THE RIESZ REPRESENTATION THEOREM FOR $L^2([0, 1])$

Let us consider $L^2([0, 1])$, the space of square-integrable real valued functions on the interval (up to equivalence almost everywhere). Let $\phi : L^2([0, 1]) \to \mathbb{R}$ be a function. We say that $\phi$ is a bounded linear functional if

$$\phi(\lambda f + g) = \lambda \phi(f) + \phi(g)$$

for all $f, g \in L^2$ and $\lambda \in \mathbb{R}$ (i.e. it is a linear function) and if

$$\sup_{\|f\|_2=1} \phi(f) = A < \infty$$

for some constant $A$. We call $A$ the norm of the linear functional.

For example, if $g \in L^2$ is any square-integrable function, then the map

$$\phi : f \mapsto \int_0^1 f(x)g(x) \, dx$$

is a bounded linear functional. In fact, the integral is well-defined by Cauchy Schwarz, which tells us further that

$$\phi(f) \leq \|f\|_2\|g\|_2 < \infty$$

so that $A = \|g\|_2$ is the norm of $\phi$ (the supremum is attained if we evaluate $\phi(g)$). Furthermore, the integral is linear in $f$ by our usual theorems on the linearity of the integral.

It turns out that this example is the only possibility.

**Theorem 1** (Riesz representation). Let $\phi : L^2([0, 1]) \to \mathbb{R}$ be a bounded linear functional with norm $A$. Then there is a square-integrable function $g \in L^2([0, 1])$ such that

$$\phi(f) = \int_0^1 f(x)g(x) \, dx$$

for all $f$. Furthermore, $A = \|g\|_2$.

A typical use of this theorem is in Exercise 6 of Homework 8. We want to show that a limit of a sequence of integrals is another integral, so we construct a bounded linear functional as a limit of the sequence and then use the representation theorem to write the functional as another integral.

In the proof of the theorem, we use the abbreviation

$$(f, g) = \int_0^1 f(x)g(x) \, dx.$$

**Proof of Theorem.** We know that $L^2([0, 1])$ has an orthonormal basis $g_k$, $k \geq 1$. For any $f \in L^2$ the sequence of partial sums

$$S_N = \sum_{k=1}^N (f, g_k)g_k$$

converges in $L^2$ to $f$. We now want to construct the element

$$g = \sum_{k=1}^{\infty} \phi(g_k)g_k.$$
It suffices to show that 
\[ \sum_{k=1}^{\infty} \phi(g_k)^2 < \infty \]
as then the sequence defining \( g \) is Cauchy in \( L^2 \) (and hence convergent because \( L^2 \) is complete). We have
\[
\sum_{k=1}^{N} \phi(g_k)^2 = \sum_{k=1}^{N} \phi(\phi(g_k)g_k) = \phi \left( \sum_{k=1}^{N} \phi(g_k)g_k \right) \leq M \| \sum_{k=1}^{N} \phi(g_k)g_k \| = M \sqrt{\sum_{k=1}^{N} \phi(g_k)^2}.
\]
Rearranging we see that 
\[
\sum_{k=1}^{N} \phi(g_k)^2 \leq M^2
\]
so the series converges as \( N \to \infty \).

We claim now that \( \phi(f) = (f, g) \). In fact, since \( \phi \) is bounded we have
\[
\phi(f) = \lim_{N \to \infty} \phi(\sum_{k=1}^{N} (f, g_k)g_k) = \lim_{N \to \infty} \sum_{k=1}^{N} \phi(g_k)(f, g_k) = \lim_{N \to \infty} (f, \sum_{k=1}^{N} \phi(g_k)g_k) = (f, g).
\]
On the last line we used the fact that \( (f, \cdot) \) is a bounded linear functional and that \( \sum_{k=1}^{N} \phi(g_k)g_k \) converges in \( L^2 \) to \( g \). The equality \( A = \| g \|_2 \) follows from Cauchy’s inequality. \( \square \)