Homework 4

Please turn these in on Monday, Nov. 16 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

1. Let T be a topological space and let $\mathcal{B}$ be a basis of open sets for T. We have shown in class that for each presheaf P on $\mathcal{B}$ that satisfies the sheaf axiom for $\mathcal{B}$, there is a sheaf $\tilde{P}$ on T and an isomorphism $\alpha : \tilde{P}_{|\mathcal{B}} \to P$. Moreover, the pair $(\tilde{P}, \alpha)$ is unique up to unique isomorphism. We call $(\tilde{P}, \alpha)$ an extension of P to a sheaf on T.
(a) Let P and Q be presheaves on $\mathcal{B}$, both satisfying the sheaf axiom for $\mathcal{B}$, and let $f : P \to Q$ be a morphism of presheaves. Let $(\tilde{P}, \alpha)$, $(\tilde{Q}, \beta)$ be extensions of P, Q to sheaves on T. Show there is a unique morphism of sheaves $\tilde{f} : \tilde{P} \to \tilde{Q}$ such that $f = \beta \circ \tilde{f}_{|\mathcal{B}} \circ \alpha^{-1}$.
(b) Define the category $\mathbf{Sh}(\mathcal{B})$ to be the full subcategory of the category $\mathbf{Psh}(\mathcal{B})$ of presheaves on $\mathcal{B}$ with objects the presheaves on $\mathcal{B}$ which satisfy the sheaf axiom for $\mathcal{B}$ (that is, the morphisms are just the morphisms of presheaves on $\mathcal{B}$). We call an object of $\mathbf{Sh}(\mathcal{B})$ a *sheaf on* $\mathcal{B}$ for short. Show that the restriction functor $\mathrm{res}_{\mathcal{B}} : \mathbf{Psh}(T) \to \mathbf{Psh}(\mathcal{B})$ sends sheaves on $\mathcal{T}$ to sheaves on $\mathcal{B}$ and defines an equivalence of categories $\mathrm{res}_{\mathcal{B}} : \mathbf{Sh}(T) \to \mathbf{Sh}(\mathcal{B})$.

2. Let R be a commutative ring. We have the set of prime ideals in R, $\operatorname{Spec} R$, with the Zariski topology, where the closed subsets are subsets of the form $V(I) := \{\mathfrak{P} \in \operatorname{Spec} R \mid \mathfrak{P} \supset I, I\}$, for $I \subset R$ an ideal.
(a) Let $X = \operatorname{Spec} R$. A *principle open subset* of X is a subset of the form $X_g := X \setminus V((g))$ for $g \in R$. Show that the set $\mathcal{B}_X$ of principle open subsets of X form a basis for the Zariski topology on X.
(b) Take $g \in R$, let $S_g := \{g^n \mid n = 0, 1, \ldots\}$ and let R_g denote the localization $S_g^{-1}R$. Giving X_g the induced topology, show that X_g is homeomorphic to $\operatorname{Spec} R_g$ (with the Zariski topology).
(c) Show that for $g, h \in R$, $X_h \subset X_g$ if and only if $h^n = ag$ for some $a \in R$ if and only if $g/1 \in R_h$ is invertible. Show that, if $X_h \subset X_g$, there is a unique ring homomorphism $\rho_{h,g} : R_g \to R_h$ making the diagram

$$\begin{tikzcd} R \arrow[r, "\rho_h"] \arrow[d, "\rho_g"'] & R_h \\ R_g \arrow[ur, "\rho_{h,g}"'] & \end{tikzcd}$$

commute, where $\rho_g : R \to R_g$, $\rho_h : R \to R_h$ are the localization homomorphisms.
(d) Show that there is a sheaf of rings $\mathcal{O}_X$ on X with $\mathcal{O}_X(X_g) = R_g$ and with $\mathrm{res}_{X_h, X_g} = \rho_{h,g}$ when $X_h \subset X_g$. Show that these properties characterize $\mathcal{O}_X$ up to unique isomorphism: if $\mathcal{O}'_X$ is another such sheaf of rings,

there is a unique isomorphism $\alpha : \mathcal{O}_X \to \mathcal{O}'_X$ of sheaves of rings such that α_{X_g} is identity on R_g for all $g \in R$.

3. Let T be a topological space and $\mathcal{O}_T$ a presheaf of rings on T. A presheaf of $\mathcal{O}_T$-modules is a presheaf of abelian groups $\mathcal{M}$ on T together with a map of presheaves (of sets) $m : \mathcal{O}_T \times \mathcal{M} \to \mathcal{M}$ such that, for each $U \subset T$ open, the map $m_U : \mathcal{O}_T(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$ defines a multiplication making $\mathcal{M}(U)$ into an $\mathcal{O}_T(U)$-module. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}_T$-modules, and $\mathcal{O}_T$-module morphism is a morphism $f : \mathcal{M} \to \mathcal{N}$ of presheaves of abelian groups.

