Homework 7

Please turn these in on Monday, Dec. 7 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

1. Verify the following properties of proper morphisms:
(a) If $f : X \to Y$ is a proper morphism and T is an algebraic set, then $f \times \mathrm{Id}_T : X \times T \to Y \times T$ is a proper morphism.
(b) if $f : X \to Y$ and $g : Z \to W$ are proper morphisms, then $f \times g : X \times Z \to Y \times W$ is a proper morphism.
(c) a projective morphism is a proper morphism.

2. Verify the following properties of projective morphisms:
(a) A composition of projective morphisms is a projective morphism
(b) if $f : X \to Y$ and $g : Z \to W$ are projective morphisms, then $f \times g : X \times Z \to Y \times W$ is a projective morphism.

3. Fix positive integers $n_1, \ldots, n_m$. For each $j = 1, \ldots, m$, let $\{X_i^{(j)}\}$ be the set of variables $X_0^{(j)}, \ldots, X_{n_j}^{(j)}$, and consider the polynomial ring

$$k[X_*^{(1)}; \ldots; X_*^{(m)}] = k[\ldots, X_i^{(j)}, \ldots].$$

An element $h = h(X_*^{(1)}; \ldots; X_*^{(m)}) \in k[X_*^{(1)}; \ldots; X_*^{(m)}]$ has multi-degree $(d_1, \ldots, d_m)$ if for each $j = 1, \ldots, m$ and each $\lambda \in k$, one has

$$h(X_*^{(1)}; \ldots; \lambda \cdot X_*^{(j)}; \ldots; X_*^{(m)}) = \lambda^{d_j} h(X_*^{(1)}; \ldots; X_*^{(j)}; \ldots; X_*^{(m)}).$$

We let $k[X_*^{(1)}; \ldots; X_*^{(m)}]_{(d_*)}$ denote the subvector space of multi-degree $(d_*) = (d_1, \ldots, d_m)$ elements. Note that

$$k[X_*^{(1)}; \ldots; X_*^{(m)}] = \oplus_{(d_*)} k[X_*^{(1)}; \ldots; X_*^{(m)}]_{(d_*)}.$$

An ideal $I \subset k[X_*^{(1)}; \ldots; X_*^{(m)}]$ is *multi-homogeneous* if

$$I = \oplus_{(d_*)} I \cap k[X_*^{(1)}; \ldots; X_*^{(m)}]_{(d_*)}.$$

(a) Show that an ideal $I \subset k[X_*^{(1)}; \ldots; X_*^{(m)}]$ is multi-homogeneous if and only if there are multi-homogeneous elements $f_i \in I$, $i = 1, \ldots, r$, with $I = (f_1, \ldots, f_r)$.
(b) Let $\mathbb{P}^{n_1}(k) \times \ldots \times \mathbb{P}^{n_m}(k)$ denote the product (in Alg_k) of the algebraic sets $\mathbb{P}^{n_1}(k), \ldots, \mathbb{P}^{n_m}(k)$ (with projection maps $p_j : \mathbb{P}^{n_1}(k) \times \ldots \times \mathbb{P}^{n_m}(k) \to \mathbb{P}^{n_j}(k)$). For $f \in k[X_*^{(1)}; \ldots; X_*^{(m)}]_{(d_*)}$, let

$$V^{mh}(f) = \{(x_1, \ldots, x_m) \in \mathbb{P}^{n_1}(k) \times \ldots \times \mathbb{P}^{n_m}(k) \mid f(x_1, \ldots, x_m) = 0\}$$

Show that the set $V^{mh}(k)$ is a well-defined closed subset of $\mathbb{P}^{n_1}(k) \times \ldots \times \mathbb{P}^{n_m}(k)$. *Hint*: Consider the set of variables $Z_{i_1,\ldots,i_m}$, $0 \leq i_j \leq n_j$ and let $k[\ldots, Z_{i_*}, \ldots]$ be the corresponding polynomial ring. Let $\mathbb{P}_{n_1,\ldots,n_m}(k)$ be

