Homework 8

Please turn these in on Monday, Dec. 14 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

1.(a) Let $(T, \mathcal{O}_T)$ be a locally ringed space, $U \subset T$ an open subset and $\mathcal{O}_U$ the restriction of $\mathcal{O}_T$ to a sheaf of rings on U: $\mathcal{O}_U(V) = \mathcal{O}_T(V)$ for $V \subset U$ open (and similarly for the restriction maps). Show that $(U, \mathcal{O}_U)$ is a locally ringed space.
(b) Let R be a commutative ring, $f \in R$. Let $\mathrm{can}_f : R \to R_f$ be the localization homomorphism, giving the map of locally ringed spaces

$$(\mathrm{Spec}\,(\mathrm{can}_f), \widetilde{\mathrm{can}_f}) : (\mathrm{Spec}\, R_f, \mathcal{O}_{\mathrm{Spec}\, R_f}) \to (\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R}).$$

Show that $(\mathrm{Spec}\,(\mathrm{can}_f), \widetilde{\mathrm{can}_f})$ defines an isomorphism of $(\mathrm{Spec}\, R_f, \mathcal{O}_{\mathrm{Spec}\, R_f})$ with $(U, \mathcal{O}_U)$ (i.e., an isomorphism in $\mathbf{LRSp}$), where $U \subset \mathrm{Spec}\, R$ is the open subset $(\mathrm{Spec}\, R)_f$.

2. Local construction of morphisms in $\mathbf{LRSp}$: Let $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ be locally ringed spaces.
(a) Let $U \subset X$ be an open subset, $j : U \to X$ the inclusion. Define a morphism of sheaves of rings

$$j^{\#} : \mathcal{O}_X \to j_* \mathcal{O}_U$$

so that $(j, j^{\#})$ defines a morphism of locally ringed spaces

$$\tilde{j} := (j, j^{\#}) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X).$$

(b) For $V \subset U \subset X$ open subsets define

$$\mathrm{res}_{V,U} : \mathrm{Hom}_{\mathbf{LRSp}}((U, \mathcal{O}_U), \mathcal{Y}) \to \mathrm{Hom}_{\mathbf{LRSp}}((V, \mathcal{O}_V), \mathcal{Y})$$

to be $\tilde{j}^*_{V,U}$, where $j : V, U : V \to U$ is the inclusion, that is $\mathrm{res}_{V,U}(f) = f \circ \tilde{j}_{V,U}$. Show that sending U to $\mathrm{Hom}_{\mathbf{LRSp}}((U, \mathcal{O}_U), \mathcal{Y})$ and $V \subset U$ to $\mathrm{res}_{V,U}$ defines a sheaf of sets on X.
(c) Suppose we have a collection of open subsets $U_\alpha \subset X$ such that $X = \cup_\alpha U_\alpha$, and a collection of morphisms (of locally ringed spaces)

$$f_\alpha : (U_\alpha, \mathcal{O}_{U_\alpha}) \to \mathcal{Y}$$

such that

$$f_\alpha \circ \tilde{j}_{U_\alpha \cap U_\beta, U_\alpha} = f_\beta \circ \tilde{j}_{U_\alpha \cap U_\beta, U_\beta}$$

for all α, β. Show there is a unique morphism $f : \mathcal{X} \to \mathcal{Y}$ such that $f \circ \tilde{j}_\alpha = f_\alpha$, where $j_\alpha : U_\alpha \to X$ is the inclusion.

3. Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a scheme, and let $U \subset X$ be an open subset. Show that $(U, \mathcal{O}_U)$ is a scheme. *Hint*: First consider the case $\mathcal{X} = \mathrm{Spec}\, R$, using 1(b) to help. The scheme $(U, \mathcal{O}_U)$ is called an *open subscheme* of X

and if $(U, \mathcal{O}_U) \cong \operatorname{Spec} R$ for some ring R, it is called an *affine open subscheme* of X.
4. To simplify the notation, we will write $\operatorname{Spec} R$ for the locally ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. Let $\mathcal{X} := (X, \mathcal{O}_X)$ be a locally ringed space, R a commutative ring. The set of *R-valued points* of $\mathcal{X}$ is the set

$$\mathcal{X}(R) := \operatorname{Hom}_{LRSp}(\operatorname{Spec} R, \mathcal{X}).$$

