PRACTICE PROBLEMS

Complete any six problems in 3 hours. Please do not work in groups or refer to your notes. After the time limit has passed, try and solve the other problems as well. These problems will not be graded.

**Problem 1.** Let us define the function \( \Gamma : \mathbb{R}^+ \to \mathbb{R} \) by the integral

\[
\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx.
\]

This function is usually called the **gamma function**.

1. Show that the integral for \( \Gamma(t) \) is well-defined for each \( 0 < t < \infty \) and that it is infinitely differentiable there.
2. Show that for each positive integer \( n \), \( \Gamma(n+1) = n! = n(n-1)(n-2) \cdots 2 \cdot 1 \).

**Solution.** The argument to the integral is non-negative so it suffices to show that the integral is finite. Splitting the integral, we have

\[
\int_0^1 x^{t-1} e^{-x} \, dx \leq \int_0^1 x^{t-1} \, dx = \frac{1}{t-1} < \infty
\]

If \( t \leq 1 \), then we have

\[
\int_1^\infty x^{t-1} e^{-x} \, dx \leq \int_1^\infty e^{-x} \, dx = e^{-1} < \infty.
\]

Otherwise, we integrate by parts \([t]\) times to get

\[
\int_1^\infty x^{t-1} e^{-x} \, dx = \text{const.} + (t-1) \cdots (t-[t]) \int_1^\infty x^{t-[t]-1} e^{-x} \, dx < \infty
\]

by the previous case.

We now claim that

\[
\Gamma^{(n)}(t) = \int_0^\infty (\log x)^n x^{t-1} e^{-x} \, dx,
\]

which is absolutely integrable by an analogous argument. We prove this by induction on \( n \), noting that \( n = 0 \) is trivial. Then we have

\[
\Gamma^{(n)}(t) = \lim_{h \to 0} \int_0^\infty (\log x)^{n-1} x^{t-1} e^{-x} \Delta(h \log x) \log x \, dx,
\]

where

\[
\Delta(y) = \frac{e^y - 1}{y}.
\]

\( \Delta \) is a nice function: it is continuous (in fact, analytic) on the whole real line, \( \Delta(0) = 1 \), and is monotone increasing. This last fact can be verified by computing its derivative, which is

\[
\Delta'(y) = \frac{ye^y - (e^y - 1)}{y^2}.
\]

We want to show that this quantity is always non-negative, or equivalently that

\[
ye^y - e^y - 1 \geq 0
\]

Replacing \( y \) with \(-y\) and rearranging it suffices to show that

\[
1 + y \leq e^y
\]

which follows since \( e^y \) is convex and \( 1 + y \) is the tangent line to \( e^y \) at \( y = 0 \).

Now we can complete the proof. It suffices to consider the integral over \((0, 1)\) and \((1, \infty)\), since if the limit exists for each integral than it exists for the sum. We can also consider positive and negative \( h \)s separately. For either case, we have a monotone sequence of functions which are either
non-negative or non-positive, which converges pointwise to the desired limit, so the result follows from the monotone convergence theorem.

Next we compute \( \Gamma(n+1) \). We have

\[
\Gamma(n+1) = \int_0^\infty t^n e^{-t} \, dt = \lim_{\epsilon \to 0} \int_\epsilon^{\infty} t^n e^{-t} \, dt
\]

by the monotone convergence theorem. The latter integral can be evaluated by parts, so that

\[
\int_\epsilon^{\infty} t^n e^{-t} \, dt = (-\epsilon^n e^{-\epsilon} + \epsilon^{-n} e^{-\epsilon}) + n \int_\epsilon^{\infty} t^{n-1} e^{-t} \, dt.
\]

If we send \( \epsilon \to 0 \), then the first term converges to zero (as \( n \) is fixed) while the second term converges to \( n \Gamma(n) \), again by the monotone convergence theorem. The result now follows by induction and the base case \( \Gamma(1) = 1 \), which can be verified directly.