the closed subset of $\mathbb{P}^N(k)$, $N = \prod_{j=1}^m (n_j+1) - 1$, defined by the ideal $I_{n_1,\ldots,n_m} \subset k[\ldots, Z_{i_*}, \ldots]$ generated by elements $Z_{i_*}Z_{j_*} - Z_{i'_*}Z_{j'_*}$ where the indices i_*, j_*, i'_*, j'_* satisfy $\{i_\ell, j_\ell\} = \{i'_\ell, j'_\ell\}$ for $1 \le \ell \le m$. Show that $\mathbb{P}^{n_1}(k) \times \ldots \times \mathbb{P}^{n_m}(k)$ is isomorphic to $\mathbb{P}_{n_1,\ldots,n_m}(k)$.
(c) For $I \subset k[X_*^{(1)}; \ldots; X_*^{(m)}]$ a multi-homogeneous ideal, let $V^{mh}(I) = \cap_f V^{mh}(f)$, as f runs over the multi-homogeneous elements of I. Show that a subset $C \subset \mathbb{P}^{n_1}(k) \times \ldots \times \mathbb{P}^{n_m}(k)$ is a closed subset if and only if there is a multi-homogeneous ideal $I \subset k[X_*^{(1)}; \ldots; X_*^{(m)}]$ with $C = V^{mh}(I)$.

4. Fix positive integers n and d. A degree d irreducible hypersurface in $\mathbb{P}^n(k)$ is a closed algebraic subset $X \subset \mathbb{P}^n(k)$ of the form $X = V^h((f))$ for some element $f \in k[X_0, \ldots, X_n]_d$, with f irreducible. A degree d (effective) divisor in $\mathbb{P}^n(k)$ is a formal finite sum $\sum_{i=1}^r n_i X_i$ where the n_i are positive integers, $X_1, \ldots, X_r$ are distinct irreducible hypersurfaces, with X_i of degree d_i, and $\sum_i n_i d_i = d$. Show that the set of degree d divisors in $\mathbb{P}^n(k)$ is in bijection with $\{f \in k[X_0, \ldots, X_n]_d | f \neq 0\}/\sim$, with $f \sim g$ if $g = \lambda f$ for some $\lambda \in k \setminus \{0\}$. This set in turn is $\mathbb{P}^{N_{n,d}}(k)$, where $N_{n,d} = \dim_k k[X_0, \ldots, X_n]_d - 1$. In fact,

$$N_{n,d} = \binom{n+d}{n} - 1.$$

A representative f for a class in $\{f \in k[X_0, \ldots, X_n]_d | f \neq 0\}/\sim$ is called a *defining equation* for the corresponding divisor X; the support of X is the closed algebraic subset $\mathrm{supp}\,(X) := V^h(f)$.
(b) Let $I_{n,d} \subset \mathbb{P}^n(k) \times \mathbb{P}^{N_{n,d}}(k)$ be the set of pairs (p, X) such that $p \in \mathrm{supp}\,(X)$. Show that $I_{n,d}$ is a closed algebraic subset of $\mathbb{P}^n(k) \times \mathbb{P}^{N_{n,d}}(k)$, in fact $I_{n,d} = V^{bh}(h)$ for h a polynomial of bi-degree $(d, 1)$.
(c) Let p be a point of $\mathbb{P}^n(k)$. Show that the set S of degree d divisors $X \in \mathbb{P}^{N_{n,d}}(k)$ with $p \in \mathrm{supp}\,(X)$ is a hyperplane (that is, a hypersurface of degree one) in $\mathbb{P}^{N_{n,d}}(k)$, and that $p \times S = I_{n,d} \cap p \times \mathbb{P}^{N_{n,d}}(k)$.
(d) Let $C \subset \mathbb{P}^n(k)$ be a closed algebraic subset. Show that the set of degree d divisors $X \subset \mathbb{P}^n(k)$ such that $C \subset \mathrm{supp}\,(X)$ forms a closed subset of $\mathbb{P}^{N_{n,d}}(k)$.

