EXERCISE SHEET 9

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Exercise 1 (Principal open subsets). Let X be a scheme and $f \in \mathcal{O}_X(X)$. Define the non-vanishing locus of f as

$$X_f := \{x \ \in X \, s.t. \, f_x \in \mathcal{O}_{X,x}^{\times}\}.$$

Show that

(1) X_f is an open subset of X;
(2) Show that f is invertible in $\in \mathcal{O}_X(X_f)$ and deduce that the inclusion $X_f \subset X$ induces a map $\mathcal{O}_X(X)_f \longrightarrow \mathcal{O}_X(X_f)$;
(3) Assume that X has a finite affine covering U_i such that $U_i \cap U_j$ has itself a finite affine covering. Show that the map constructed above is an isomorphism. [Hint: use the sheaf condition with respect to a finite open affine cover.]

Exercise 2 (Criterion for Affineness). (1) Let $X \longrightarrow Y$ be a map of schemes. Show that f is an isomorphism iff there exists an affine open covering $\{V_i\}_i$ of Y the map $X \times_Y V_i \longrightarrow V_i$ is an isomorphism. In other words being an isomorphism is a Zariski-local condition on the target.
(2) Let X be a scheme. show that X is affine iff there exists a collection of elements $f_1, \ldots, f_r \in \mathcal{O}_X(X)\}$ such that the X_{f_i}'s are affine and $(f_1, \ldots, f_r) = \mathcal{O}_X(X)$.

Exercise 3 (Reduced structure). Let R be a ring and $N(R) := \{x \in R \, s.t. \, x^n = 0 \text{ for some } n \in \mathbb{N}\}$. Recall that a ring is defined to be reduced if $N(R) = 0$. Show that

(1) if S is a multiplicative closed subset of R, then $S^{-1}(N(R)) \simeq N(S^{-1}R)$;
(2) the following are equivalent:
   (a) R is reduced;
   (b) $R_{\mathfrak{p}}$ is reduced for every $\mathfrak{p} \in \operatorname{Spec} R$;
   (c) $R_{\mathfrak{m}}$is reduced for every $\mathfrak{m} \in \operatorname{Specmax} R$;
(3) the forgetful functor $\mathbf{CRng}^{\mathrm{red}} \longrightarrow \mathbf{CRng}$ has a left adjoint. (Here $\mathbf{CRng}^{\mathrm{red}}$ denotes the category of reduced commutative rings and morphisms of rings).

Let's globalize these notions to schemes.

Definition 3.1. A scheme X is defined to be reduced at $x \in X$ if the ring $\mathcal{O}_{X,x}$ is reduced. X is defined to be reduced if it's reduced at each of its points.

Show that:

(1) The following are equivalent:
   (a) X is reduced;
   (b) for every affine covering $\{U_i = \operatorname{Spec} A_i\}_i$ of X, the A_i's are reduced;
   (c) for every open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced.
(2) The assignement $U \mapsto \mathcal{N}(U) := \{x \in \mathcal{O}_X(U) \, s.t. \, x \in N(\mathcal{O}_{X,x}) \ \forall x \in U\}$ together with the restrictions induced by those of $\mathcal{O}_X$, defines a sheaf of $\mathcal{O}_X$-modules on X; if U is an affine open subset of X, then $N(\mathcal{O}_X(U)) = \mathcal{N}(U)$.
(3) Let's show that the locally ringed space $(X, \mathcal{O}_X/\mathcal{N})$ is a scheme:
   (a) Show that you can reduce to the case of X is an affine scheme;
   (b) In the case $X = \operatorname{Spec} A$: describe $\mathcal{O}_X/\mathcal{N}$ in terms of the functor $\tilde{}$ of Exercise 4 of the exercise sheet 4 and conclude;
(4) Denote $(X, \mathcal{O}_X/\mathcal{N})$ by $(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$ the scheme constructed at the previous point. Show that there is a natural map $r : X^{\mathrm{red}} \longrightarrow X$ and this data satisfy the following universal property: for every map $f : Y \longrightarrow X$ with Y reduced, f factors uniquely through r. In a picture, given the solid diagram (i.e. the non dashed part) there exists a unique dashead arrow making the diagram

$$\begin{tikzcd}
& X^{\mathrm{red}} \arrow[d, "r"] \\
Y \arrow[ur, dashed] \arrow[r, "f"'] & X
\end{tikzcd}$$

commutative.

