^1_0(\Omega_\ell).
\end{cases}
\]

(4)

The solution \((u_\ell, p_\ell)\) of (4) is called the weak solution to the Stokes problem

\[
\begin{cases}
-\mu \Delta u_\ell + \nabla p_\ell = f \quad \text{in} \quad \Omega_\ell \\
\text{div} u_\ell = 0 \quad \text{in} \quad \Omega_\ell \\
u_\ell = 0 \quad \text{on} \quad \partial \Omega_\ell
\end{cases}
\]

(5)

The Stokes problem (5) describes the stationary motion of an incompressible fluid of viscosity \( \mu \) in \( \Omega_\ell \) under the action of an external force \( f \). The velocity is assumed to vanish on the border of the cylinder \( \Omega_\ell \). We refer the reader to [11], [17], [18] for further details.

First we consider the case where the applied force \( f \) is independent of the variable \( x_1 \):

**Theorem 1** Assume that \( f = f(x') \) and \( f \in (L^2(\omega))^n \). Then, for some positive constants \( \alpha, \beta \) depending only on \( \omega \), the solution \((u_\ell, p_\ell)\) to the problem (4) satisfies

\[
\|\nabla (u_\ell - u_\infty)\|_{2,\Omega_\ell/2} + \|p_\ell - p_\infty\|_{2,\Omega_\ell/2} \leq \alpha e^{-\beta \ell} \|f\|_{2, \omega}
\]

(6)

as \( \ell \) goes to \( +\infty \), where \( u_\infty \) is the solution to the variational equation

\[
\begin{cases}
u_\infty \in \mathbb{V}(\omega) \\
\mu \int_\omega \nabla' u_\infty \cdot \nabla' v \, dx = \int_\omega f \cdot v \, dx \quad \text{for all} \quad v \in \mathbb{V}(\omega)
\end{cases}
\]

(7)

and \( p_\infty \in \hat{L}^2_{\text{loc}}(\mathbb{R} \times \omega) \) satisfies

\[-\mu \Delta u_\infty + \nabla p_\infty = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times \omega).\]

(8)
Remark 2 1. In the inequality (6) and the equation (8) the function \( u_\infty \) is understood as the extension of the solution to the problem (7) which is constant in the direction \( e_1 \), i.e., \( u_\infty(x_1, x') = u_\infty(x') \) on \( \mathbb{R} \times \omega \).

2. The existence and uniqueness of the solution to the problem (7) follow easily from the Lax-Milgram Theorem. The existence of \( p_\infty \in \tilde{L}^2_{\text{loc}}(\mathbb{R} \times \omega) \) satisfying (8) is part of the theorem.

3. The condition \( \int_\omega u_\infty^1 \, dx' = 0 \) (i.e., \( u_\infty^1 \in \tilde{H}^1_0(\omega) \)) is necessary, since \( u_\infty^1(x_1, \cdot) \in \tilde{H}^1_0(\omega) \) for \( L^1 \)-a.e. \( x_1 \in (-\ell, \ell) \) (see Lemma 8). As noticed by Kosugi [13], one can also see from a counterexample that one cannot replace the space \( \mathcal{V}(\omega) \) by the space \( H^1_0(\omega) \times \tilde{H}^1_0(\omega) \) in problem (7). Indeed, when \( f = (1, 0, \ldots, 0) \), the solutions to problems (4) are in this case \( u_\ell = 0 \) and \( p_\ell = x_1 \) for all \( \ell > 0 \), while the equation satisfied by \( u_\infty^1 \) would be (if \( \mathcal{V}(\omega) \) is replaced by \( H^1_0(\omega) \times \tilde{H}^1_0(\omega) \))
\(- \Delta u_\infty^1 = 1 \) in \( H^{-1}(\omega) \) (hence \( u_\infty^1 \neq 0 \)). Moreover, under these assumptions we get \( p_\infty = 0 \).

An obvious consequence of Theorem 1 is the following

Corollary 3 Let \( \ell_0 > 0 \) be fixed and assume that \( f = f(x') \) and \( f \in (L^2(\omega))^n \). Then there exist two positive constants \( \alpha, \alpha', \) depending only on \( \omega \), such that the solution \((u_\ell, p_\ell)\) to the problem (4) satisfies the inequality
\[ ||\nabla(u_\ell - u_\infty)||_{2,\Omega_{\ell_0}} + ||p_\ell - p_\infty||_{2,\Omega_{\ell_0}} \leq \alpha e^{-\alpha'\ell}||f||_{2,\omega} \]
as \( \ell \) goes to \( +\infty \), where \( u_\infty \) and \( p_\infty \) are the solutions to the equations (7) and respectively (8).

The Corollary 3 states that the pairs \((u_\ell, p_\ell)\) converge (in the \( H^1_0 \times \tilde{L}^2 \)-norm) to the pair \((u_\infty, p_\infty)\) on any fixed finite cylinder included in \( \mathbb{R} \times \omega \) and the convergence is exponential. Moreover, the rate of convergence is independent of the length of the fixed cylinder.

Before considering the periodic case, let us make some observations on the limit \((u_\infty, p_\infty)\) defined in the statement of Theorem 1. We first remark that the equation (7) is equivalent to the following system of equations in the unknowns \( u'_\infty \) and \( u_\infty^1 \) (recall that \( u_\infty = (u_\infty^1, u'_\infty) \)):

\[
\left\{
\begin{aligned}
\mu \int_\omega \nabla u_\infty^1 \cdot \nabla v' \, dx &= \int_\omega f' \cdot v' \, dx \quad \text{for all } v' \in H^1_0(\omega) \\
\mu \int_\omega \nabla u_\infty^1 \cdot \nabla w \, dx &= \int_\omega f \, w \, dx \quad \text{for all } w \in H^1_0(\omega)
\end{aligned}
\right.
\]

(9)

\[
\left\{
\begin{aligned}
\mu \int_\omega \nabla u_\infty^1 \cdot \nabla v' \, dx &= \int_\omega f' \cdot v' \, dx \quad \text{for all } v' \in H^1_0(\omega) \\
\mu \int_\omega \nabla u_\infty^1 \cdot \nabla w \, dx &= \int_\omega f \, w \, dx \quad \text{for all } w \in H^1_0(\omega)
\end{aligned}
\right.
\]

(10)
In order to derive (9) and (10), we consider particular test functions of the form \((v^1, v') = (0, v^2, \ldots, v^n)\) and \((v^1, v') = (w, 0, \ldots, 0)\) respectively. One should remark that for \(n = 2\), \(u'_\infty = 0\).

In terms of partial differential equations, (9) is equivalent to the problem

\[
\begin{aligned}
(u'_\infty, p_\omega) &\in \bar{H}^1_0(\omega) \times \hat{L}^2(\omega) \\
-\mu \Delta' u'_\infty + \nabla' p_\omega &= f' \quad \text{in} \quad H^{-1}(\omega).
\end{aligned}
\]  

(11)

The pressure \(p_\omega\) is uniquely determined in the space \(\hat{L}^2(\omega)\) by the equation (11). This follows from the fact that \(u'_\infty\) satisfies the variational equation (9) (see, e.g., [17], [18], [11] or [1, Lemma 2.7]). Note that the problem above is the \((n-1)\)-dimensional Stokes problem in the domain \(\omega\).

The equation (10) is equivalent to the problem

\[
\begin{aligned}
(u^1_\infty, k) &\in \bar{H}^1_0(\omega) \times \mathbb{R} \\
\mu \Delta' u^1_\infty + f^1 &= k \quad \text{in} \quad H^{-1}(\omega).
\end{aligned}
\]  

(12)

This is an immediate consequence of the following property: if \(f \in \mathcal{D}'(\omega)\) satisfies \(\langle f, \varphi \rangle = 0\) for all \(\varphi \in \mathcal{D}(\omega)\) with \(\int_\omega \varphi \, dx = 0\), then \(f\) is a constant.

Using the linearity of the operator \(\Delta'\), one can see that

\[ u^1_\infty = w^1 - k\bar{u}, \]

where \(w^1, \bar{u} \in H^1_0(\omega)\) are the solutions to

\[
\begin{aligned}
-\mu \Delta' w^1 &= f^1 \quad \text{in} \quad H^{-1}(\omega) \\
-\mu \Delta' \bar{u} &= 1 \quad \text{in} \quad H^{-1}(\omega).
\end{aligned}
\]

From the constraint \(\int_\omega u^1_\infty \, dx = 0\), one derive the value of \(k\):

\[ k = \frac{\int_\omega w^1 \, dx}{\int_\omega \bar{u} \, dx}. \]

Note that, by the weak maximum principle, \(\bar{u} \geq 0 \ \mathcal{L}^{n-1}\)-a.e. in \(\omega\), hence \(\int_\omega \bar{u} \, dx > 0\) (since \(\bar{u} \neq 0\), which is obvious from the equation \(-\mu \Delta' \bar{u} = 1\)).

We can now express \(p_\infty\) in terms of the pressure \(p_\omega\), which is the pressure of the solution to the \((n-1)\)-dimensional Stokes problem (11). Indeed, one has

\[ p_\infty(x_1, x') = p_\omega(x') + kx_1. \]  

(13)

It is trivial to check that \(p_\infty\) defined by the above formula satisfies the equation (8).
Remark 4 1. In (13), \( p_\infty \) is to be understood as the class (in the space \( \hat{L}^2_{\text{loc}}(\mathbb{R} \times \omega) \)) of the function \( (x_1, x') \in \mathbb{R} \times \omega \mapsto p_\omega(x') + kx_1 \).

2. If \( f^1 = 0 \), then \( u^1 = 0 \) which implies \( k = 0 \) and \( u^1_\infty = 0 \). Consequently, if in addition to the hypotheses of Theorem 1, we assume that the applied forces are orthogonal to the axis of the cylinder (i.e. \( f^1 = 0 \)), then the solution to the \( n \)-dimensional Stokes problem (5) converges to the solution of the \( (n - 1) \)-dimensional Stokes problem (11).

We now consider the case where the force is periodic in the direction \( e_1 \).

Theorem 5 Assume that \( f(x) = f(x + e_1) \) for \( \mathcal{L}^n \)-a.e. \( x \in \mathbb{R} \times \omega \) and \( f \in L^2(Q) \). Then, for some positive constants \( \alpha, a \) depending only on \( \omega \), the solution \( (u_\ell, p_\ell) \) to the problem (4) satisfies
\[
\| \nabla (u_\ell - u_\infty) \|_{2,\Omega_{\ell/2}} + \| p_\ell - p_\infty \|_{2,\Omega_{\ell/2}} \leq \alpha e^{-a\ell} \| f \|_{2,Q}
\] (14)
as \( \ell \) goes to \( +\infty \), where \( u_\infty \) is the solution to
\[
\begin{aligned}
u_\infty &\in V_{\text{per}}(Q) \\
\mu \int_Q \nabla u_\infty \cdot \nabla v \, dx = \int_Q f \cdot v \, dx &\quad \text{for all } v \in V_{\text{per}}(Q)
\end{aligned}
\] (15)
and \( p_\infty \in \hat{L}^2_{\text{loc}}(\mathbb{R} \times \omega) \) satisfies
\[
-\mu \Delta u_\infty + \nabla p_\infty = f \quad \text{in } D'(\mathbb{R} \times \omega).
\] (16)

Remark 6 1. In the inequality (14) and the equation (16) the function \( u_\infty \) is of course the periodic extension in the direction \( e_1 \) of the solution to the problem (15), i.e. \( u_\infty(x) = u_\infty(x + e_1) \) a.e. on \( \mathbb{R} \times \omega \).