If $\mathcal{O}_T$ is a sheaf of rings, a sheaf of $\mathcal{O}_T$-modules is a presheaf of $\mathcal{O}_T$-modules $\mathcal{M}$ such that as a presheaf of abelian groups, $\mathcal{M}$ is a sheaf; morphisms of sheaves of $\mathcal{O}_T$ modules are morphisms of presheaves. The category of sheaves of $\mathcal{O}_T$-modules is denoted $\mathcal{O}_T - \mathbf{Mod}$ and the category of presheaves of $\mathcal{O}_T$-modules is denoted $\mathcal{O}_T - \mathbf{Mod}^p$.
(a) Show that the sheafification functor for presheaves of abelian groups defines a functor $a_{\mathcal{O}_T} : \mathcal{O}_T - \mathbf{Mod}^p \to \mathcal{O}_T - \mathbf{Mod}$, left adjoint to the inclusion functor $i : \mathcal{O}_T - \mathbf{Mod} \to \mathcal{O}_T - \mathbf{Mod}^p$.
(b) Show that $\mathcal{O}_T - \mathbf{Mod}^p$ is an abelian category, where the kernel and cokernel of a morphism $f : \mathcal{M} \to \mathcal{N}$ is given by $(\ker^p f)(U) = \ker(f_U : \mathcal{M}(U) \to \mathcal{N}(U))$, $(\operatorname{coker}^p f)(U) = \operatorname{coker}(f_U : \mathcal{M}(U) \to \mathcal{N}(U))$ for every $U \subset T$ open.
(c) Show that $\mathcal{O}_T - \mathbf{Mod}$ is an abelian category, where for a morphism $f : \mathcal{M} \to \mathcal{N}$, the kernel and cokernel of f are given by $\ker f = \ker^p f$ and $\operatorname{coker} f = a_{\mathcal{O}_T}(\operatorname{coker}^p f)$.

4. (a) Let R be a commutative ring, M an R-module, $X = \operatorname{Spec} R$. For $g \in R$, let M_g be the localization $S_g^{-1}M$. M_g is thus an R_g-module and via the localization map $\rho_g : R \to R_g$, M_g is also an R-module. Show that for each $g, h \in R$ with $X_h \subset X_g$ there is a unique R-module homomorphism $\phi_{h,g} : M_g \to M_h$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\phi_h"] \arrow[d, "\phi_g"'] & M_h \\ M_g \arrow[ur, "\phi_{h,g}"'] & \end{tikzcd}$$

commute, where $\phi_g : M \to M_g$, $\phi_h : M \to M_h$ are the localization homomorphisms.
(b) Let $\{f_\alpha \mid \alpha \in A\}$ be a collection of elements of R which generate the unit ideal, and let M be an R-module. Show that the sequence

$$M \xrightarrow{\text{res}} \prod_{\alpha \in A} M_{f_\alpha} \underset{\text{res}^2}{\overset{\text{res}^1}{\rightrightarrows}} \prod_{\alpha, \beta \in A} M_{f_\alpha f_\beta}$$

is exact. Here the maps are defined by $\mathrm{res}(m)_\alpha = \phi_{f_\alpha}(m)$, $\mathrm{res}^1((m_\alpha))_{\alpha,\beta} = \phi_{f_\alpha f_\beta, f_\alpha}(m_\alpha)$ and $\mathrm{res}^2((m_\alpha))_{\alpha,\beta} = \phi_{f_\alpha f_\beta, f_\beta}(m_\beta)$.
(c) Show that there is a sheaf of $\mathcal{O}_X$-modules $(\tilde{M}, m : \mathcal{O}_X \times \tilde{M} \to \tilde{M})$ with $\tilde{M}(X_g) = M_g$ and with $m_{X_g} : \mathcal{O}_X(X_g) \times \tilde{M}(X_g) \to \tilde{M}(X_g)$ the multiplication $R_g \times M_g \to M_g$ that makes M_g an R_g-module.
(d) Show that the assignment $M \mapsto \tilde{M}$ extends to a functor $\sim: R-\mathbf{Mod} \to \mathcal{O}_X - \mathbf{Mod}$
(e) Show that the functor $\sim$ is fully faithful. Is $\sim$ an equivalence of categories?

5. Let $U \subset k^n$ be the open subset $k^n \setminus \{0\}$. Show that the restriction map $\mathcal{O}_{k^n}(k^n) \to \mathcal{O}_{k^n}(U)$ is an isomorphism for $n \geq 2$, and is not an isomorphism for $n = 1$.

6. Let $X \subset \mathbb{P}^n(k)$ be a closed algebraic subset, and let $\{f_\alpha \mid \alpha \in A\}$ be a set of homogeneous elements in $k[X]$.
(a) For $f, g \in k[X]$ homogeneous, show that $X_g \subset X_f$ if and only if $g^n = af$ for some $n > 0$ and some homogeneous $a \in k[X]$.
(b) Take $f \in k[X]$ homogeneous and assume that $X_{f_\alpha} \subset X_f$ for each α. Let $d_\alpha = \deg f_\alpha$, $d = \deg f$. Show that $X_f = \cup_\alpha X_{f_\alpha}$ if and only if $\{f_\alpha^d / f^{d_\alpha}, \alpha \in A\}$ generates the unit ideal in $k[X]_f^0$.