(5) Assume that X admits a finite covering by open affine subschemes: show that the following sequence is exact:

$$0 \longrightarrow N(\mathcal{O}_X(X)) \longrightarrow \mathcal{O}_X(X) \overset{r_\#(X)}{\longrightarrow} r_*\mathcal{O}_X^{\mathrm{red}}(X).$$

(6) Show that if Z is a closed subset of X then Z has a canonical scheme structure. [Hint: Cover X by affine open subschemes $U_i = \operatorname{Spec} A_i$ where $Z \cap U_i =: Z_i$ writes as $V(I_i)$ for some ideal $I_i \subseteq A_i$ and define the *reduced* scheme structure on the Z_i's. Show that with such scheme structures the Z_i's glue.]

Exercise 4 (Noetherian schemes).

Definition 4.1. A scheme X is defined to be *Locally Noetherian* if it has an open affine cover $\{U_i = \operatorname{Spec} A_i\}_i$ where A_i is a noetherian ring. A scheme is *Noetherian* if it is quasi-compact (i.e. the underlying space is quasi-compact; in other words every open cover of X can be refined to a finite open cover) and locally noetherian.

Let X be a locally noetherian scheme. Show that

(1) Any open or closed subscheme of X is locally noetherian;
(2) For every point $x \in X$ the ring $\mathcal{O}_{X,x}$ is noetherian;
(3) An affine scheme $Y = \operatorname{Spec} A$ is noetherian iff A is noetherian. (In particular, if X is noetherian and $U \subseteq X$ is an affine subscheme, then $\mathcal{O}_X(U)$ is a noetherian ring.) In order to achieve this point observe that one implication is obvious; for the oher we can follow these steps:
   (a) Prove that if Z is any scheme, $U, V \subseteq X$ are open affine subschemes and $z \in U \cap V$, then there exists an open subscheme $z \in W \subseteq U \cap V$ such that W is a principal open subscheme of both V and U.
   (b) Prove that you can cover $X = \operatorname{Spec} A$ by principal open subsets $\{D(f_i)\}_{i \in \{1,\ldots,n\}}$ where $f_i \in A$ and A_{f_i} is noetherian.
   (c) Suppose that A is any ring and that $\{f_1, \ldots f_n\} \subset A$ generate the unit ideal of A. Then for every A-module M, we have that M is finitely generated iff M_{f_i} is finitely generated for every i.
   (d) Conclude.

   Definition 4.2. A space X is called noetherian if it satisfies the "d.c.c" on closed subsets, i.e. any decreasing chain of closed subsets of X stabilizes. Show that:

(4) if X is a noetherian space and $Y \subseteq X$ is a subspace, then X is quasi-compact and Y is noetherian;
(5) the following are equivalent:
   (a) X is noetherian;
   (b) Every open subspace of X is quasi-compact;
   (c) Every subspace $Y \subseteq X$ is quasi-compact;
(6) if A is a noetherian ring, then $X = \operatorname{Spec} A$ is noetherian space;
(7) Find a non-noetherian ring A such that $\operatorname{Spec} A$ is a noetherian space;
(8) Let A be a ring such that $\operatorname{Spec} A$ is noetherian: show that the set of prime ideals satisfis the "a.c.c". Does the converse hold?
   Let's get back to schemes. Let X be a scheme.
(9) If X is noetherian, then the underlying topological space (usually denoted by $|X|$) is a noetherian space; the converse is false.

Exercise 5 (Varieties).

Definition 5.1. Let S be any scheme. A *scheme over* S is a pair (X, p_X) where X is a scheme and $p_X : X \longrightarrow S$ is a morphism of schemes (sometimes p_X is called *structure map* and, with abuse, is often omitted from the notation). A morphism of schemes over S (sometimes called S-morphism) is a map $f : X \longrightarrow Y$ compatible with the structure maps; in other words it's a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & S & \end{tikzcd} .$$

Schemes over S and S-morphisms form a category denoted by $\mathbf{Sch}_S$. When $S = \operatorname{Spec} R$ we simply write $\mathbf{Sch}_R$. Note that if $S = \operatorname{Spec} \mathbb{Z}$ then $\mathbf{Sch}_{\mathbb{Z}} = \mathbf{Sch}$: why? Note also that the functor Spec restricts to $\mathbf{CAlg}_R \longrightarrow \mathbf{Sch}_R$. Given a scheme over S, call it X, a T point of X over S is an S-map $f : T \longrightarrow X$; they are denoted by $X_S(T)$; in particular the S-points of X over S, denoted by $X_S(S)$, are those maps $\sigma : S \longrightarrow X$ such that $p_X \circ \sigma = id_S : S \longrightarrow S$. If no confusion arises one removes the subscript S form the notation. If $T = \operatorname{Spec} B$ then it's customary to call T-points "B-points".