2. The existence and uniqueness of the solution to the problem (15) follow easily from the Lax-Milgram Theorem. The existence of \( p_\infty \in \hat{L}^2_{\text{loc}}(\mathbb{R} \times \omega) \) satisfying (16) is part of the theorem.

3. The condition \( \int_Q u^1_\infty \, dx = 0 \) (which is one of the properties of the elements of \( V_{\text{per}}(Q) \)) is equivalent to \( \int_\omega u^1_\infty(x_1, x') \, dx' = 0 \) for \( \mathcal{L}^1 \)-a.e. \( x_1 \in (0, 1) \), since the last integral is constant on \( (0, 1) \) (in order to prove it, one can argue as in the proof of Proposition 8). In other words, the flux of the solution \( u_\infty \) vanish. One can see that this is a necessary condition, since the functions \( u^1_\ell \) satisfy the same condition (again by Proposition 8).

As for Theorem 1, we have the following obvious consequence of Theorem 5:

Corollary 7 Let \( \ell_0 > 0 \) be fixed and assume that \( f(x) = f(x + e_1) \) for \( \mathcal{L}^n \)-a.e. \( x \in \mathbb{R} \times \omega \) and \( f \in L^2(Q) \). Then there exist two positive constants \( \alpha, a, \)
depending only on \( \omega \), such that the solution \((u_\ell, p_\ell)\) to the problem (4) satisfies
\[
\| \nabla (u_\ell - u_\infty) \|_{2, \Omega_0} + \| p_\ell - p_\infty \|_{2, \Omega_0} \leq \alpha e^{-a\ell} \| f \|_{2, Q}
\]
as \( \ell \) goes to \(+\infty\), where \( u_\infty \) and \( p_\infty \) are the solutions to the equations (15) and respectively (16).

3 Proofs

To a large extent, the proofs of theorems 1 and 5 are very similar. In this section we will give a detailed proof of Theorem 1 and we will only point out the differences occurring in the proof of Theorem 5.

First, let us prove a very useful property of vector fields in \( \hat{H}_0^1(O) \).

**Proposition 8** Let \( O \subset \mathbb{R}^n \) be a bounded open set and \( u \in \hat{H}_0^1(O) \). Then for \( \mathcal{L}^1 \)-a.e. \( x \in \mathbb{R} \),
\[
\int_{\mathbb{R}^{n-1}} u^1(x_1, x') \, dx' = 0
\]
(17)
(u is extended by 0 outside \( O \)).

**PROOF.** Let \( A > 0 \) such that \( O \subset [-A, A]^n \). We still have \( u \in \hat{H}_0^1((-A, A)^n) \). For all \( x_1 \in (-A, A) \), let \( A_{x_1} := (-A, x_1) \times (-A, A)^{n-1} \). Then, by the divergence formula, we have that
\[
0 = \int_{A_{x_1}} \text{div} \, u \, dx = \int_{\partial A_{x_1}} u \cdot \nu \, d\sigma = \int_{(-A, A)^{n-1}} u^1(x_1, x') \, dx' = \int_{\mathbb{R}^{n-1}} u^1(x_1, x') \, dx',
\]
since \( u = 0 \) on \( \partial A_{x_1} \setminus \{x_1\} \times (-A, A)^{n-1} \). In the above formula the function \( u^1(x_1, \cdot) \) in the last two integrals is the trace of \( u^1 \) on the section \( \{x_1\} \times (-A, A)^{n-1} \). However, for \( \mathcal{L}^1 \)-a.e. \( x_1 \in (-A, A) \), this trace is \( \mathcal{L}^{n-1} \)-a.e. equal to the restriction of \( u^1 \) to \( \{x_1\} \times (-A, A)^{n-1} \) and the result follows.

The statement is obvious for \( x_1 \notin (-A, A) \).

In our proofs, we first estimate the velocity \( u_\ell - u_\infty \), then we find the estimate for the pressure \( p_\ell - p_\infty \) from the velocity estimate and the equation satisfied by \( u_\ell - u_\infty \) and \( p_\ell - p_\infty \). The following lemma will be useful for the estimate of the pressure.
Lemma 9  Let $\ell \geq 1$ and $g \in L^2(\Omega_\ell)$ such that $\int_{\Omega_\ell} g \, dx = 0$. Then there exists $u \in \mathbb{H}_0^1(\Omega_\ell)$ such that

$$\begin{cases}
\text{div } u = g \quad \text{in } \Omega_\ell \\
\|\nabla u\|_{2,\Omega_\ell} \leq C\ell\|g\|_{2,\Omega_\ell},
\end{cases}$$

where $C$ is a constant depending only on $\omega$.

PROOF. We use a scaling argument. First, we recall the following result (see, e.g., [11], [3], [1]):

Let $\tilde{Q} := (-1,1) \times \omega$. For any $\tilde{g} \in L^2(\tilde{Q})$ satisfying $\int_{\tilde{Q}} \tilde{g} \, dx = 0$, there exists a vector field $\tilde{u} \in \mathbb{H}_0^1(\tilde{Q})$ such that

$$\begin{align}
\text{div } \tilde{u} &= \tilde{g} \quad \text{in } \tilde{Q}, \\
\|\nabla \tilde{u}\|_{2,\tilde{Q}} &\leq C\|\tilde{g}\|_{2,\tilde{Q}},
\end{align}$$

where the constant $C$ depends only on $\tilde{Q}$ (hence on $\omega$).

Second, we remark that $(u,g) \in \mathbb{H}_0^1(\Omega_\ell) \times L^2(\Omega_\ell)$ and satisfies

$$\int_{\Omega_\ell} g \, dx = 0 \quad \text{and } \text{div } u = g \quad \text{in } \Omega_\ell$$

if and only if $(\tilde{u},\tilde{g}) \in \mathbb{H}_0^1(\tilde{Q}) \times L^2(\tilde{Q})$ and satisfies

$$\int_{\tilde{Q}} \tilde{g} \, dx = 0 \quad \text{and } \text{div } \tilde{u} = \tilde{g} \quad \text{in } \tilde{Q},$$

where

$$\tilde{u}(x_1,x') = \left(\frac{1}{\ell} u^1(\ell x_1,x'), u'(\ell x_1,x') \right) \quad \text{and } \tilde{g}(x_1,x') = g(\ell x_1,x')$$  \hspace{1cm} (20)

We have

$$\int_{\tilde{Q}} |\tilde{g}|^2 \, dx = \int_{\tilde{Q}} |g(\ell x_1,x')|^2 \, dx = \frac{1}{\ell} \int_{\Omega_\ell} |g|^2 \, dx.$$

Hence

$$\|\tilde{g}\|_{2,\tilde{Q}}^2 = \frac{1}{\ell} \|g\|_{2,\Omega_\ell}^2$$  \hspace{1cm} (21)

Similarly, we obtain
From the inequalities above, we see that
\[ \| \nabla \tilde{u} \|_{2,\tilde{Q}}^2 \geq \frac{1}{\ell} \| \nabla u \|_{2,\Omega}^2, \]
since \( \ell \geq 1 \).

Then we construct \( u \) in the following way: for \( g \in L^2(\Omega_\ell) \) with average 0, we construct \( \tilde{g} \in L^2(\tilde{Q}) \) as in (20), then for this \( \tilde{g} \) we find \( \tilde{u} \in H_0^1(\tilde{Q}) \) satisfying (18) and (19), and we retrieve \( u \) from \( \tilde{u} \) as in (20).

We obtain
\[ \frac{1}{\ell^3} \| \nabla u \|_{2,\Omega}^2 \leq \| \nabla \tilde{u} \|_{2,\tilde{Q}}^2 \leq C^2 \| \tilde{g} \|_{2,\tilde{Q}}^2 = \frac{C^2}{\ell} \| g \|_{2,\Omega}^2, \]
which ends the proof.

Proof of Theorem 1. We cast the proof into seven steps.

(i) For all \( \ell > 0 \), one has that
\[ \mu \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_\ell} f \cdot v \, dx \quad \text{for all} \quad v \in \hat{H}_0^1(\Omega_\ell). \] (22)

Let \( v \in \hat{H}_0^1(\Omega_\ell) \). Since \( u_\infty \) is independent of \( x_1 \) we have \( \nabla u_\infty \cdot \nabla v = \nabla' u_\infty \cdot \nabla' v \) and thus
\[ \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_\ell} \nabla' u_\infty \cdot \nabla' v \, dx = \int_{\omega} \int_{-\ell}^{\ell} \nabla' u_\infty \cdot \nabla' v \, dx_1 \, dx' \]
\[ = \int_{\omega} \nabla' u_\infty(x') \cdot \left( \int_{-\ell}^{\ell} \nabla' v(x_1, x') \, dx_1 \right) \, dx' \] (23)

Let us consider the following vector field on \( \omega \):
\[ \tilde{u}(x') := \int_{-\ell}^{\ell} v(x_1, x') \, dx_1. \]
Then \( \tilde{v} \in \mathbb{V}(\omega) \). Indeed, if \((v_k)_{k \in \mathbb{N}} \subset (D(\Omega_\ell))^n\) is an approximating sequence of \( v \) in \( H^1_0(\Omega_\ell) \), then it is easy to check that the vector fields

\[
x' \in \omega \mapsto \tilde{v}_k(x') := \int_{-\ell}^{\ell} v_k(x_1, x') \, dx_1
\]

belong to \( (D(\omega))^n \) and converge to \( \tilde{v} \) in the \( (H^1_0(\omega))^n \)-norm. Indeed, it is enough to notice that for all \( k \in \mathbb{N} \),

\[
\nabla' \tilde{v}_k(x') = \int_{-\ell}^{\ell} \nabla' v_k(x_1, x') \, dx_1 \quad \text{for all} \ x' \in \omega
\]

and to use the Fubini Theorem and the fact that \( v_k \to v \) in \( H^1_0(\Omega_\ell) \). Moreover, making \( k \) go to \( +\infty \) in the equality above gives

\[
\nabla' \tilde{v}(x') = \int_{-\ell}^{\ell} \nabla' v(x_1, x') \, dx_1 \quad \text{for} \ L^{n-1}\text{-a.e. } x' \in \omega. \quad (24)
\]

For any \( k \in \mathbb{N} \),

\[
\text{div}' \tilde{v}_k = \int_{-\ell}^{\ell} (\text{div}' v_k)(x_1, x') \, dx_1 = \int_{-\ell}^{\ell} (\text{div} v_k)(x_1, x') \, dx_1 - \int_{-\ell}^{\ell} \partial_1 v_k^1(x_1, x') \, dx_1 = \int_{-\ell}^{\ell} (\text{div} v_k)(x_1, x') \, dx_1.
\]

Since \( v_k \to v \) in \( H^1_0(\Omega_\ell) \), we have that

\[
\int_{-\ell}^{\ell} (\text{div} v_k)(x_1, x') \, dx_1 \to \int_{-\ell}^{\ell} (\text{div} v)(x_1, x') \, dx_1 = 0 \quad \text{in } L^2(\omega).
\]

On the other hand \( \text{div}' \tilde{v}_k \to \text{div}' \tilde{v} \) in \( L^2(\omega) \), thus \( \text{div}' \tilde{v} = 0 \) in \( \omega \).