(a) For a fixed locally ringed space $\mathcal{X}$, define an extension of the assignment $R \mapsto \mathcal{X}(R)$ to a functor

$$\mathcal{X}(-) : \mathbf{ComRing} \to \mathbf{Sets}.$$

Similarly, show how sending $\mathcal{X}$ to $\mathcal{X}(-)$ extends to a functor

$$R - pts : \mathbf{LRSp} \to \mathbf{Fun}(\mathbf{ComRing}, \mathbf{Sets}).$$

Here $\mathbf{Fun}(\mathcal{A}, \mathcal{B})$ is a category of functors $F : \mathcal{A} \to \mathcal{B}$, where a morphism of functors $\vartheta : F \to G$ is a natural transformation.
(c) Recall the *Yoneda lemma*: Let $\mathcal{A}$ be a category. For an object $a \in \mathcal{A}$, we have the so-called representable functor

$$h_a : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$$

$h_a = \operatorname{Hom}_{\mathcal{A}}(-, a)$, that is $h_a(b) = \operatorname{Hom}_{\mathcal{A}}(b, a)$ and for a morphism $f : b \to c$ in $\mathcal{A}$ $h_a(f) = f^* : \operatorname{Hom}_{\mathcal{A}}(c, a) \to \operatorname{Hom}_{\mathcal{A}}(b, a)$. For $g : a \to a'$ a morphism in $\mathcal{A}$, we have the natural transformation $g_* : h_a \to h_{a'}$, i.e., $(g_*)_b : h_a(b) \to h_{a'}(b)$ is the map

$$g_* : \operatorname{Hom}_{\mathcal{A}}(b, a) \to \operatorname{Hom}_{\mathcal{A}}(b, a').$$

The Yoneda lemma says that this map

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}(a, a') &\to \operatorname{Hom}_{\mathbf{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Sets})}(h_a, h_{a'}) \\ g &\mapsto g_* \end{aligned}$$

is a bijection. For A and B commutative rings, show that sending $\phi : A \to B$ to $(\operatorname{Spec} \phi)_*$ defines a bijection

$$\operatorname{Hom}_{\mathbf{ComRing}}(A, B) \to \operatorname{Hom}_{\mathbf{Fun}(\mathbf{ComRing}, \mathbf{Sets})}(\operatorname{Spec} B(-), \operatorname{Spec} A(-)).$$

(d) Let X and Y be schemes. Show that sending a morphism of schemes $f : X \to Y$ to the natural transformation $f_* : X(-) \to Y(-)$ defines a bijection

$$\operatorname{Hom}_{\mathbf{Sch}}(X, Y) \to \operatorname{Hom}_{\mathbf{Fun}(\mathbf{ComRing}, \mathbf{Sets})}(X(-), Y(-)).$$

In other words, a scheme X is determined by its functor $R \mapsto X(R)$. *Hint*: take a covering of X by affine open subschemes U_α and for each α, β a cover of $U_\alpha \cap U_\beta$ be affine open subschemes.
(e) Let R be a commutative ring. We have the canonical homomorphism $\phi_R : \mathbb{Z} \to R$ sending $1 \in \mathbb{Z}$ to $1 \in R$; let $f_R \in R[X_1, \ldots, X_n]$ be the image of f under the extension of ϕ_R to

$$\phi_R : \mathbb{Z}[X_1, \ldots, X_n] \to R[X_1, \ldots, X_n]$$

sending X_i to X_i. For $r_* := r_1, \ldots, r_n \in R$, we have the evaluation homomorphism $ev_{r_*} : \mathbb{Z}[X_1, \ldots, X_n] \to R$ sending X_i to r_i; for $f \in \mathbb{Z}[X_1, \ldots, X_n]$, write simply $f(r_1, \ldots, r_n)$ for $ev_{r_*}(f)$.

Let $I \subset \mathbb{Z}[X_1, \ldots, X_n]$ be an ideal and let $A = \mathbb{Z}[X_1, \ldots, X_n]/I$. Define a bijection of $\operatorname{Spec} A(R)$ with the subset $V_R(I) \subset R^n$

$$V_R(I) := \{(r_1, \ldots, r_n) \in R^n \mid f(r_1, \ldots, r_n) = 0 \text{ for all } f \in I\}.$$