Let k be a field, X be a scheme over k and Y be any scheme. Show that:

(1) The set of k-points over k of X is in bijection with $\{x \in X \; s.t. \; \kappa(x) \simeq k\}$. Here $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(2) Assume K is any field: then $X(K)$ is in bijection wtth the set of pairs (y, i_y), where $y \in Y$ and $i_y : \kappa(y) \longrightarrow K$.
(3) Let $Z \subseteq Y$ be a open (or closed) subschme of Y then $Z(K) = Y(K) \cap Z$. [Hint: compare the residue fields in Y and in Z of a K-point].
(4) Let A be a finitely generated k-algebra and $X = \operatorname{Spec} A$ the associated k-scheme. In particular $A \simeq k[X_1, \dots, X_n]/(f_1, \dots f_r)$. Let $K \supset k$ be a field extension. Describe $X_k(k)$ and $X_k(K)$.
(5) Assume that X is a k-scheme admitting a (possibly infinite) open cover by affine schemes $\operatorname{Spec} A_i$ where A_i is a finitely generated k-algebra for every i (in such a case we say that p_X is a map of finite type). Show that the following are equivalent:
   (a) $x \in X$ is a closed point (i.e. $\{x\}$ is a closed subset of X);
   (b) the natural map $k \longrightarrow \kappa(x)$ is a finite extension of fields;
   (c) the natural map $k \longrightarrow \kappa(x)$ is an algebraic extension of fields;
   Deduce that $x \in X$ is a closed point iff there exists an open affine subscheme $\operatorname{Spec} A \subseteq X$ such that x is closed in $\operatorname{Spec} A$.

   Remark 5.2. Given a topological space T, a subspace $Z \subseteq T$ is closed iff the exists an open cover (and then for every open cover) T_i of T it holds that $Z \cap T_i$ is closed in T_i.

(6) Find a scheme Y with an open subset U and a point $x \in U$ such that x is closed in U but x is not closed in Y.

Exercise 6 (Glueing). (1) Let R be a ring; mimic the construction of $\mathbb{P}^1_R$ to construct the scheme $\mathbb{P}^n_R$:
- Start with $n+1$ affine spaces $\mathbb{A}^n_R \simeq \operatorname{Spec} R[\widehat{X_i}] =: X_i$, where $R[\widehat{X_i}] := R[X_0, \dots, \widehat{X_i}, \dots, X_n]$ is a free polynomial ring in n indeterminates, $\widehat{X_i}$ means that X_i is missing. Define open subschemes $X_{i,j} := \operatorname{Spec} R[\widehat{X_i}]_{X_j}$.
- Define adequate isomoprphisms $\phi_{i,j} : X_{i,j} \longrightarrow X_{j,i}$ that satisfy the cocycle condition, so that you actually get a scheme by gluing the X_i's. Be careful in the definition of the $\phi_{i,j}$ so that you don't axcidentally glue lines with double origins.
- Observe that this produces a scheme over $\operatorname{Spec} R$, and that $\mathbb{P}^n_R \simeq \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$ in the category $\mathbf{Sch}_{\operatorname{Spec} R}$.
- Let k be a field: show that $\mathbb{P}^n_k(k)$ coincides with the set of non-zero $n+1$-tuples modulo invertible scalars.

(2) Consider the polynomial $F := X_0 X_2^2 = X_1(X_1 - X_0)(X_1 + X_0) \in \mathbb{Z}[X_0, X_1, X_2]$ and let $E_i := \mathbb{Z}[\widehat{X_i}]/F_i$ where $F_0 := F(1, X_1, X_2)$ and similiarly for F_1 and F_2. Show that the gluing data for $\mathbb{P}^2_{\mathbb{Z}}$ induce gluing data for the E_i's, giving rise to a scheme E.
(3) Let R be any ring and show that the previous argument applies to any homonegenous ideal of $R[X_0, \dots, X_n]$.
(4) Let k be a field, $R := k[T]$, and consider the polynomial $F := X_0 X_2^2 = X_1(X_1 - X_0)(X_1 + T X_0) \in R[X_0, X_1, X_2]$: by applying the construction above we get a map $f : E \longrightarrow \operatorname{Spec} R$.
- Compute the fibres of the map f.
- Construct a scheme and a map $\bar{f} : \bar{E} \longrightarrow \mathbb{P}^1_k$ that extends the map f; [in other words if $i : \mathbb{A}^1_k \simeq U_0 \simeq \operatorname{Spec} R \longrightarrow \mathbb{P}^1_k$ is the map missing ∞ we want a cartesian diagram as follows.