Finally,

\[
\int_{\omega} \tilde{v}^1 \, dx' = \int_{\omega} \int_{-\ell}^{\ell} v^1(x_1, x') \, dx_1 \, dx' = \int_{-\ell}^{\ell} \int_{\omega} v^1(x_1, x') \, dx' \, dx_1 = 0,
\]

since by Proposition 8, \( \int_{\omega} v^1(x_1, x') \, dx' = 0 \) for \( L^1\text{-a.e. } x_1 \in (-\ell, \ell) \).

Therefore we can use \( \tilde{v} \) as a test function in equation (7). Then, from (23) and (24) we derive

\[
\mu \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = \mu \int_{\omega} \nabla' u_\infty \cdot \left( \int_{-\ell}^{\ell} \nabla' v \, dx_1 \right) \, dx'
\]

\[
= \mu \int_{\omega} \nabla' u_\infty \cdot \nabla' \tilde{v} \, dx' = \int_{\omega} f \cdot \tilde{v} \, dx'
\]

\[
= \int_{\omega} f(x') \cdot \left( \int_{-\ell}^{\ell} v(x_1, x') \, dx_1 \right) \, dx'
\]

\[
= \int_{\Omega_\ell} f \cdot v \, dx
\]

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There exists \( p_\infty \in L^2_{\text{loc}}(\mathbb{R} \times \omega) \) satisfying (8).

The easiest way to prove this statement is to take \( p_\infty \) given by the formula (13). We however prefer to give a proof which remains valid in the periodic case, since it only uses the equation (22) without any assumption on \( f \), other than to belong to the space \( L^2_{\text{loc}}(\mathbb{R} \times \omega) \). Consequently, this step will be identical in the proof of Theorem 5.

It is enough to find a representative function for \( p_\infty \), which for simplicity will also be denoted by \( p_\infty \).

The equation (22) can be written in the form

\[
\langle \mu \Delta u_\infty + f, v \rangle = 0 \quad \text{for all} \quad v \in \hat{H}^1_0(\Omega),
\]

where the duality \( \langle \cdot, \cdot \rangle \) is the \( \langle \mathcal{H}^{-1}(\Omega), \mathcal{H}^1_0(\Omega) \rangle \)-duality. Then (see, e.g., [1, Lemma 2.7]) there exists \( p_{\infty, \ell} \in L^2(\Omega_\ell) \) satisfying \( \mu \Delta u_\infty + f = \nabla p_{\infty, \ell} \) in \( \mathcal{H}^{-1}(\Omega_\ell) \). We construct \( p_\infty \) on each \( \Omega_k (k \in \mathbb{N}^*) \) in the following way.

For \( k = 1 \) we take \( p_{\infty, 1} = p_{\infty, 1} \) on \( \Omega_1 \), where \( p_{\infty, 1} \) is obtained as above. Next, we construct \( p_\infty \) on \( \Omega_k, k \geq 2 \) by induction.

Assume that we have constructed \( p_\infty \) on \( \Omega_{k-1} \). Using the argument described above, there exists a \( p_{\infty, k} \in L^2(\Omega_k) \) satisfying \( \mu \Delta u_\infty + f = \nabla p_{\infty, k} \) in \( \Omega_k \). In particular \( \nabla (p_{\infty, k} - p_\infty) = 0 \) in \( \Omega_{k-1} \) and since \( \Omega_{k-1} \) is connected, there exists a constant \( c_k \) such that

\[
p_{\infty, k} - p_\infty = c_k \quad \text{on} \quad \Omega_{k-1}.
\]

Then we define \( p_\infty \) on \( \Omega_k \setminus \Omega_{k-1} \) by taking \( p_\infty := p_{\infty, k} - c_k \). Combining this last relation with (25) we obtain in fact that \( p_\infty = p_{\infty, k} - c_k \) on the whole set \( \Omega_k \), which obviously imply that \( p_\infty \) also satisfies \( \mu \Delta u_\infty + f = \nabla p_\infty \) in \( \Omega_k \).

(iii) There exists a positive constant \( C \) depending only on \( \omega \) such that

\[
\int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq \frac{C}{1 + C} \int_{\Omega_{\ell_1+1}} |\nabla (u_\ell - u_\infty)|^2 \, dx,
\]

for all \( \ell_1 \leq \ell - 1 \).

From equation (4) we deduce

\[
\mu \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} f \cdot v \, dx \quad \text{for all} \quad v \in \hat{H}^1_0(\Omega_\ell).
\]
By substracting (22), we obtain (since $\mu \neq 0$):
\[
\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx = 0 \quad \text{for all } v \in \hat{H}^1_0(\Omega_\ell).
\] (27)

Denote then by $\rho$ the function defined on the real line whose graph is depicted below ($\rho = 1$ on $[-\ell_1, \ell_1]$, $\rho = 0$ on $\mathbb{R} \setminus (-\ell_1 - 1, \ell_1 + 1)$ and $\rho(x_1) = \ell_1 + 1 - |x_1|$ on $(-\ell_1 - 1, -\ell_1) \cup (\ell_1, \ell_1 + 1)$).

\[
\begin{array}{c}
\hline
& 1 \\
\hline
-\ell_1 & \ell_1 \\
\hline
\end{array}
\]

Since $\ell_1 \leq \ell - 1$, one has clearly $\rho(x_1)(u_\ell - u_\infty) \in \hat{H}^1_0(\Omega_\ell)$. Moreover, since $\text{div} (u_\ell - u_\infty) = 0$ in $\Omega_\ell$, we have
\[
\text{div} \left( \rho(u_\ell - u_\infty) \right) = (\partial_1 \rho)(u^1_\ell - u^1_\infty) \quad \text{in } \Omega_\ell.
\]

By a classical result that we already mentioned (see, e.g., [11], [3], [1]) there exists a vector field $\beta$ such that
\[
\beta \in \hat{H}^1_0(D_{\ell_1}), \quad \text{where } D_{\ell_1} := \Omega_{\ell_1 + 1} \setminus \Omega_{\ell_1}
\]
\[
\text{div} \beta = (\partial_1 \rho)(u^1_\ell - u^1_\infty) \quad \text{in } D_{\ell_1}
\]
\[
\|\nabla \beta\|_{2,D_{\ell_1}} \leq C\|u^1_\ell - u^1_\infty\|_{2,D_{\ell_1}}
\] (28) (29)

for some constant $C$ depending only on $\omega$, hence independent of $\ell$ and $\ell_1$. Indeed, one can remark that we have the required compatibility condition, that is to say
\[
\int_{(\ell_1, \ell_1 + 1) \times \omega} (\partial_1 \rho)(u^1_\ell - u^1_\infty) \, dx = \int_{(\ell_1, \ell_1 + 1) \times \omega} -(u^1_\ell - u^1_\infty) \, dx
\]
\[
= \int_{\ell_1}^{\ell_1 + 1} \int_\omega u^1_\infty \, dx' \, dx_1 - \int_{\ell_1}^{\ell_1 + 1} \int_\omega u^1_\ell \, dx' \, dx_1
\]
\[
= 0,
\]
the last equality being a consequence of Proposition 8 and of the fact that $\int_\omega u^1_\infty \, dx' = 0$ (since $u^1_\infty \in \hat{H}^1_0(\omega)$).
Since the divergence is invariant by translation, one can argue on \((0, 1) \times \omega\) and construct the desired field \(\beta\) separately on the two connected components of \(D_{\ell_1} \cap (\ell_1, \ell_1 + 1) \times \omega\) and \((-\ell_1 - 1, -\ell_1) \times \omega\).

We extend \(\beta\) by 0 outside \(D_{\ell_1}\) and we remark that we still have \(\beta \in H^1_0(\Omega_{\ell})\) and \(\text{div} \, \beta = (\partial_1 \rho)(u_\ell - u_\infty)\) in \(\Omega_{\ell}\). Therefore

\[ \rho(u_\ell - u_\infty) - \beta \in H^1_0(\Omega_{\ell}) \]

and from (27) we get

\[ \int_{\Omega_{\ell}} \nabla(u_\ell - u_\infty) \cdot \nabla(\rho(u_\ell - u_\infty) - \beta) \, dx = 0. \]

This implies

\[
\int_{\Omega_{\ell}} \rho|\nabla(u_\ell - u_\infty)|^2 \, dx = -\int_{D_{\ell_1}} \partial_1(u_\ell - u_\infty) \cdot (\partial_1 \rho)(u_\ell - u_\infty) \, dx \\
+ \int_{D_{\ell_1}} \nabla(u_\ell - u_\infty) \cdot \nabla \beta \, dx \\
\leq \int_{D_{\ell_1}} |\nabla(u_\ell - u_\infty)||u_\ell - u_\infty| \, dx + \int_{D_{\ell_1}} |\nabla(u_\ell - u_\infty)||\nabla \beta| \, dx.
\]

Using the Young inequality in the last two integrals we obtain

\[
\int_{\Omega_{\ell}} \rho|\nabla(u_\ell - u_\infty)|^2 \, dx \leq \int_{D_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \, dx + \frac{1}{2} \int_{D_{\ell_1}} |u_\ell - u_\infty|^2 \, dx + \frac{1}{2} \int_{D_{\ell_1}} |\nabla \beta|^2 \, dx.
\]

By (29) we derive for some constant \(C\) (depending only on \(\omega\)) that

\[
\int_{\Omega_{\ell}} \rho|\nabla(u_\ell - u_\infty)|^2 \, dx \leq \int_{D_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \, dx + C \int_{D_{\ell_1}} |u_\ell - u_\infty|^2 \, dx \tag{30}
\]

Since \(u_\ell - u_\infty\) vanishes on the lateral boundary of the cylinder \(\Omega_{\ell}\), we have a Poincaré inequality (see, e.g., [4]) of the type

\[
\int_{D_{\ell_1}} |u_\ell - u_\infty|^2 \, dx \leq C \int_{D_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \tag{31}
\]

where \(C\) depends only on \(\omega\) and is independent of \(\ell\) and \(\ell_1\). Thus, from (30), we deduce that for some constant depending only on \(\omega\), we have

\[
\int_{\Omega_{\ell}} \rho|\nabla(u_\ell - u_\infty)|^2 \, dx \leq C \int_{D_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \, dx
\]

Since \(\rho\) is nonnegative, this leads to

\[
\int_{\Omega_{\ell_1 + 1}} |\nabla(u_\ell - u_\infty)|^2 \, dx = \int_{\Omega_{\ell_1}} \rho|\nabla(u_\ell - u_\infty)|^2 \, dx \leq \int_{\Omega_{\ell}} \rho|\nabla(u_\ell - u_\infty)|^2 \, dx \\
\leq C \left\{ \int_{\Omega_{\ell_1 + 1}} |\nabla(u_\ell - u_\infty)|^2 \, dx - \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \, dx \right\}
\]

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which finally implies (26).

(iv) We have the inequality

\[ \int_{\Omega_{\ell/2}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq \gamma e^{-\frac{\ln \ell}{2}} \int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 \, dx, \quad (32) \]

where \( \gamma := \frac{C + 1}{C} \) and \( C \) is the constant appearing in the inequality (26).

The inequality (26) is valid for any \( \ell_1 \leq \ell - 1 \). Applying it \( \left\lfloor \frac{\ell}{2} \right\rfloor \) times starting from \( \frac{\ell}{2} \) (where \( \left\lfloor \frac{\ell}{2} \right\rfloor \) denotes the integer part of \( \frac{\ell}{2} \)) we obtain

\[ \int_{\Omega_{\ell/2}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq \left( \frac{C}{1 + C} \right)^{\frac{1}{2}} \int_{\Omega_{\ell/2+\left\lfloor \frac{\ell}{2} \right\rfloor}} |\nabla (u_\ell - u_\infty)|^2 \, dx. \]

Noting that \( \frac{\ell}{2} - 1 \leq \left\lfloor \frac{\ell}{2} \right\rfloor \leq \frac{\ell}{2} \) and \( \frac{C}{C + 1} < 1 \), we have \( \Omega_{\ell/2+\left\lfloor \frac{\ell}{2} \right\rfloor} \subset \Omega_\ell \) and consequently,

\[ \int_{\Omega_{\ell/2}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq \left( \frac{C}{1 + C} \right)^{\frac{1}{2}-1} \int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 \, dx. \]

and the inequality (32) follows.