**Problem 2.** Fix some \( \delta \in \mathbb{R} \) and let \( f : [0, \infty) \to \mathbb{R} \) be given by the equation

\[
f(x) = \frac{\sin(x^2)}{x} + \frac{\delta x}{1+x}.
\]

Show that

\[
\lim_{n \to \infty} \int_0^a f(nx) \, dx = a\delta
\]

for each \( a > 0 \).

**Solution.** Consider the sequence of functions \( f_n(x) = f(nx) \). For each fixed \( 0 \leq x \leq a \), \( f_n(x) = f(nx) \to \delta \) as \( n \to \infty \). Furthermore, the sequence is uniformly bounded by some \( M > 0 \). Since we are working on a finite interval \([0,a]\) we have

\[
\lim_{n \to \infty} \int_0^a f(n(x)) \, dx = \lim_{n \to \infty} \int_0^a f_n(x) \, dx = \int_0^a \lim_{n \to \infty} f_n(x) \, dx = \delta a
\]

by the dominated convergence theorem, with dominating functions \( g(x) = M1_{0 \leq x \leq a} \).

**Problem 3.** Show that

\[
\lim_{n \to \infty} \left( \log n - \sum_{k=1}^n \frac{1}{k} \right) = \lim_{n \to \infty} \int_0^n \left( 1 - \frac{x}{n} \right)^n \log x \, dx = \int_0^\infty e^{-x} \log x \, dx.
\]

**Solution.** Let \( f_n(x) = (1-xn^{-1})^n 1_{0 \leq x \leq n} \). Then \( 0 \leq f_n(x) \) and \( f_n(x) \leq e^{-x} \) by the convexity of \( e^{-x} \). By a theorem of Euler we have \( f_n(x) \to e^{-x} \) for each \( x \), so since

\[
\int_0^\infty e^{-x} \log x \, dx < \infty
\]

(say, by bounding the integral on \((0,1)\) by the integral of \( \log \) and the integral on \((1,\infty)\) by integrating by parts) we have by the dominated convergence theorem

\[
\lim_{n \to \infty} \int_0^\infty f_n(x) \log x \, dx = \int_0^\infty e^{-x} \log x \, dx.
\]

Now we evaluate the sequence of integrals. We have, using the substitution \( u = 1 - xn^{-1} \),

\[
\int_0^n (1-xn^{-1})^n \log x \, dx = n \int_0^1 u^n \log(n(1-u)) \, du
\]

Of course \( \log(n(1-u)) = \log n + \log(1-u) \) so the above integral is equal to

\[
n \int_0^1 u^n \log n \, du + n \int_0^1 u^n \log(1-u) = \frac{n}{n+1} \log n + n \int_0^1 u^n \log(1-u).
\]

Let us study the latter integral. The integrand is bounded by \( \log(1-u) \) which is integrable. \( \log(1-u) \) has the power series expansion

\[
\log(1 - u) = -\sum_{k=1}^{\infty} \frac{u^k}{k}
\]
which converges uniformly in compact subsets of \((-1, 1)\). Furthermore, the partial sums of the series are monotone decreasing in \(k\), so by the monotone convergence theorem

\[
\int_0^1 u^n \log(1 - u) = -\sum_{k=1}^{\infty} \int \frac{u^{k+n}}{k} \, du = -\sum_{k=1}^{\infty} \frac{1}{k(n+k+1)}.
\]

We observe that

\[
\frac{1}{k(n+k+1)} = \frac{1}{n+1} \left( \frac{1}{k} - \frac{1}{n+k+1} \right)
\]

so that the series is telescoping and

\[
\int_0^1 u^n \log(1 - u) = -\frac{1}{n+1} \sum_{k=1}^{n+1} \frac{1}{k}.
\]

We have therefore shown that

\[
\int_0^n \left(1 - \frac{x}{n}\right)^n \log x \, dx = \frac{n}{n+1} \log n + \frac{n}{n+1} \sum_{k=1}^{n+1} \frac{1}{k}.
\]

The right hand side differs from the sequence \(\log n - \sum_{k=1}^{n+1} k^{-1}\) by a quantity bounded by a constant times \(n^{-1} \log n\), so the result follows.