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "f"'] & \bar{E} \arrow[d, "\bar{f}"] \\ \mathbb{A}^1_k \arrow[r, "i"] & \mathbb{P}^1_k \end{tikzcd}$$

  In particular $j : E \longrightarrow \bar{E}$ needs to be an open subscheme and $f = \bar{f}_{|E}$].
- Compute the fibres of $\bar{f}$: you just need to compute the fibre at infinity.
- Construct the projections onto the three "axes" $p_0, p_1, p_2 : E \longrightarrow \mathbb{P}^1$ as maps of schemes.

Exercise 7 (Fibred products). (1) Let A, B be two rings: describe and briefly prove the universal property of $B \otimes_{\mathbb{Z}} C$ in the category $\mathbf{CRng}$ of commutative rings with morphisms of rings (maps of rings respect 1). Assume now that, in addition, A, B are R-algebras. Describe the universal property of $A \otimes_R B$ in the category of $\mathbf{CAlg}_R$ of commutative R-algebras and in the category of $\mathbf{CAlg}_{\mathbb{Z}} = \mathbf{CRng}$. Deduce that $\mathbf{CAlg}_R$ has coproducts and pushouts.
(2) Le A be a commutative ring; show that:
- $A[x] \otimes_A A[Y] \simeq A[X, Y]$;
- If I is an ideal of A, then $A/I \otimes_A B \simeq B/BI$;
- If I, J are ideals of A, then $A/I \otimes_A A/J \simeq B/(I + J)$;
- Compute the following tensor products:

- $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m$ $(m, n \geq 2 \in \mathbb{N})$;
- $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ where p is a prime;
- $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$;
- $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$;
- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X]/(X^2+1)$;
- $A[X]/(X) \otimes_A A[X]/(X-1)$;
- $A[X]/(X) \otimes_{A[X]} A[X]/(X-1)$;
- $A[X,Y] \otimes_{A[X]} A[X]/(X-1)$.

(3) Let S be a scheme and X, Y, Z be schemes over S. Show that:
- The following natural maps are isomorphisms of S-schemes: $X \times_S S \longrightarrow X$, $S \times_S Y \longrightarrow Y$, $X \times_S Y \longrightarrow Y \times_S X$, and $(X \times_S Y) \times_S Z \longrightarrow X \times_S (Y \times_S Z)$.
- Assume that the map $Z \longrightarrow S$ making Z an S-scheme factors as $Z \longrightarrow Y \longrightarrow S$. Then the canonical map $(X \times_S Y) \times_Y Z \longrightarrow X \times_S Z$ is an isomorphism.
- Given S-morphisms $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ there exists a unique morphism over S, denoted by $f \times g : X \times_S X' \longrightarrow Y \times_S Y'$ such that the following diagram commutes:

$$\begin{tikzcd}
& X\times_S Y \arrow[rr, "f\times g"] \arrow[ddl] \arrow[dr] & & X'\times_S Y' \arrow[ddl] \arrow[dr] & \\
& & Y \arrow[rr, "g"] \arrow[ddl] & & Y' \arrow[ddl] \\
X \arrow[rr, "f"] \arrow[dr] & & X' \arrow[dr] & & \\
& S \arrow[rr, equal] & & S &
\end{tikzcd}$$

- Show that $(X \times_S Y)(S) \simeq X(S) \times Y(S)$ and that $\mathrm{Hom}_X(X, Y_X) \simeq \mathrm{Hom}_S(X, Y)$, where $Y_X := Y \times_S X$ is a X-scheme via the second projection.
- Assume that the right hand side square, in the following diagram, is cartesian. Then the outer suqare is cartesian iff the left hand side square is.