(v) For any \( \ell > 0 \),

\[ \int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq C\ell \| f \|_{2,\omega}^2, \quad (33) \]

where \( C \) is a constant depending only on \( \omega \).

Taking \( v = u_\ell \) in (4) we have

\[ \mu \int_{\Omega_\ell} |\nabla u_\ell|^2 \, dx = \int_{\Omega_\ell} f \cdot u_\ell \, dx \leq \| f \|_{2,\Omega_\ell} \| u_\ell \|_{2,\Omega_\ell} \]

\[ \leq C \| f \|_{2,\Omega_\ell} \| \nabla u_\ell \|_{2,\Omega_\ell} \]

for some constant \( C \) depending only on \( \omega \) (we used the same Poincaré type inequality than in (31)). Thus

\[ \| \nabla u_\ell \|_{2,\Omega_\ell} \leq \frac{C}{\mu} \| f \|_{2,\Omega_\ell}. \]
Since $f$ is independent of $x_1$ we derive easily
\[ \|\nabla u_{\ell}\|_{2,\Omega_{\ell}}^2 \leq \left( \frac{C}{\mu} \right)^2 \|f\|_{2,\Omega_{\ell}}^2 = \left( \frac{C}{\mu} \right)^2 \ell \int_{\ell}^{\ell} \int_{\omega} |f|^2 \, dx \, dx_1 = 2 \left( \frac{C}{\mu} \right)^2 \ell \|f\|_{2,\omega}^2. \tag{34} \]

Similarly, taking $v = u_{\infty}$ in the equation (7) we get
\[ \|\nabla' u_{\infty}\|_{2,\omega} \leq \frac{C}{\mu} \|f\|_{2,\omega}. \]

Taking the square and integrating in the variable $x_1$ we derive
\[ \|\nabla u_{\infty}\|_{2,\Omega_{\ell}}^2 = \|\nabla' u_{\infty}\|_{2,\Omega_{\ell}}^2 \leq 2 \left( \frac{C}{\mu} \right)^2 \ell \|f\|_{2,\omega}^2. \tag{35} \]

Combining (34) and (35) we get the inequality (33).

(vi) Estimate for the pressure: there exists a constant depending only on $\omega$ such that
\[ \|p_{\ell} - p_{\infty}\|_{2,\Omega_{\ell}/2} \leq C\ell \|\nabla(u_{\ell} - u_{\infty})\|_{2,\Omega_{\ell}/2} \tag{36} \]
for $\ell$ large enough.

By substracting the equations (5) and (8) we obtain in $H^{-1}(\Omega_{\ell})$:
\[ -\nabla (p_{\ell} - p_{\infty}) = -\mu \Delta(u_{\ell} - u_{\infty}). \]

This is equivalent to
\[ \int_{\Omega_{\ell}} (p_{\ell} - p_{\infty}) \text{div} v \, dx = \mu \int_{\Omega_{\ell}} \nabla(u_{\ell} - u_{\infty}) \cdot \nabla v \, dx \quad \text{for all} \quad v \in H^1_0(\Omega_{\ell}). \tag{37} \]

For $p_{\ell} - p_{\infty}$ belonging to $L^2(\Omega_{\ell})$, we choose the representative, for the simplicity also denoted by $p_{\ell} - p_{\infty}$, which satisfies
\[ \int_{\Omega_{\ell}/2} (p_{\ell} - p_{\infty}) \, dx = 0. \]

Then, by Lemma 9, there exists $v \in H^1_0(\Omega_{\ell}/2)$ satisfying
\[
\begin{cases} 
\text{div} v = p_{\ell} - p_{\infty} \quad \text{in} \quad \Omega_{\ell/2} \\
\|\nabla v\|_{2,\Omega_{\ell}/2} \leq C\ell \|p_{\ell} - p_{\infty}\|_{2,\Omega_{\ell}/2}
\end{cases}
\]
for some constant $C$ depending only on $\omega$ (and independent of $\ell$).
Extending $v$ by 0 outside $\Omega_{\ell/2}$ and using it as a test function in (37), we get

$$\int_{\Omega_{\ell/2}} (p_\ell - p_\infty)^2 \, dx = \int_{\Omega_\ell} (p_\ell - p_\infty) \, \text{div} \, v \, dx = \mu \int_{\Omega_\ell} \nabla (u_\ell - u_\infty) \cdot \nabla v \, dx$$

$$= \mu \int_{\Omega_{\ell/2}} \nabla (u_\ell - u_\infty) \cdot \nabla v \, dx$$

This leads to

$$\|p_\ell - p_\infty\|_{2, \Omega_{\ell/2}}^2 \leq \mu \|\nabla (u_\ell - u_\infty)\|_{2, \Omega_{\ell/2}} \|\nabla v\|_{2, \Omega_{\ell/2}}$$

$$\leq C \ell \|\nabla (u_\ell - u_\infty)\|_{2, \Omega_{\ell/2}} \|p_\ell - p_\infty\|_{2, \Omega_{\ell/2}}.$$ 

Finally, from the definition of the $\hat{L}^2$-norm (see also (3)), we have that

$$\|p_\ell - p_\infty\|_{2, \Omega_{\ell/2}} = \|p_\ell - p_\infty\|_{2, \Omega_{\ell/2}} \leq C \ell \|\nabla (u_\ell - u_\infty)\|_{2, \Omega_{\ell/2}}.$$ 

(vii) **Conclusion of the proof.** Combining the inequalities (32) and (33) we obtain

$$\|\nabla (u_\ell - u_\infty)\|_{2, \Omega_{\ell/2}} \leq C \ell^{1/2} e^{-\frac{\ln \gamma}{4} \ell} \|f\|_{2, \omega},$$

where $C$ is a constant depending only on $\omega$. From (38) and (36) we obtain

$$\|p_\ell - p_\infty\|_{2, \Omega_{\ell/2}} \leq C \ell^{3/2} e^{-\frac{\ln \gamma}{4} \ell} \|f\|_{2, \omega}.$$ 

Then by adding the last two inequalities we obtain the desired inequality (6) with a constant $a$ that can be any positive real number smaller than $\frac{\ln \gamma}{4}$.

\[\square\]

**Remark 10** 1. One can consider more general domains than $\Omega_\ell$. More specifically, one can consider the equation (5) on domains $\Omega_\ell'$, where $\Omega_\ell'$ are bounded Lipschitz domains satisfying for instance

$$\Omega_\ell \subset \Omega_\ell' \subset \Omega_{\ell \eta},$$

for some $\eta \geq 1$. The proof is identical in all points except for a variant in step (v), where for the estimate of $\|\nabla u_\ell\|_{2, \Omega_\ell}$ we have

$$\|\nabla u_\ell\|_{2, \Omega_\ell}^2 \leq \|\nabla u_\ell\|_{2, \Omega_\ell'}^2 \leq \left( \frac{C}{\mu} \right)^2 \|f\|_{2, \Omega_\ell'}^2$$

$$\leq \left( \frac{C}{\mu} \right)^2 \|f\|_{2, \Omega_\ell}^2 = \left( \frac{C}{\mu} \right)^2 \int_{\ell}^{\ell'} \int_{\omega} \|f\|^2 \, dx \, dx_1$$

$$= 2 \left( \frac{C}{\mu} \right)^2 \ell^n \|f\|_{2, \omega}^2.$$
Hence, instead of (33), we will obtain

\[ \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \, dx \leq C\ell^\gamma \|f\|_{2,\omega}^2, \]

for some constant \(C\) depending only on \(\omega\).

2. One can also consider weaker regularity assumptions for \(f\). More precisely, one can take \(f\) to be a functional of the following form:

\[ \langle f, v \rangle := \langle f_\omega, \int_{-\ell}^\ell v(x_1, \cdot) \, dx_1 \rangle \text{ for all } v \in \hat{H}^1_0(\Omega_\ell), \quad (40) \]

where \(f_\omega\) is a distribution in \(\mathbb{H}^{-1}(\omega)\). Following the arguments in the first step of the proof, one can easily check that the functional \(f\) defined above belongs to \(\mathbb{H}^{-1}(\Omega_\ell)\) for all \(\ell > 0\). In fact, the formula (40) defines a distribution on \(\mathbb{R} \times \omega\) independent of the variable \(x_1\) whose “restriction” to the \((n-1)\)-dimensional set \(\omega\) is \(f_\omega\).

**Proof of Theorem 5.** The proof follows the same steps as in the proof of Theorem 1. We will only describe the parts of the proof different from the previous one.

(i) For all \(\ell > 0\), one has that

\[ \mu \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_\ell} f \cdot v \, dx, \text{ for all } v \in \hat{H}^1_0(\Omega_\ell). \]

Note first that, since \(u_\infty \in \mathbb{H}^1(Q)\), \(u_\infty = 0\) on \((0,1) \times \partial\omega\) and \(u_\infty(0, \cdot) = u_\infty(1, \cdot)\), we have that the periodic extension of \(u_\infty\) in the direction \(e_1\), for simplicity also denoted by \(u_\infty\), belongs to \(\mathbb{H}^1(\Omega_\ell)\) for all \(\ell > 0\) and satisfies \(u_\infty = 0\) on \(\mathbb{R} \times \partial\omega\) and

\[ \nabla u_\infty(x + ie_1) = \nabla u_\infty(x) \text{ for } \mathcal{L}^n\text{-a.e. } x \in Q \text{ and all } i \in \mathbb{Z}. \quad (41) \]

An immediate consequence of (41) is that \(\text{div } u_\infty = 0\) in \(\mathbb{R} \times \omega\).