**Problem 4.** Let \(\phi \in L^1([0,1])\). Define the function \(f : \mathbb{R} \to \mathbb{R}\) by the integral

\[
f(t) = \int_0^1 |\phi(x) - t| \, dx.
\]

1. Show that \(f\) is a continuous function.
2. Show that if \(m(\{\phi = t\}) = 0\) for all \(t\) in an interval then \(f\) is continuously differentiable on that interval.
3. Show that if \(f\) is continuously differentiable on an interval then \(m(\{\phi = t\}) = 0\) for all \(t\) in the interval.

**Solution.**

1. We shall in fact show that \(f\) is \(1\)-Lipschitz. For, for any \(s\) and \(t \in \mathbb{R}\),

\[
|f(t) - f(s)| \leq \int_0^1 ||\phi(x) - t| - |\phi(x) - s|| \, dx \leq \int_0^1 |t - s| \, dx \leq |t - s|
\]

by the triangle inequality, and the result follows.

2. Fix \(t\) in the interior of the interval. We claim that if \(E_t = \{x \mid \phi(x) > t\}\) then \(f'(t) = 2m(E_t) - 1\); Note that the above is a continuous function by the monotone convergence theorem for measures and the hypothesis that \(m(\{\phi = t\}) = 0\).

Let \(\delta > 0\) and split the integral for \(f\) for \(s\) near \(t\), so that

\[
f(s) = \int_{\phi \leq t-\delta} + \int_{t-\delta < \phi \leq t+\delta} + \int_{t+\delta < \phi} |\phi(x) - s| \, dx.
\]

Now let us consider difference quotients for \(f\) with \(h < \delta\). We have

\[
\frac{f(t+h) - f(t)}{h} = A + B + C
\]

where

\[
A = -m(\phi \leq t - \delta), \quad C = m(\phi > t + \delta),
\]

and

\[
B = \int_{t-\delta < \phi \leq t+\delta} \frac{|\phi(x) - t - h| - |\phi(x) - t|}{h} \, dx.
\]

The latter integrand is bounded by \(1\) by the triangle inequality, so the integral is bounded by \(m(t - \delta < \phi \leq t + \delta)\). If we send \(\delta \to 0\), then \(A\) converges to \(-1 + m(E_t)\), \(C\) converges to \(m(E_t)\), and \(B\) converges to zero, all by the monotone convergence theorem for measures.
We note that we can assume (without hypothesis) by the monotone convergence theorem. For \( B \), we further split the integral

\[
B = \int_{t-\delta < \phi < t} \frac{|\phi(x) - t - h| - |\phi(x) - t|}{h} \, dx + \int_{\phi = t} \frac{|\phi(x) - t - h| - |\phi(x) - t|}{h} \, dx + \int_{t < \phi < t + \delta} \frac{|\phi(x) - t - h| - |\phi(x) - t|}{h} \, dx.
\]

By the dominated convergence theorem, the first and third of these integrals converge to zero as \( \delta \to 0 \). Thus we see that the limit \( f'(t) \) exists if and only if

\[
\lim_{h \to 0} \int_{\phi = t} \frac{|\phi(x) - t - h| - |\phi(x) - t|}{h} \, dx
\]

exists. However, the integrand is identically equal to \(|h|h^{-1} = \text{sign} h\), so the limit exists if and only if \( m(\phi = t) = 0 \).

**Problem 5.** Suppose \( f : X \to (0, \infty) \) is a measurable function so that

\[
\int_X f(x) \, d\mu(x) < \infty \quad \text{and} \quad \int_X \frac{1}{f(x)} \, d\mu(x) < \infty.
\]

Show that \( \mu(X) < \infty \).

**Solution.** We have \( 1 = \sqrt{f(x)f(x)^{-1}} \) so

\[
\mu(X) = \int_X d\mu(x) \leq \int_X \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} \, d\mu(x) \leq \sqrt{\int_X f(x) \, d\mu(x)} \sqrt{\int_X \frac{1}{f(x)} \, d\mu(x)} < \infty,
\]

by Cauchy-Schwarz.