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\
S'' \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

Exercise 8 (Fibres). (1) Let $\phi : A \longrightarrow B$ ba a map of commutative rings, $\mathfrak{p} \in \operatorname{Spec} A$, and $f = \operatorname{Spec}(\phi)$. Show that under the homoeomorphic identification $\operatorname{Spec} A_{\mathfrak{p}}$ with the subspace (with induced topology) $\{\mathfrak{p} \in \operatorname{Spec} A \, s.t.\, \mathfrak{p} \cap (A \setminus \mathfrak{p} = \emptyset)\} \subseteq \operatorname{Spec} A$, the restriction of f to $\operatorname{Spec} A_{\mathfrak{p}}$ coincides with the map $\operatorname{Spec} \phi_{\mathfrak{p}} : \operatorname{Spec} A_{\mathfrak{p}} \longrightarrow \operatorname{Spec} B_{\mathfrak{p}}$.

(2) Let $\phi : A \longrightarrow B$ ba a map of commutative rings, let I be an ideal of A, and $f = \operatorname{Spec}(\phi)$. Show that under the homeomorphic indentification $\operatorname{Spec} A/I$ with the subspace $\{\mathfrak{p} \in \operatorname{Spec} A \, s.t.\, \mathfrak{p} \supseteq I\} \subseteq \operatorname{Spec} A$, the restriction of f to $\operatorname{Spec} A/I$ coincides with the map $\operatorname{Spec} \bar{f} : \operatorname{Spec} A/I \longrightarrow \operatorname{Spec} B/IB$.

(3) By combining the previous points and observing that we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_{\mathfrak{p}} \arrow[r] \arrow[d] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[d] \\
B \arrow[r] & B_{\mathfrak{p}} \arrow[r] & B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}
\end{tikzcd}$$

show that the fibre of the map $f = \operatorname{Spec} \phi : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ in $\mathfrak{p} \in \operatorname{Spec} A$, denoted as $f^{-1}(\mathfrak{p})$ is identified, as topological space, to $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ where $\kappa(\mathfrak{p}) := A\mathfrak{p}/\mathfrak{p}A_{\mathfrak{p}}$.

Definition 8.1. Let $f : Y \longrightarrow X$ be a map of schemes and let $x \in X$ be a point of X. We call $\mathcal{O}_{X,x}/\mathfrak{m}_x =: \kappa(x)$ residue field at x: it is endowed with a natural map $\operatorname{Spec}(\kappa(x)) \longrightarrow X$ (see exercise 5). The scheme-theoretic fibre of f in x is the $\kappa(x)$-scheme $Y \times_X \operatorname{Spec}(\kappa(x))$, where the structure of $\kappa(x)$-scheme is induced by the projection on the second factor. It's customary to make an abuse of notation and denoted it by Y_x, omitting the map f from the notation.

(4) Show that the second projection $Y_x \longrightarrow Y$ identifies the underlying topological space of Y_x with the subspace $f^{-1}(x) \subset Y$ (with the induced topology).

(5) Explicitely compute some example taking inspiration from 7.

(6) Some computations:
   (a) Consider the map $f : \operatorname{Spec} \mathbb{R}[T] \longrightarrow \operatorname{Spec} \mathbb{R}[T]$ induced by $T \mapsto T^2$. Compte the fibres of f in $(X-1), (X), (X+1), (0)$.

(b) Let k be a field, $X = \operatorname{Spec} k[T]$, $Y = \operatorname{Spec}(k[X, Y, T]/(X^2 - Y^2 - T))$ and $f : Y \longrightarrow X$ the map induced by the inclusion $k[T] \subset k[X, Y, T]/(X^2 - Y^2 - T)$. Observe that if x is the point of X corresponding to the prime ideal $(X - a)$, then the map $i_x : \operatorname{Spec} k \longrightarrow X$ is induced by the projection $\pi_x : k[T] \longrightarrow k[T]/(T - a)$. Show that, for $x = (X - a)$, we have $Y_x \simeq \operatorname{Spec} k[X, Y]/(X^2 - Y^2 - a)$. Compute the fibre over the point $(0) \in X$. Prove that the map of rings $k[T] \longrightarrow k[X, Y, T]/(X^2 - Y^2 - T)$ is flat: what can we deduce about the fibres of the map?

(c) Let $Y := \operatorname{Spec} \mathbb{Z}[i]$, $X := \operatorname{Spec} \mathbb{Z}$, and $f : Y \longrightarrow X$ the map induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$. Compute the fibres of the map at the points $(2), (3), (5), (0) \in \operatorname{Spec} \mathbb{Z}$. Try to give a meaning to the *imprecise* sentence "For every $x \in X$ the fibre of f in x has two points".