Let \(v \in \hat{H}^1_0(\Omega_\ell)\). We extend \(v\) by 0 outside \(\Omega_\ell\). Then we have

\[ \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = \int_{\mathbb{R} \times \omega} \nabla u_\infty \cdot \nabla v \, dx = \sum_{i \in \mathbb{Z}} \int_{Q_i} \nabla u_\infty \cdot \nabla v \, dx, \quad (42) \]

where

\[ Q_i := Q + ie_1. \]
By (41) and a change of variable we have
\[
\int_{Q_i} \nabla u_{\infty} \cdot \nabla v \, dx = \int_{Q} \nabla u_{\infty}(x + ie_1) \cdot \nabla v(x + ie_1) \, dx \\
= \int_{Q} \nabla u_{\infty}(x) \cdot \nabla v(x + ie_1) \, dx.
\]
Combining this with (42) this leads to
\[
\int_{\Omega_{\ell}} \nabla u_{\infty} \cdot \nabla v \, dx = \int_{Q} \nabla u_{\infty}(x) \cdot \sum_{i \in \mathbb{Z}} \nabla v(x + ie_1) \, dx, \tag{43}
\]
Let us consider the following vector field on \(Q\):
\[
x \in Q \mapsto \tilde{v}(x) := \sum_{i \in \mathbb{Z}} v(x + ie_1).
\]
Note that since \(v = 0\) outside \(\Omega_{\ell}\), only a finite number of terms do not vanish in the above sum.
We claim that \(\tilde{v} \in \mathbb{V}_{\text{per}}(Q)\). Indeed, it is obvious that \(\tilde{v} \in H^1(Q)\), that \(\tilde{v} = 0\) on \((0,1) \times \partial \omega\) and that
\[
\nabla \tilde{v}(x) = \sum_{i \in \mathbb{Z}} \nabla v(x + ie_1), \tag{44}
\]
An immediate consequence of (44) is that \(\text{div} \, \tilde{v} = 0\) in \(Q\). We also have that
\[
\tilde{v}(0, x') = \tilde{v}(1, x') = \sum_{i \in \mathbb{Z}} v(i, x') \text{ for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \omega.
\]
Finally
\[
\int_{Q} \tilde{v}^1 \, dx = \int_{Q} \sum_{i \in \mathbb{Z}} v^1(x + ie_1) \, dx = \sum_{i \in \mathbb{Z}} \int_{Q_i} v^1 \, dx = \int_{\mathbb{R} \times \omega} v^1 \, dx \\
= \int_{\mathbb{R}} \int_{\omega} v^1 \, dx' \, dx_1 = 0
\]
the last equality being a consequence of Proposition 8.
Therefore, we can use \(\tilde{v}\) as a test function in the equation (15). Together with
and (44), this gives
\[
\mu \int_{\Omega_t} \nabla u_\infty \cdot \nabla v \, dx = \mu \int_Q \nabla u_\infty \cdot \nabla \tilde{v} \, dx = \int_Q f \cdot \tilde{v} \, dx \\
= \int_Q f(x) \cdot \sum_{i \in \mathbb{Z}} v(x + ie_1) \, dx \\
= \sum_{i \in \mathbb{Z}} \int_Q f(x + ie_1) \cdot v(x + ie_1) \, dx \\
= \sum_{i \in \mathbb{Z}} \int_{Q_i} f \cdot v \, dx = \int_{\mathbb{R} \times \omega} f \cdot v \, dx \\
= \int_{\Omega_t} f \cdot v \, dx.
\]

This completes the proof of the step (i).

The step (ii) is identical.

The step (iii) is almost identical, only the argument allowing to construct the field $\beta$ on $D_{t_1}$ is slightly different, that is to say we have now
\[
\int_{(t_1, t_1 + 1) \times \omega} (\partial_1 \rho) (u^1_\ell - u^1_\infty) \, dx = \int_{(t_1, t_1 + 1) \times \omega} -(u^1_\ell - u^1_\infty) \, dx \\
= \int_{(t_1, t_1 + 1) \times \omega} u^1_\infty \, dx - \int_{(t_1, t_1 + 1) \times \omega} u^1_\ell \, dx \\
= \int_Q u^1_\infty \, dx - \int_{t_1}^{t_1 + 1} \int_\omega u^1_\ell \, dx \, dx_1 \\
= 0,
\]
by Proposition 8, the periodicity of $u^1_\infty$ and of the fact that $u^1_\infty \in \bar{H}^1(Q)$.

The step (iv) is identical.

(v) For any $\ell \geq 1$,
\[
\int_{\Omega_t} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq C \ell \|f\|^2_{2,Q}, \tag{45}
\]
where $C$ is a constant depending only on $\omega$.

Taking $v = u_\ell$ in (4) we have
\[
\mu \int_{\Omega_t} |\nabla u_\ell|^2 \, dx = \int_{\Omega_t} f \cdot u_\ell \, dx \leq \|f\|_{2,\Omega_t} \|u_\ell\|_{2,\Omega_t} \\
\leq C \|f\|_{2,\Omega_t} \|\nabla u_\ell\|_{2,\Omega_t}
\]
for some constant $C$ depending only on $\omega$ (we used the same Poincaré type inequality as in (31)). Thus, using the periodicity of $f$,

\[
\|\nabla u_\ell\|_{2,\Omega_\ell}^2 \leq \left(\frac{C\mu}{\mu}\right)^2 \|f\|_{2,\Omega_\ell}^2 \leq \frac{1}{2}\|f\|_{2,\Omega_{[\ell]+1}}^2 \leq 2([\ell] + 1)\left(\frac{C\mu}{\mu}\right)^2 \|f\|_{2,Q}^2 \leq 4\left(\frac{C\mu}{\mu}\right)^2 \ell \|f\|_{2,Q}^2.
\]

Similarly, taking $v = u_\infty$ in the equation (15) we get

\[
\|\nabla u_\infty\|_{2,Q} \leq \frac{C\mu}{\mu} \|f\|_{2,Q}
\]

Hence,

\[
\|\nabla u_\infty\|_{2,\Omega_\ell}^2 \leq \|\nabla u_\infty\|_{2,\Omega_{[\ell]+1}}^2 = 2([\ell] + 1)\|\nabla u_\infty\|_{2,Q}^2 \leq 4\left(\frac{C\mu}{\mu}\right)^2 \ell \|f\|_{2,Q}^2.
\]

Combining (46) and (47) we get the inequality (45).

The last two steps are identical to the ones of the proof of Theorem 1.

□

In the periodic case, we also have a relation between the pressure $p_\infty$ defined on $\mathbb{R} \times \omega$ by the equation (16) and the pressure $p_Q$ defined on $Q$ by

\[
-\mu \Delta u_\infty + \nabla p_Q = f \quad \text{in} \quad H^{-1}(Q).
\]

As a consequence of Proposition 8, we have that $\hat{H}^1_0(Q) \subset V_{\text{per}}(Q)$. Consequently, there exists a unique $p_Q \in L^2(Q)$ satisfying (48) (see, e.g., [18], [1]). In fact, the pair $(u_\infty, p_Q)$ satisfies the following equation on $Q$:

\[
\mu \int_Q \nabla u_\infty \cdot \nabla v\, dx = \int_Q p_Q \text{div} \, v\, dx
\]

\[
= \int_Q f \cdot v\, dx - k \int_\omega \int v^1(0, x') \, dx' \quad \text{for all} \quad v \in \mathbb{H}^1_{\text{per}}(Q),
\]

where $k$ is a constant depending only on $u_\infty$.

In order to prove (49), we consider (for an arbitrary $v \in \mathbb{H}^1_{\text{per}}(Q)$) the vector
fields $\bar{v}$ and $\tilde{v}$ defined by
\[
\bar{v}(x_1, x') := \left(\int_\omega v^1(0, x') \, dx'\right) \left(\rho(x'), 0, \ldots, 0\right),
\] (50)
where $\rho \in H^1_0(\omega)$ is a function that satisfies $\int_\omega \rho \, dx = 1$; and
\[
\begin{cases}
\tilde{v} \in H^1(\Omega) \\
\quad \text{div } \tilde{v} = 0 \quad \text{in } \Omega \\
\quad \tilde{v}(0, x') = \tilde{v}(1, x') = (v - \bar{v})(0, x') \quad \text{and } \tilde{v} = 0 \quad \text{on } (0, 1) \times \partial \omega.
\end{cases}
\]
That a function satisfying the conditions above exists follows from the fact that the following compatibility condition is satisfied (see, e.g., [11], [1]):
\[
\int_{\partial \Omega} \tilde{v} \cdot \nu \, d\sigma = 0.
\]
Note also that by the divergence formula, we have that, for $L^1$-a.e. $x_1 \in (0, 1)$,
\[
0 = \int_{(0, x_1) \times \omega} \text{div } \tilde{v} \, dx = \int_{\partial (0, x_1) \times \omega} \tilde{v} \cdot \nu \, d\sigma = \int_\omega \tilde{v}^1(x_1, x') \, dx' - \int_\omega \tilde{v}^1(0, x') \, dx' = \int_\omega \tilde{v}^1(x_1, x') \, dx',
\]
since
\[
\int_\omega \tilde{v}^1(0, x') \, dx' = \int_\omega v^1(0, x') \, dx' - \int_\omega \tilde{v}^1(0, x') \, dx' = \int_\omega v^1(0, x') \, dx' - \int_\omega v^1(0, x') \, dx' \int_\omega \rho(x') \, dx' = 0.
\]
Therefore $\tilde{v} \in V_{\text{per}}(Q)$ and $v - \bar{v} - \tilde{v} \in H^1_0(Q)$. We then obtain (49) by using $(v - \bar{v} - \tilde{v})$ as a test function in (48):
\[
\mu \int_Q \nabla u_\infty \cdot \nabla (v - \bar{v} - \tilde{v}) \, dx - \int_Q p_Q \, \text{div } (v - \bar{v} - \tilde{v}) \, dx = \int_Q f \cdot (v - \bar{v} - \tilde{v}) \, dx.
\] This is equivalent to
\[
\mu \int_Q \nabla u_\infty \cdot \nabla v \, dx - \mu \int_Q \nabla u_\infty \cdot \nabla \bar{v} \, dx - \int_Q p_Q \, \text{div } v \, dx = \int_Q f \cdot v \, dx - \int_Q f \cdot \bar{v} \, dx
\] since $\text{div } \bar{v} = \text{div } \tilde{v} = 0$ in $Q$ and
\[
\mu \int_Q \nabla u_\infty \cdot \nabla \tilde{v} \, dx = \int_Q f \cdot \tilde{v} \, dx
\]
by (15). Using the expression of $\tilde{v}$, we obtain
\[
\int_Q \nabla u_\infty \cdot \nabla \bar{v} \, dx = \int_Q \nabla' u^1_\infty \cdot \nabla' \tilde{v}^1 \, dx = \left(\int_\omega v^1(0, x') \, dx'\right) \int_Q \nabla' u^1_\infty \cdot \nabla' \rho \, dx
\]
and
\[ \int_Q f \cdot \tilde{v} \, dx = \int_Q f^1 \tilde{v}^1 \, dx = \left( \int_{\omega} v^1(x', x') \, dx' \right) \int_Q f^1 \rho \, dx. \]
Hence the constant \( k \) is given by
\[ k := -\mu \int_Q \nabla' u^1_{\infty} \cdot \nabla' \rho \, dx + \int_Q f^1 \rho \, dx. \quad (51) \]

Let us now deduce an intrinsic formula (involving only the data of the problem, i.e., the function \( f \)) for the constant \( k \). Introducing \( \pi \) the solution to
\[ \begin{cases} \pi \in \hat{H}^1_{\text{per}}(Q) \\ \mu \int_Q \nabla \pi \cdot \nabla v \, dx = \int_Q v^1 \, dx \quad \text{for all } v \in \hat{H}^1_{\text{per}}(Q) \end{cases} \quad (52) \]
one sees by taking \( v = \pi \) in (49) that
\[ k = \frac{\int_Q f \cdot \pi \, dx}{\int_{\omega} \pi^1(0, x') \, dx'}. \quad (53) \]
In fact, by the uniqueness of the solution to the problem (52), one has that \( \pi(x_1, x') = (u^1(x'), 0, \ldots, 0) \), where \( u^1 \in H^1_0(\omega) \) is the solution to
\[ -\mu \Delta' u^1 = 1 \quad \text{in} \quad H^{-1}(\omega). \]
Indeed, \( \pi \in \hat{H}^1_{\text{per}}(Q) \) and
\[ \mu \int_Q \nabla \pi \cdot \nabla v \, dx = \mu \int_Q \nabla' u^1 \cdot \nabla' v^1 \, dx = \mu \int_{\omega} \nabla' u^1 \cdot \left( \int_0^1 \nabla' v^1 \, dx_1 \right) \, dx' \]
\[ = \mu \int_{\omega} \nabla' u^1 \cdot \nabla' \left( \int_0^1 v^1 \, dx_1 \right) \, dx' = \int_{\omega} \int_0^1 v^1 \, dx_1 \, dx' = \int_Q v^1 \, dx, \]
for all \( v \in \hat{H}^1_{\text{per}}(Q) \), since the function \( x' \in \omega \mapsto \int_0^1 v^1(x_1, x') \, dx_1 \) belongs to \( H^1_0(\omega) \) and \( \int_0^1 \nabla' v^1 \, dx_1 = \nabla' \left( \int_0^1 v^1 \, dx_1 \right) \) (see step (i) in the proof of Theorem 1).