**Problem 6.** (1) Suppose \( \mu(X) < \infty \). Show that if \( f \in L^q(X, \mu) \) for some \( 1 \leq q < \infty \) then \( f \in L^r(X, \mu) \) for \( 1 \leq r \leq q \).

(2) Give an example of a measure space \((X, \mu)\) with \( \mu(X) = \infty \) and a function \( f \in L^2(X, \mu) \) where \( f \notin L^1(X, \mu) \).

**Solution.** (1) Let \( f : X \to \mathbb{R} \) with \( \int_X |f(x)|^q \, dx < \infty \). Let \( 1 \leq r \leq q \) so that \( q/r \geq 1 \). Let \( p \) be the dual exponent to \( q/r \). Then by Hölder’s inequality

\[
\int_X |f(x)|^r \, d\mu(x) \leq \left( \int_X |f(x)|^q \right)^{r/q} \left( \int_X 1^p \, d\mu(x) \right)^{1/p} < \infty,
\]

as required.

(2) Let \( X = [1, \infty) \) with Lebesgue measure and \( f : X \to \mathbb{R} \) the function \( f(x) = x^{-1} \). Then \( \int_X f(x)^2 \, dx = \int_1^\infty x^{-2} \, dx = 1 < \infty \) while \( \int_X f(x) \, dx = \infty \).

**Problem 7.** Suppose \( \mu(X) < \infty \). Show that if \( f \in L^\infty(X, \mu) \) then \( f \in L^p(X, \mu) \) and \( \|f\|_p \overset{p \to \infty}{\to} \|f\|_\infty \).

**Solution.** We note that we can assume \( \|f\|_\infty > 0 \), as otherwise \( f = 0 \) a.e. and the result is trivial.

We first show that \( \limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty \). Let \( E = \{x \in X \mid |f(x)| > \|f\|_\infty \} \). Then \( \mu(E) = 0 \) and so

\[
\int_X |f(x)|^p \, d\mu(x) = \int_{E^c} |f(x)|^p \, d\mu(x) \leq \|f\|_\infty^p \int_{E^c} d\mu(x) = \|f\|_\infty^p \mu(X).
\]

Taking \( p \)-th roots we have

\[
\|f\|_p \leq \|f\|_\infty \mu(X)^{1/p}
\]

so that

\[
\limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty.
\]

Next we establish the lower bound. Let \( \epsilon > 0 \) be arbitrary and set \( F = \{x \in X \mid |f(x)| \geq \|f\|_\infty - \epsilon \} \). If \( \epsilon < \|f\|_\infty \) then \( \mu(F) > 0 \) and

\[
\int_X |f(x)|^p \, d\mu(x) \geq \int_F (\|f\|_\infty - \epsilon)^p \, d\mu(x) = (\|f\|_\infty - \epsilon)^p \mu(F).
\]
Taking $p$th roots,
\[ \|f\|_p \geq (\|f\|_\infty)^{1/p} \]
and so
\[ \liminf_{p \to \infty} \|f\|_p \geq (\|f\|_\infty - \epsilon) \]
since $\mu(F)^{1/p} \to 1$ as $\mu(F) > 0$. But since $\epsilon > 0$ was arbitrary the result follows.

**Problem 8.** Show that the infinitely differentiable functions are dense in $L^p(\mathbb{R}, dx)$ for $1 \leq p < \infty$.

**Solution.** By the monotone convergence theorem, we see that bounded and compactly supported functions are dense in $L^p$ (apply monotone convergence to $|1_{|f|<M}|1_{|x|<M}|f - f|$ as $M \to \infty$). By the definition of integrability we can approximate this function with a simple function. Since each simple function is the sum of a finite number of indicator functions, it suffices to approximate each indicator function by a smooth function.

