Homework 6

Please turn these in on Monday, Nov. 30 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

1. Let X be an affine algebraic set. Let Y be one of the following algebraic sets: $\mathbb{P}^n(k)$, $n \geq 1$, $\mathbb{P}^n(k) \setminus \{[0 : \ldots : 0 : 1]\}$, $n \geq 2$. Let $f : Y \to X$ be a morphism. Show that $f(x) = f(y)$ for all $x, y \in Y$.

2. Let M be an R-module, $f_1, \ldots, f_n \in R$ that generate the unit ideal. We write M_i for the localization M_{f_i}, M_{ij} for the localization $M_{f_i f_j}$ and M_{ijk} for the localization $M_{f_i f_j f_k}$. Suppose we have elements $m_{ij} \in M_{ij}$, $1 \leq i, j \leq n$, such that for all triples (i, j, k), $1 \leq i, j, k \leq n$, we have

$$x_{jk} - x_{ik} + x_{ij} = 0 \qquad (*)$$

in M_{ijk}. Show there are elements $x_i \in M_i$, $i = 1, \ldots, n$, such that $x_{ij} = x_j - x_i$, $1 \leq i, j \leq n$. *Hint*: 1st note that there is an $N >> 0$ and $a_{ij} \in M_i$ such that the $f_j^N x_{ij} = a_{ij}/1$ in M_{ij} and

$$a_{jk}/1 - a_{ik}/1 + f_k^N x_{ij} = 0$$

in M_{ij}. Then use an argument like in HW 4, problem 4(b).

3. Let R be a commutative ring, M a flat R module. Let

$$0 \to M'' \to M' \to M \to 0$$

be an exact sequence of R-modules, and let N be an R-module. Show that

$$0 \to M'' \otimes_R N \to M' \otimes_R N \to M \otimes_R N \to 0$$

is exact. *Hint*: Let

$$\cdots \to F_2 \to F_1 \to F_0 \to N \to 0$$

be a free resolution of N. Show that

$$0 \to M'' \otimes_R F_n \to M' \otimes_R F_n \to M \otimes_R F_n \to 0$$

is exact for all n, and that

$$\cdots \to M \otimes_R F_2 \to M \otimes_R F_1 \to M \otimes_R F_0 \to M \otimes_R N \to 0$$

$$M' \otimes_R F_1 \to M' \otimes_R F_0 \to M' \otimes_R N \to 0$$

and

$$M'' \otimes_R F_1 \to M'' \otimes_R F_0 \to M'' \otimes_R N \to 0$$

are all exact. Then do a diagram chase on the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & M' \otimes_R F_2 & \longrightarrow & M \otimes_R F_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'' \otimes_R F_1 & \longrightarrow & M' \otimes_R F_1 & \longrightarrow & M \otimes_R F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'' \otimes_R F_0 & \longrightarrow & M' \otimes_R F_0 & \longrightarrow & M \otimes_R F_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M'' \otimes_R N & \longrightarrow & M' \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

4. Let $(R, \mathfrak{m})$ be a noetherian local ring, M a finitely generated R-module.
(a) Show that M is a flat R-module if and only if M is a free R-module. *Hint*: Let $k = R/\mathfrak{m}$. Lift a k-basis of $M \otimes_R k$ to generators $m_1, \ldots, m_n$ for M, i.e., a surjection $R^n \to M$, giving the exact sequence

$$0 \to N \to R^n \to M \to 0$$

Then use (3) and Nakayama's lemma.
(b) Let A be a noetherian ring, N a finitely generated A-module. Show that N is flat if and only if N is projective. *Hint*: Suppose M is flat. Use (a) to show that there are $f_1, \ldots, f_n$ in A, generating the unit ideal, such that N_{f_i} is free for each i. Then consider a surjection $\pi : A^m \to N$. Let $N' := \ker \pi$ and consider the A-module $H := Hom_A(N, N')$. Chose A_{f_i}-linear maps $g_i : N_{f_i} \to A^m_{f_i}$ with $\pi \circ g_i = \mathrm{Id}_{N_{f_i}}$ and let $g_{ij} = g_j/1 - g_i/1$ in $Hom_A(N_{f_i f_j}, N'_{f_i f_j}) = H_{f_i f_j}$. Then use (2).

4. Let $S_{n,d}$ be the set of indices $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $\sum_j i_j = d$, let $N_{n,d} = \#S_{n,d} - 1$. (Note that $\#S_{n,d} = \binom{n+2}{d+1}$). For $I = (i_0, \ldots, i_n) \in S_{n,d}$, let X^I denote the monomial $X_0^{i_0} \cdots X_n^{i_n}$. Let $\{Z_I \mid I \in S_{n,d}\}$ be a set of variables indexed by $S_{n,d}$, and let $k[\ldots Z_I \ldots]$ be the polyonomial ring on the Z_I. Let

$$\phi^*_{n,d} : k[\ldots Z_I \ldots] \to k[X_0, \ldots, X_n]$$

be the k-algebra homomorphism with $\phi^*_{n,d}(Z_I) = X^I$.
(a) Show that $I_{n,d} := \ker \phi^*_{n,d}$ is a homogeneous ideal, and that $\prod_{I_i} Z_{I_i} - \prod_{J_j} Z_{J_j}$ is in $I_{n,d}$ if $\sum_i I_i = \sum_j J_j$ in $\mathbb{N}^n$. Let $X_{n,d} = V^h(I_{n,d}) \subset \mathbb{P}^N(k)$.
(b) Let $\phi_{n,d} : \mathbb{P}^n(k) \to \mathbb{P}^N(k)$ be the map of sets

$$\phi_{n,d}([x_0 : \ldots : x_n]) = [\ldots : \phi_{n,d}([x_0 : \ldots : x_n])_I : \ldots]$$

with $\phi_{n,d}([x_0 : \ldots : x_n])_I = x^I$. Show that $\phi_{n,d}(\mathbb{P}^n(k)) \subset X_{n,d}$ and that $\phi_{n,d}$ is injective (as a map of sets).
(c) Show that the restriction of $\phi_{n,d}$ to $U_j \subset \mathbb{P}^n(k)$ gives an isomorphism of U_j with $X_{n,d} \cap U_I$, with $I = (i_0, \ldots, i_n)$, $i_j = d$, $i_\ell = 0$ for $\ell \neq j$. Conclude that $\phi_{n,d} : \mathbb{P}^n(k) \to X_{n,d}$ is an isomorphism of algebraic sets.
(d) Let $f \in k[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree $d > 0$. Show that the open subset $\mathbb{P}^n(k)_f$ of $\mathbb{P}^n(k)$ is affine. *Hint* Write $f = \sum_{I \in S_{n,d}} a_I X^I$, and show that $\mathbb{P}^n(k)_f$ is isomorphic to $X_{n,d} \setminus V(\sum_I a_I Z_I)$.

5. Let X and Y be affine algebraic sets. We have defined in class the product $X \times Y$ in algebraic sets, which has as set of points the product of sets $X \times Y$.
(a) for $U \subset X$, $V \subset Y$ open subsets, show that the subset $U \times V$ of $X \times Y$ is an open subset.
(b) Show by examples that the Zariski topology on $X \times Y$ is strictly finer that the product topology, that is, there is an open subset $W \subset X \times Y$ that is not a union of product open subsets $U \times V$, $U \subset X$, $V \subset Y$ open. *Hint*: consider the case $X = Y = k^1$.