Taking into account the special form of \( \pi \), we get
\[ k = \frac{\int_{\omega} \tilde{f} u^1 \, dx'}{\int_{\omega} u^1 \, dx'}, \quad (54) \]
where
\[ \tilde{f}(x') := \int_0^1 f^1(x_1, x') \, dx_1. \quad (55) \]

We can now derive the relation between \( p_{\infty} \) and \( p_Q \) from the equation (49). For a fixed (but otherwise arbitrary) \( \ell > 0 \), let \( v \in \hat{H}^1_0(\Omega_{\ell}) \) which we consider
extended by 0 outside $\Omega_\ell$. We also extend $p_Q$ by periodicity in the direction $e_1$, i.e.,
\[ p_Q(x + ie_1) := p_Q(x) \quad \text{for all} \quad x \in Q, \, i \in \mathbb{Z} . \]

By the same computations as in part (i) of the proof of Theorem 5, we obtain that
\[
\mu \int_{\mathbb{R} \times \omega} \nabla u_\infty \cdot \nabla v \, dx - \int_{\mathbb{R} \times \omega} p_Q \div v = \mu \int_Q \nabla u_\infty(x) \cdot \sum_{i \in \mathbb{Z}} \nabla v(x + ie_1) \, dx
- \int_Q p_Q \sum_{i \in \mathbb{Z}} \div v(x + ie_1) \, dx ,
\]
where $v$ is an arbitrary function in $H^1_0(\Omega_\ell)$ (for some $\ell > 0$).

Arguing as in the proof of Theorem 5, one can see that the function
\[ x \in Q \mapsto \tilde{v}(x) := \sum_{i \in \mathbb{Z}} v(x + ie_1) . \]
belongs to $H^1_{\text{per}}(Q)$ and that
\[
\nabla \tilde{v}(x) = \sum_{i \in \mathbb{Z}} \nabla v(x + ie_1) .
\]

Hence we can use $\tilde{v}$ as a test function in (49). Then from (56) and (57) we derive
\[
\mu \int_{\mathbb{R} \times \omega} \nabla u_\infty \cdot \nabla v \, dx - \int_{\mathbb{R} \times \omega} p_Q \div v = \mu \int_Q \nabla u_\infty(x) \cdot \nabla \tilde{v} \, dx - \int_Q p_Q \div \tilde{v} \, dx
- \int_Q f \cdot \tilde{v} \, dx - k \int_{\omega} \tilde{v}(0, x') \, dx' = \mu \int_Q \nabla u_\infty(x) \cdot \nabla v \, dx - \int_Q p_Q \div v \, dx
- \int_Q f \cdot v \, dx - k \sum_{i \in \mathbb{Z}} \int_{\omega} v^1(i, x') \, dx' = \sum_{i \in \mathbb{Z}} \int_{Q_i} f \cdot v \, dx - k \sum_{i \in \mathbb{Z}} \int_{\omega} v^1(i, x') \, dx' = \int_{\mathbb{R} \times \omega} f \cdot v \, dx - k \sum_{i \in \mathbb{Z}} \int_{\omega} v^1(i, x') \, dx' ,
\]
where $Q_i := Q + ie_1$. We claim that (58) implies
\[
\mu \int_{\mathbb{R} \times \omega} \nabla u_\infty \cdot \nabla v \, dx - \int_{\mathbb{R} \times \omega} p_\infty \div v \, dx = \int_{\mathbb{R} \times \omega} f \cdot v \, dx ,
\]
where $p_\infty$ is given by
\[ p_\infty(x + ie_1) := p_Q(x) + ki \quad \text{for all} \quad x \in Q, \, i \in \mathbb{Z} . \]
In other words \( p_{\infty} = p_Q + k\zeta \), where \( \zeta \) is the step function \( \zeta := \sum_{i \in \mathbb{Z}} i 1_Q \).

Thus, in order to prove (59), it is enough to note that

\[
\int_{\mathbb{R} \times \omega} \zeta \text{div} \, v \, dx = \sum_{i \in \mathbb{Z}} i \int_{Q_i} \text{div} \, v \, dx = \sum_{i \in \mathbb{Z}} i \int_{\partial Q_i} v \cdot \nu \, d\sigma
= \sum_{i \in \mathbb{Z}} i \left( \int_{\omega} v^1(i + 1, x') \, dx' - \int_{\omega} v^1(i, x') \, dx' \right)
= \sum_{i \in \mathbb{Z}} (i - 1) \int_{\omega} v^1(i, x') \, dx' - \sum_{i \in \mathbb{Z}} i \int_{\omega} v^1(i, x') \, dx'
= -\sum_{i \in \mathbb{Z}} \int_{\omega} v^1(i, x') \, dx' .
\]

Thus we proved that (59) holds for all \( v \in \bigcup_{\ell > 0} \mathbb{H}^1_0(\Omega_\ell) \) (extended by 0 outside \( \Omega_\ell \)), in particular for any \( \varphi \in \mathcal{D}(\mathbb{R} \times \omega) \). Therefore, the pressure \( p_{\infty} \) satisfying the equation (16) is given by the formula (60), the constant \( k \) being given by (54). This is a consequence of the uniqueness of the solution to the equation (in the unknown \( p \)) \( \nabla p = \mu \Delta u_{\infty} + f \) in the space \( \dot{L}^2(\mathbb{R} \times \omega) \).

**Remark 11**

1. If \( \tilde{f} = 0 \) (where \( \tilde{f} \) is given by (55)), then \( k = 0 \). Consequently, if in addition to the hypotheses of Theorem 1, we assume that the applied forces satisfy \( \int_0^1 f^1(x_1, x') \, dx_1 = 0 \) for \( \mathcal{L}^{n-1} \)-a.e. \( x' \in \omega \), then the limit pressure \( p_{\infty} \) is also periodic. In particular, the limit \( (u_{\infty}, p_{\infty}) \) is periodic if the applied forces are orthogonal to the axis of the cylinder (i.e. if \( f^1 = 0 \)).

2. We can also take a more general \( f \) in the periodic case. More precisely, if \( f_Q \in (\mathbb{H}^1_{\text{per}}(Q))^\prime \), one can consider

\[
\langle f, v \rangle := \langle f_Q, \sum_{i \in \mathbb{Z}} v(\cdot + ie_1) \rangle \quad \text{for all} \quad v \in \mathbb{H}^1_0(\Omega_\ell). \tag{61}
\]

Note that \( f \) belongs to \( \mathbb{H}^{-1}(\Omega_\ell) \) for all \( \ell > 0 \). In fact, \( f \) defined by (61) is the periodical extension in the direction \( e_1 \) of \( f_Q \).

3. Instead of simple cylinders, one can consider more general periodic domains. For instance the figure below gives an example of what domain can be considered.
The general setting depicted above is the following: $Q$ is an arbitrary bounded Lipschitz domain (a domain is an open connected set) included in $(0, 1) \times \mathbb{R}^{n-1}$ such that the sets

$$
\Omega_k := \text{int} \bigcup_{i=-k}^{k-1} \overline{Q}_i, \ k \in \mathbb{N}^*,
$$

are also Lipschitz domains.

The connectedness of the sets $\Omega_k$ is in this case equivalent to the condition $\Omega_k \cap (\{0\} \times \mathbb{R}^{n-1}) \neq \emptyset$. In other words, there exist $y' \in \mathbb{R}^{n-1}$ and $r > 0$ such that $B_n((0, y'), r) \subset \Omega_k$. We define the set $\omega \subset \mathbb{R}^{n-1}$ by

$$
\{0\} \times \omega := \Omega_k \cap (\{0\} \times \mathbb{R}^{n-1})
$$

Note that $\omega$ is an open subset of $\mathbb{R}^{n-1}$. The set $\{0, 1\} \times \omega$ is the periodic part of $\partial Q$.

The periodicity condition on $Q$ is now the following:

$$
v(0, x') = v(1, x') \text{ for } L^{n-1}\text{-a.e. } x' \in \omega.
$$

The condition $v = 0$ on $(0, 1) \times \partial \omega$ in the definition of $\mathcal{H}^1_{\text{per}}(Q)$ is replaced by

$$
v = 0 \text{ on } \partial_{\text{lat}} Q := \partial Q \setminus (\{0, 1\} \times \omega)
$$

Thus,

$$
\mathcal{H}^1_{\text{per}}(Q) := \{v \in \mathcal{H}^1(Q) ; \ v = 0 \text{ on } \partial_{\text{lat}} Q \text{ and } v(0, \cdot) = v(1, \cdot) \text{ on } \omega\}.
$$

The definitions of the spaces $\mathcal{H}^1_{\text{per}}(Q)$ and $\mathcal{V}_{\text{per}}(Q)$ are exactly the same as in section 1.

As in the previous case, one can consider the equation (5) on domains $\Omega'_\ell$, where $\Omega'_\ell$ are Lipschitz domains satisfying

$$
\Omega_\ell \subset \Omega'_\ell \subset \Omega_{[\ell]}.
$$
for some \( \eta \geq 1 \).

Finally, the set \( \Omega_{\ell/2} \) appearing in the norms of the left hand side term of (14) must be replaced by the set \( \Omega_{[\ell/2]} \) and the set \( \mathbb{R} \times \omega \) appearing in the statement of Theorem 5 must be replaced by the set \( \bigcup_{k \in \mathbb{N}^*} \Omega_k \). The proof of Theorem 5 is in this case practically the same, but one needs a new version of Lemma 9 suited to this situation. We present this lemma now.

In the following lemma, \( Q \) is a bounded Lipschitz domain included in \( (0,1) \times \mathbb{R}^{n-1} \) such that the sets \( \Omega_k \) defined by (62) are connected.

**Lemma 12** Let \( k \in \mathbb{N}^* \) and \( g \in L^2(\Omega_k) \) such that \( \int_{\Omega_k} g \, dx = 0 \). Then there exists \( u \in H^1_0(\Omega_k) \) such that

\[
\begin{cases}
\text{div} \, u = g \text{ in } \Omega_k \\
\|\nabla u\|_{2,\Omega_k} \leq C_k \|g\|_{2,\Omega_k},
\end{cases}
\]

where \( C \) is a constant depending only on \( Q \).