We can approximate an indicator function $1_E$ with a step function as follows. Let $E \subseteq U$ with $m(U \setminus E) < \epsilon$ and $U$ open. Now
\[ \int |1_U - 1_E| = \int |1_U - 1_E|^p < \epsilon. \]
$U$ is the union of an at-most countable number of open intervals, call them $U_k$. Then
\[ U = \bigcup_{k=1}^{\infty} U_k \]
so by monotone convergence we must have $N$ large so that
\[ m\left( \bigcup_{k=N+1}^{\infty} \right) < \epsilon \]
and so
\[ \int |\sum_{k=1}^{N} 1_{U_k} - 1_E| = \int \left| \sum_{k=1}^{N} 1_{U_k} - 1_E \right|^p < 2\epsilon. \]
It therefore suffices to approximate each $1_{U_k}$ by a smooth function. Consider
\[ f(x) = \begin{cases} e^{-1/(x^2-1)}, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases} \]
This function is clearly smooth away from $\pm 1$. It suffices to show that the derivatives of this function converge to zero as $x \to \pm 1$ from inside the interval. In fact, we have
\[ \frac{d^n}{dx^n} e^{-1/(x^2-1)} = R_n(x)e^{-1/(x^2-1)} \]
where $R_n$ is an (explicit) rational function in $x$. But this converges to zero at $\pm 1$ so the function is smooth on the whole real line.

Let $C = \int f(y) \, dy$ (which is finite as $f$ is smooth and compactly supported) and define $g(x) = C^{-1} \int_0^x f(y) \, dy$. Clearly $g = 0$ for $x < -1$ and $g = 1$ for $x > 1$, and $g$ is a smooth function by the fundamental theorem of calculus. We can then approximate the indicator function of an interval (say $[0, 1]$) arbitrarily closely by
\[ h_\epsilon(x) = g(\epsilon^{-1}x) - g(\epsilon^{-1}(x-1)). \]
which is a smooth function such that
\[ \int_{\mathbb{R}} |1_{[0,1]}(x) - h_\epsilon(x)| \, dx \leq 2\epsilon. \]

**Problem 9.** Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. Show that $f$ is also Lebesgue integrable and that $f$ is continuous a.e.
Solution. Let \( P_k \) be a sequence of nested partitions of \([0, 1]\) such that
\[
\lim_{k \to \infty} \int_0^1 U(f, P_k, x) \, dx = \lim_{k \to \infty} \int_0^1 L(f, P_k, x) \, dx.
\]
Each \( U(f, P_k, \cdot) \) and \( L(f, P_k, \cdot) \) as a sequence of functions in \( k \) is strictly decreasing (resp. increasing). Let \( U(f, x) = \inf_{k \geq 1} U(f, P_k, x) \) and \( L(f, x) = \sup_{k \geq 1} L(f, P_k, x) \). These functions are clearly measurable and
\[
\int_0^1 U(f, x) - L(f, x) \, dx \leq \liminf_{k \to \infty} \int_0^1 U(f, P_k, x) - L(f, P_k, x) \, dx = 0.
\]
Since \( U(f, \cdot) - L(f, \cdot) \geq 0 \) the difference must be zero a.e..

Let \( E \) be the set of points where the difference is zero and not contained in any of the partitions \( P_k \). Clearly \( m(E) = 1 \) as there are a countable number of points in any \( P_k \). On \( E \) we have \( L(f, P_k, x) \leq f \leq U(f, P_k, x) \) for all \( k \) so that \( L(f, x) \leq f \leq U(f, x) \); since \( L(f, x) = U(f, x) \), we conclude that \( f = U \), say, on \( E \), hence \( f \) is measurable. It is integrable since it is bounded above and below by integrable functions, so it suffices to show that it is continuous on \( E \).

Since \( x \) is not contained on any of the points in any \( P_k \), we have a sequence of intervals \( a_k < x < b_k \) such that \( a_k \to x \) and \( b_k \to x \). Since \( x \in E \), we must have for \( k \) sufficiently large,
\[
U(f, P_k, x) - \epsilon \leq L(f, P_k, x) = \inf_{a_k < y < b_k} f(y) \leq f(x) \leq \sup_{a_k < y < b_k} f(y) = U(f, P_k, x)
\]
so \( |f(x) - f(y)| < \epsilon \) for all other \( y \) in the interval. As \( \epsilon \) was arbitrary, \( f \) must be continuous at \( x \), hence all of \( E \) as required.