5. Fix a further integer m. Let $I_{n,d,m} \subset \mathbb{P}^n(k) \times \mathbb{P}^{N_{n,d}}(k)^m$ be the set of tuples $(p, X_1, \ldots, X_m)$ such that $p \in \cap_{i=1}^m \mathrm{supp}\,(X_i)$.
(a) Show that $I_{n,d,m}$ is a closed algebraic subset of $\mathbb{P}^n(k) \times \mathbb{P}^{N_{n,d}}(k)^m$.
(b) Let $C \subset \mathbb{P}^n(k)$ be a closed algebraic subset. Show that the set of m-tuples $(X_1, \ldots, X_m)$, with X_i a degree d divisor in $\mathbb{P}^n(k)$, and with $C \cap \cap_{i=1} m \mathrm{supp}\,(X_i) \neq \emptyset$, is a closed algebraic subset of $\mathbb{P}^{N_{n,d}}(k)^m$. *Hint*: 1st show that

$$\{(p, X_1, \ldots, X_m) \mid p \in C \cap \cap_{i=1}^m \mathrm{supp}\,(X_i)\}$$

is a closed subset of $\mathbb{P}^n(k) \times \mathbb{P}^{N_{n,d}}(k)^m$.
(c) Let $C \subset \mathbb{P}^{N,d_1}(k) \times \ldots \times \mathbb{P}^{N,d_r}(k)$ be a closed subset. Show that the set of r-tuples of divisors

$$\{(X_1, \ldots, X_r) \in C \mid \cap_{i=1}^r \operatorname{supp}(X_i) \neq \emptyset\}$$

is a closed subset of C. *Hint*: use the hint in (b).

6. Define as for $\mathbb{P}^n(k)$ the irreducible hypersurfaces $X \subset k^n$ as the algebraic closed subsets $V(f)$ for $f \in k[x_1, \ldots, x_n]$ with f irreducible, and a divisor X in k^n as a finite formal sum $\sum_{i=1}^r n_i X_i$, $n_i > 0$, $X_1, \ldots, X_r$ distinct irreducible hypersurfaces. If we write $X_i = V(f_i)$, the polynomial $f = \prod_i f_i^{n_i}$ is called a defining equation for X (uniquely defined up to a non-zero constant) and $\operatorname{supp}(X) := V(f)$. A point $x \in \operatorname{supp}(X)$ is called a *smooth point* of the divisor X if X has defining equation f and $(\partial f/\partial x_i)(x) \neq 0$ for some $i = 1, \ldots, n$. $x \in \operatorname{supp}(X)$ is a *singular point* if x is not a smooth point.
(a) Let $X \subset \mathbb{P}^n(k)$ be a degree d divisor with defining equation f. For each i, we have the open subset $U_i \subset \mathbb{P}^n(k)$, $U_i = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n(k) | x_i \neq 0\}$, which we identify with k^n by the bijection

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_n)$$

Dehomogenizing f with respect to X_i gives us the polynomial

$$f^{(i)} \in k[x_1, \ldots, x_n],$$

with $x_j = X_{j-1}$ for $j = 1, \ldots i$, $x_j = X_j$ for $j = i+1, \ldots, n$, and we let $X_i := X \cap U_i$ be the divisor with defining equation $f^{(i)}$ Call a point $x \in \operatorname{supp}(X)$ a smooth/singular point of X if x is a smooth/singular point of X_i for some i. Show that $x \in X$ is a singular point if and only if $(\partial f/\partial X_j)(x) = 0$ for all $j = 0, \ldots, n$. Conclude that if $x \in X$ is in $U_i \cap U_j$, then x is a singular point of X_i if and only if x is a singular point of X_j. *Hint*: first show that for $f \in k[X_0, \ldots, X_n]_d$, we have

$$\sum_{j=0}^{n} X_j \cdot (\partial f/\partial X_j) = d \cdot f,$$

so if $f(x) = 0$ and $(\partial f/\partial X_j)(x) = 0$ for all $j \neq i$, then $x_i \cdot (\partial f/\partial X_i)(x) = 0$.
(b) Take $p \in \mathbb{P}^n(k)$ Show that the set of degree d divisors X such that $p \in \operatorname{supp}(X)$ is a singular point of X is a closed subset of $\mathbb{P}^{N_{n,d}}(k)$, in fact, a *linear subset*, that is, defined by linear homogeneous equations.
(c) Call a divisor X in $\mathbb{P}^n(k)$ *singular* if there is a singular point in $\operatorname{supp}(X)$. Show that the set of singular divisors of degree d in $\mathbb{P}^n(k)$ forms a closed subset of $\mathbb{P}^{N_{n,d}}(k)$.
Remark: This set is actually an hypersurface, called the *discriminant locus*.