**PROOF.** Let \( r \leq \frac{1}{2} \) and \( y = (0, y') \) such that the open ball \( B := B_n(y,r) \) is included in \( \Omega_k \). Let \( \varphi \in D(B) \) such that

\[
\int_{B'} \varphi(0, x') \, dx' = 1,
\]

where \( B' := B_{n-1}(y', r) \). We define a function \( \tilde{\varphi} \) on \( \Omega_k \) in the following way:

\[
\tilde{\varphi} := \sum_{j=-k}^{k-1} \left\{ \left( \int_{\bigcup_{i=-k}^{-1} Q_i} g \, dx \right) \varphi(\cdot - je_1) \right\}.
\]

Note that \( \tilde{\varphi} \in D(\Omega_k) \) and that for all \( j \in \{-k, \ldots, k\} \),

\[
\tilde{\varphi}(x) = \left( \int_{\bigcup_{i=-k}^{-1} Q_i} g \, dx \right) \varphi(x - je_1) \text{ for all } x \in \left( j - \frac{1}{2}, j + \frac{1}{2} \right) \times \mathbb{R}^{n-1} \quad (64)
\]

with the convention \( \bigcup_{i=-k}^{-1} = \emptyset \). We emphasize that for \( j = k \) we have by the assumption on \( g \),

\[
\int_{\bigcup_{i=-k}^{k-1} Q_i} g \, dx = \int_{\Omega_k} g \, dx = 0.
\]

For each \( j \in \{-k, \ldots, k-1\} \), we solve the following problem

\[
\begin{cases}
\text{div} \, u_j \in H^1_0(Q_j) \\
\text{div} \, u_j = g - \partial_1 \tilde{\varphi} \text{ in } Q_j \\
\|\nabla u_j\|_{2, Q_j}^2 \leq \tilde{C}' \|g - \partial_1 \tilde{\varphi}\|_{2, Q_j}^2,
\end{cases}
\]

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where $\tilde{C}$ is a constant depending only on $Q$. The compatibility condition allowing to solve the problem above is satisfied. Indeed,

\[
\int_{Q_j} (g - \partial_1 \tilde{\varphi}) \, dx = \int_{Q_j} g \, dx - \int_{Q_j} \partial_1 \tilde{\varphi} \, dx = \int_{Q_j} g \, dx - \int_{\partial Q_j} \tilde{\varphi} \nu_1 \, d\sigma
\]

\[
= \int_{Q_j} g \, dx - \left( \int_{B'} \tilde{\varphi}(j + 1, x') \, dx' - \int_{B'} \tilde{\varphi}(j, x') \, dx' \right)
\]

\[
= \int_{Q_j} g \, dx - \left( \sum_{l = -k}^{l = -1} g \, dx - \int_{\cup_{l = -k}^{l = -1} Q_l} g \, dx \right) \int_{B'} \varphi(0, x') \, dx'
\]

\[
= \int_{Q_j} g \, dx - \int_{Q_j} g \, dx = 0,
\]

where in the third equality we have used (64).

Now we take

\[
u = \sum_{j = -k}^{j = k-1} u_j 1_{Q_j} + (\tilde{\varphi}, 0, \ldots, 0).
\]

Obviously, $u \in H_0^1(\Omega_k)$ and $\operatorname{div} u = g$ in $\Omega_k$. Finally

\[
\|\nabla u\|^2_{2, \Omega_k} = \sum_{j = -k}^{j = k-1} \|\nabla u\|^2_{2, Q_j} \, dx \leq 2 \sum_{j = -k}^{j = k-1} (\|\nabla u_j\|^2_{2, Q_j} + \|\nabla \tilde{\varphi}\|^2_{2, Q_j})
\]

\[
\leq 2 \sum_{j = -k}^{j = k-1} (\tilde{C}' \|g - \partial_1 \tilde{\varphi}\|^2_{2, Q_j} + \|\nabla \tilde{\varphi}\|^2_{2, Q_j})
\]

\[
\leq 2 \sum_{j = -k}^{j = k-1} (2\tilde{C}' \|g\|^2_{2, Q_j} + (2\tilde{C}' + 1) \|\nabla \tilde{\varphi}\|^2_{2, Q_j})
\]

\[
= 4\tilde{C}' \|g\|^2_{2, \Omega_k} + (4\tilde{C}' + 2) \|\nabla \tilde{\varphi}\|^2_{2, \Omega_k}
\]

\[
= 4\tilde{C}' \|g\|^2_{2, \Omega_k} + (4\tilde{C}' + 2) \sum_{j = -k}^{j = k-1} \left( \int_{\cup_{l = -k}^{l = -1} Q_l} g \, dx \right)^2 \|\nabla \varphi(\cdot - je_1)\|^2_{2, \Omega_k}
\]

\[
\leq 4\tilde{C}' \|g\|^2_{2, \Omega_k} + (4\tilde{C}' + 2) \sum_{j = -k}^{j = k-1} \left( \int_{\Omega_k} |g| \, dx \right)^2 \|\nabla \varphi\|^2_{2, B}
\]

\[
\leq 4\tilde{C}' \|g\|^2_{2, \Omega_k} + (4\tilde{C}' + 2)(2k + 1) \|g\|^2_{2, \Omega_k} L^n(\Omega_k) \|\nabla \varphi\|^2_{2, B}
\]

\[
\leq C^2 k^2 \|g\|^2_{2, \Omega_k},
\]

where in the third equality we have used (64) on each set $\Omega_k \cap \left( \left( j - \frac{1}{2}, j + \frac{1}{2} \right) \times \mathbb{R}^{n-1} \right)$ and the following partition of the set $\Omega_k$:

\[
\Omega_k = \bigcup_{j = -k}^{j = k} \left( \Omega_k \cap \left( \left( j - \frac{1}{2}, j + \frac{1}{2} \right) \times \mathbb{R}^{n-1} \right) \right).
\]
Another slight difference with the case of cylinders is that, while the pressure $p_\infty$ is still given by (60), the constant $k$ appearing in this formula is no longer given by (54), because we no longer have the formula $\mathbf{u}(x_1, x') = (u_1(x'), 0, \ldots, 0)$ for the solution $\mathbf{u}$ to the problem (52). However, the formula (53) remains valid. We also have a formula of type (51) for $k$. More specifically, we have

$$k := -\mu \int_Q \nabla u_\infty \cdot \nabla \tilde{\varphi} \, dx + \int_Q p_Q \partial_1 \tilde{\varphi} + \int_Q f_1 \tilde{\varphi} \, dx,$$

where

$$\tilde{\varphi}(x) := \varphi(x) + \varphi(x - e_1) \quad \text{for all } x \in Q$$

and $\varphi$ is a function as in the proof of Lemma 12 (note that it is possible to have $Q$ such that it contains no cylinder $(0, 1) \times \omega', \omega' \subset \mathbb{R}^{n-1}$). In order to derive (65) and to prove that $(u_\infty, p_Q)$ satisfies the equation (49), we follow the same steps as in the case of cylinders, but with a function $\tilde{v}$ defined by

$$\tilde{v} := \left( \int_\omega v_1(0, x') \, dx' \right) \left( \tilde{\varphi}, 0, \ldots, 0 \right),$$

where $\omega$ is defined by (63) and $v$ is an arbitrary function in $H^1_{per}(Q)$.

4 A more general point of view

In this section the only assumption on the applied forces is that they satisfy some $L^2$-polynomial growth property. Under these general hypotheses, we prove that the solutions to the Stokes problems (4) converge to the solution to a Stokes problem in the infinite cylinder $\mathbb{R} \times \omega$. In particular, we can see the Theorems 1 and 5 as consequences of the following general result:

**Theorem 13** Let $f \in L^2_{loc}(\mathbb{R} \times \omega)$ and assume that there exist $\beta, C \geq 0$ such that

$$\|f\|_{2, \Omega_\ell} \leq C\ell^\beta \quad \text{for all } \ell > 0.$$

Then, for some positive constants $\alpha$, depending only on $\omega$, the solution $(u_\ell, p_\ell)$ to the problem (4) satisfies

$$\|\nabla (u_\ell - u_\infty)\|_{2, \Omega_{\ell/2}} + \|p_\ell - p_\infty\|_{2, \Omega_{\ell/2}} \leq \alpha \ell^{-\alpha}$$

as $\ell$ goes to $+\infty$, where the pair $(u_\infty, p_\infty) \in H^1_{loc}(\mathbb{R} \times \omega) \times \mathbf{L}^2_{loc}(\mathbb{R} \times \omega)$ is the
unique solution to the following Stokes problem in the infinite cylinder $\mathbb{R} \times \omega$:

\[
\begin{align*}
-\mu \Delta u_\infty + \nabla p_\infty &= f \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times \omega) \\
\text{div} \ n_\infty &= 0 \quad \text{in} \quad \mathbb{R} \times \omega \\
\n_\infty &= 0 \quad \text{on} \quad \mathbb{R} \times \partial \omega \\
\int_\omega u_\infty(x_1, x') \, dx' &= 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } x_1 \in \mathbb{R} \\
\|\nabla u_\infty\|_{2, \Omega \ell} &\leq C_\infty \ell^\gamma \quad \text{for all } \ell > 0, \text{ for some constants } \gamma, C_\infty \geq 0.
\end{align*}
\]

(68)

**Remark 14** 1. The existence and uniqueness of the solution to (68) is obvious by the Lax-Milgram Theorem if $\beta = 0$, but is nontrivial if $\beta > 0$. As we will see in the proof, $u_\infty$ satisfies the inequality in (68) with $\gamma = \beta$. However, the uniqueness result remains true even if we allow $\gamma$ to be any nonnegative constant.

2. The Theorems 1 and 5 correspond to the case $\beta = \frac{1}{2}$. Moreover, in the settings of these theorems, the limits $(u_\infty, p_\infty)$ given by Theorem 13 coincide with the ones described in the statements of Theorems 1 and 5, which are derived from the solutions to the equations (7), respectively (15).

3. The integral equality in the problem (68) says that the flux of the fluid vanishes. One can consider a problem with a prescribed flux which does not vanish, i.e.,

\[
\int_\omega u_\infty(x_1, x') \, dx' = \delta \quad \text{for } \mathcal{L}^1\text{-a.e. } x_1 \in \mathbb{R},
\]

where $\delta$ is a real constant. Then the Theorem 13 still holds true, provided that one modifies the boundary conditions in problems (4) with some nonhomogeneous conditions on $\{-\ell, \ell\} \times \omega$ having the same flux. For example, one can take

\[
u(-\ell, x') = u(\ell, x') = g(x') \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \omega,
\]

where $g \in (H_0^1(\omega))^n$ and $\int_\omega g^1 \, dx' = \delta$.

The problem of finding solutions with a prescribed flux for the Navier-Stokes system in infinite cylinders has been studied by Pileckas in [16].

**PROOF.** We use the same ideas as in the proof of Theorem 1, the main difference being that we first prove that $(u_\ell)_{\ell \geq 0}$ is a Cauchy "sequence" in order to prove the existence of a limit. The proof is divided into four steps.

(i) For all $\ell \geq 1$, $r \in [0, 1]$ one has

\[
\|\nabla (u_{\ell+r} - u_\ell)\|_{2, \Omega_{\ell/2}} \leq \eta e^{-\alpha \ell}
\]

(69)
for some positive constants \( \eta, a \) depending only on \( \omega \).

It is enough to remark that \((u_{\ell+r} - u_\ell)\) satisfy

\[
\int_{\Omega_\ell} \nabla(u_{\ell+r} - u_\ell) \cdot \nabla v \, dx = 0 \quad \text{for all } v \in H^1_0(\Omega_\ell),
\]

then to follow the same arguments as in steps (iii) and (iv) of the proof of Theorem 1 in order to obtain

\[
\|\nabla(u_{\ell+r} - u_\ell)\|_{2,\Omega_{\ell/2}} \leq ce^{-a\ell}\|\nabla(u_{\ell+r} - u_\ell)\|_{2,\Omega_\ell}
\]

(70)

for some positive constants \( c, a' \) depending only on \( \omega \).

Using \( u_\ell \) as a test function in (4), we get

\[
\|\nabla u_\ell\|_{2,\Omega_\ell} \leq \frac{C}{\mu} \|f\|_{2,\Omega_\ell} \leq C\ell^\beta,
\]

(71)

for all \( \ell > 0 \), for some constants \( C \) depending only on \( \omega \). Therefore, we have the estimate

\[
\|\nabla(u_{\ell+r} - u_\ell)\|_{2,\Omega_\ell} \leq \|\nabla u_{\ell+r}\|_{2,\Omega_\ell} + \|\nabla u_\ell\|_{2,\Omega_\ell}
\leq \|\nabla u_{\ell+r}\|_{2,\Omega_{\ell+r}} + \|\nabla u_\ell\|_{2,\Omega_\ell} \leq C\left((\ell + r)^\beta + \ell^\beta\right)
\leq C\ell^\beta \left(\left(\frac{\ell + r}{\ell}\right)^\beta + 1\right) \leq (2^\beta + 1)C\ell^\beta.
\]

(72)

Combining (70) with (72) we obtain (69) with a constant \( a \) that can be any positive constant smaller than \( a' \).

(ii) There exist \( C, a > 0 \) depending only on \( \omega \) such that

\[
\|\nabla(u_{\ell+t} - u_\ell)\|_{2,\Omega_{\ell/2}} \leq Ce^{-a\ell}
\]

(73)

for all \( \ell \geq 1 \) and \( t \geq 0 \).
Indeed,

\[ \| \nabla (u_{\ell+t} - u_{\ell}) \|_{2, \Omega_{\ell}/2} \leq \left( \sum_{i=0}^{[\ell]-1} \| \nabla (u_{\ell+i+1} - u_{\ell+i}) \|_{2, \Omega_{\ell}/2} \right) + \| \nabla (u_{\ell+t} - u_{\ell+[\ell]}) \|_{2, \Omega_{\ell}/2} \]

\[ \leq \left( \sum_{i=0}^{[\ell]-1} \| \nabla (u_{\ell+i+1} - u_{\ell+i}) \|_{2, \Omega_{\ell+[\ell]}/2} \right) + \| \nabla (u_{\ell+t} - u_{\ell+[\ell]}) \|_{2, \Omega_{\ell+[\ell]}/2} \]

\[ \leq \sum_{i=0}^{[\ell]} \eta e^{-a(\ell+i)} = \eta e^{-a} \sum_{i=0}^{[\ell]} e^{-ai} \]

\[ \leq \frac{\eta}{1 - e^{-a}} e^{-a\ell} . \]

Hence we get (73) for \( C \) given by \( \eta \frac{1}{1 - e^{-a}} \).

(iii) There exists \( u_{\infty} \in H^1_{\text{loc}}(\mathbb{R} \times \omega) \) such that for all \( \ell_0 > 0 \), \( u_{\ell} \to u_{\infty} \) in \( H^1(\Omega_{\ell_0}) \) and \( u_{\infty} \) satisfies the last four properties of (68).

A trivial consequence of (73) is that for a fixed (but otherwise arbitrary) \( \ell_0 > 0 \),

\[ \| \nabla (u_{\ell+t} - u_{\ell}) \|_{2, \Omega_{\ell_0}} \leq Ce^{-a\ell} \]

for all \( \ell \) large enough. This implies that \( (u_{\ell})_{\ell>0} \) is a Cauchy “sequence” with respect to the \( H^1(\Omega_{\ell_0}) \)-norm. Thus there exists \( u_{\infty,\ell_0} \in \mathbb{H}^1(\Omega_{\ell_0}) \) such that \( u_{\ell} \to u_{\infty,\ell_0} \) in \( \mathbb{H}^1(\Omega_{\ell_0}) \). Since \( u_{\ell} = 0 \) on \( (-\ell_0, \ell_0) \times \partial \omega \), \( \text{div} \ u_{\ell} = 0 \) in \( \Omega_{\ell_0} \) and \( \int_{\omega} u_{\ell}^1(x_1, x') \, dx' = 0 \) for \( L^1 \)-a.e. \( x_1 \in (-\ell_0, \ell_0) \) (by Proposition 8), we derive the same properties for the limit \( u_{\infty,\ell_0} \). By letting \( \ell_0 \) vary in \( \mathbb{N}^* \) we construct a function \( u_{\infty} \in H^1_{\text{loc}}(\mathbb{R} \times \omega) \) such that

\[ u_{\ell} \to u_{\infty} \text{ in } H^1(\Omega_{\ell_0}) \text{ for all } \ell_0 > 0 . \]

Moreover \( u_{\infty} = 0 \) on \( \mathbb{R} \times \partial \omega \), \( \text{div} \ u_{\infty} = 0 \) in \( \mathbb{R} \times \omega \) and \( \int_{\omega} u_{\infty}^1(x_1, x') \, dx' = 0 \) for \( L^1 \)-a.e. \( x_1 \in \mathbb{R} \).

For any \( \ell \geq 1 \), we derive

\[ \| \nabla (u_{\ell} - u_{\infty}) \|_{2, \Omega_{\ell}/2} \leq Ce^{-a\ell} \]  

(74)

by keeping \( \ell \) fixed and letting \( t \) go to \(+\infty\) in (73). Thus, combining (74) with (71), we get

\[ \| \nabla u_{\infty} \|_{2, \Omega_{\ell}} \leq \| \nabla (u_{\infty} - u_{2\ell}) \|_{2, \Omega_{\ell}} + \| \nabla u_{2\ell} \|_{2, \Omega_{\ell}} \leq C \left( e^{-2a\ell} + (2\ell)^{\beta} \right) \leq C_{\infty} \ell^{\beta} . \]
(iv) Conclusion of the proof.

Let $\ell > 0$ be fixed. It is obvious that for all $\ell' \geq \ell$, $u_{\ell'}$ satisfies the following variational equation

$$
\mu \int_{\Omega_{\ell}} \nabla u_{\ell'} \cdot \nabla v \, dx = \int_{\Omega_{\ell}} f \cdot v \, dx \quad \text{for all } v \in \hat{H}^1_0(\Omega_{\ell}) \tag{75}
$$

Making $\ell' \to +\infty$ in (75), we get

$$
\mu \int_{\Omega_{\ell}} \nabla u_{\infty} \cdot \nabla v \, dx = \int_{\Omega_{\ell}} f \cdot v \, dx \quad \text{for all } v \in \hat{H}^1_0(\Omega_{\ell}).
$$

Arguing as in the step (ii) of the proof of Theorem 1, we find $p_{\infty} \in \hat{L}^2_{\text{loc}}(\mathbb{R} \times \omega)$ such that

$$
-\mu \Delta u_{\infty} + \nabla p_{\infty} = f \quad \text{in } D'(\mathbb{R} \times \omega).
$$

To find the estimate

$$
\|p_{\ell} - p_{\infty}\|_{2, \Omega_{\ell}/2} \leq \alpha e^{-a'l}
$$

for $\ell$ large enough, we argue exactly as in step (vi) of the proof of Theorem 1.

Finally, let us prove the uniqueness of the solution $u_{\infty}$ satisfying (68). Assume there exist $u_{\infty,1}, u_{\infty,2}$ solutions to (68), where the inequality in (68) is satisfied for $\gamma_1$, respectively $\gamma_2$. Then, for all $\ell > 0$, we have

$$
\int_{\Omega_{\ell}} \nabla(u_{\infty,1} - u_{\infty,2}) \cdot \nabla v \, dx = 0 \quad \text{for all } v \in \hat{H}^1_0(\Omega_{\ell}).
$$

Using again the arguments of (iii) and (iv) from the proof of Theorem 1, one gets

$$
\|\nabla(u_{\infty,1} - u_{\infty,2})\|_{2, \Omega_{\ell}/2} \leq ce^{-a'\ell}\|\nabla(u_{\infty,1} - u_{\infty,2})\|_{2, \Omega_{\ell}}
$$

$$
\leq ce^{-a'\ell}(\|\nabla u_{\infty,1}\|_{2, \Omega_{\ell}} + \|\nabla u_{\infty,2}\|_{2, \Omega_{\ell}})
$$

$$
\leq ce^{-a'\ell}(\ell^{\gamma_1} + \ell^{\gamma_2})
$$

for some positive constants $c$, $a'$ depending only on $\omega$. Making $\ell \to +\infty$ in (76), we deduce $\|\nabla(u_{\infty,1} - u_{\infty,2})\|_{2, \mathbb{R} \times \omega} = 0$, hence $u_{\infty,1} = u_{\infty,2}$.

**Remark 15** One can see from the proof that we can even consider an exponential growth for $\|f\|_{2, \Omega_{\ell}}$. More specifically, the Theorem 13 still holds true if $f$ satisfies

$$
\|f\|_{2, \Omega_{\ell}} \leq Ce^{\tau\ell} \quad \text{for all } \ell > 0,
$$

where $\tau$ is any positive constant smaller than the constant $a'$ appearing in the inequality (70). Similarly, an exponential growth (with the same kind of exponent) for the norm $\|\nabla u_{\infty}\|_{2, \Omega_{\ell}}$ also insures the uniqueness of the solution to (68).
An obvious consequence of Theorem 13 is the following

**Corollary 16** Let \( \ell_0 > 0 \) be fixed and assume that \( f \in L^2_{\text{loc}}(\mathbb{R} \times \omega) \) satisfies

\[
\|f\|_{2,\Omega_0} \leq C\ell^\beta \quad \text{for all} \quad \ell > 0,
\]

for some constants \( C, \beta \geq 0 \). Then there exist two positive constants \( \alpha, \alpha \), depending only on \( \omega \), such that the solution \((u_\ell, p_\ell)\) to the problem (4) satisfies the inequality

\[
\| \nabla (u_\ell - u_\infty) \|_{2,\Omega_0} + \| p_\ell - p_\infty \|_{2,\Omega_0} \leq \alpha e^{-\alpha \ell}
\]

as \( \ell \) goes to \(+\infty\), where the pair \((u_\infty, p_\infty)\) is the solution to the problem (68).

As a byproduct of Theorem 13, one obtains the solution of the Stokes problem (68) in the infinite cylinder \( \mathbb{R} \times \omega \) as a limit of the solutions to the problems (1). As we have noticed, a direct proof of the existence and uniqueness of the solution to (68) cannot be achieved by the simple application of the Lax-Milgram Theorem. Another approach for the problem on the infinite cylinder is to work with weighted Sobolev spaces. Note that the function \( f \) satisfying (66) belongs to any weighted Sobolev space \( L^{2, -\beta - \varepsilon}_{\text{loc}}(\mathbb{R} \times \omega) \) for \( \varepsilon > 0 \), where

\[
L^2_{\tau}(\mathbb{R} \times \omega) := \left\{ f \in L^2_{\text{loc}}(\mathbb{R} \times \omega) : \| f \|^2_{L^2_{\tau}(\mathbb{R} \times \omega)} := \int_{\mathbb{R} \times \omega} (1 + |x|^2)^\tau |f|^2 \, dx < +\infty \right\}.
\]

In fact the hypotheses (66) and \( f \in L^2_{-\beta}(\mathbb{R} \times \omega) \) are almost equivalent since in the opposite sense, the inequality (66) is satisfied for any \( f \in L^2_{-\beta}(\mathbb{R} \times \omega) \).

For more details regarding the Stokes and Navier-Stokes problems in infinite domains and the applications of weighted Sobolev spaces, see, e.g., the works of Nazarov and Pileckas in [14] and [15].

**Acknowledgement**

The authors have been supported by the Swiss National Science Foundation under the contracts #20-111543/1 and #20-117614/1.

**References**


