Homework 3

Please turn these in on Monday, Nov. 9 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

First some background. Recall that a category $\mathcal{C}$ is an *additive* category if

(i) The Hom sets are abelian groups, and the composition $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is bilinear.
(ii) There is an object 0 which is both initial and terminal
(ii) For objects X and Y of $\mathcal{C}$, a bi-product $X \oplus Y$ exists. This means: there is an object $X \oplus Y$ with morphisms $i_X : X \to X \oplus Y$, $i_Y : Y \to X \oplus Y$, $p_X : X \oplus Y \to X$, $p_Y : X \oplus Y \to Y$ such that $p_X i_X = \mathrm{Id}_X$, $p_Y i_Y = \mathrm{Id}_Y$, $p_Y i_X = 0$, $p_X i_Y = 0$ and $i_X p_X + i_Y p_Y = \mathrm{Id}_{X \oplus Y}$.

Note that this implies that $(X \oplus Y, i_X, i_Y)$ is a categorical co-product and $(X \oplus Y, p_X, p_Y)$ is a categorical product. If $\mathcal{C}$ and $\mathcal{D}$ are additive categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is additive if for each pair of objects in $\mathcal{C}$, the map on the Hom sets given by F is a group homomorphism. This implies that F is compatible with bi-products.

An additive category $\mathcal{C}$ is an *abelian category* if for each morphism $f : X \to Y$, $\ker f$ and $\operatorname{coker} f$ both exist and the canonical morphism $\alpha : \operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ is an isomorphism. Recall that $\ker f$ is really a morphism $i : \ker f \to X$ such that for all objects Z, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z, \ker f) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{C}}(Z, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(Z, Y)$$

is an exact sequence of abelian groups, and $\operatorname{coker} f$ is really a morphism $j : B \to \operatorname{coker} f$ such that such that for all objects Z, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\operatorname{coker} f, Z) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is an exact sequence of abelian groups. Letting $p : X \to \operatorname{coker}(\ker f)$ and $q : \ker(\operatorname{coker} f) \to Y$ be the canonical morphisms, the morphism α is the unique morphism with $q \circ \alpha \circ p = f$.

The morphism $p : X \to \operatorname{coker}(\ker f)$ is called the *co-image* of f and $q : \ker(\operatorname{coker} f) \to Y$ is called the *image* of f (even if α is not an isomorphism).

We fix a topological space T.

1. Let $\mathcal{C}$ be an additive category. Show that the category of presheaves $\mathbf{Psh}^{\mathcal{C}}(T)$ is an additive category where:

(i) for $f, g : P \to Q$ morphisms of presheaves, $f \pm g : P \to Q$ is the morphism with $(f \pm g)_U : P(U) \to Q(U)$ the map $f_U \pm g_U$, for each open $U \subset T$.
(ii) For presheaves P, Q, the biproduct is given by $(P \oplus Q)(U) := P(U) \oplus Q(U)$. You need to define the restriction maps res_{VU}, the maps i_P, i_Q, p_P, p_Q: these are all induced by the restriction maps for P and Q and the maps $i_{P(U)}, i_{Q(U)}, p_{P(U)}, p_{Q(U)}$ for $U \subset T$ open.

2. Show that the sheaf category $\mathbf{Sh}^{\mathcal{C}}(T)$ is an additive subcategory of $\mathbf{Psh}^{\mathcal{C}}(T)$, i.e., the bi-product of sheaves is a sheaf and the initial object 0 is a sheaf.

3. Let $\mathcal{A}$ be an abelian category. Show that the presheaf category $\mathbf{Psh}^{\mathcal{C}}(T)$ is an abelian category, where for a morphism $f : P \to Q$ of presheaves, the presheaf kernel $i : \ker^p f \to P$ evaluated at an open $U \subset T$ is the kernel of $f_U : P(U) \to Q(U)$, and similarly for the presheaf cokernel $j : Q \to \mathrm{coker}^p f$.

4. Show that the sheaf category $\mathbf{Sh}^{\mathcal{A}}(T)$ is an abelian category, where for $f : P \to Q$
(i) The sheaf kernel $\ker f$ is the presheaf kernel $\ker^p f$.
(ii) The sheaf cokernel $\mathrm{coker}\, f$ is the sheafification of the presheaf cokernel $\mathrm{coker}^p f$.
Hint: For (i) the main point is to show that $\ker^p f$ is a sheaf. For (ii) use the universal property of sheafification.

5. In this problem we work is the categories of presheaves/sheaves of abelian groups.
(i) for S a sheaf, $U \subset T$ open, $s \in S(U)$, then $s = 0$ if and only if $s_x = 0$ in S_x for all $x \in U$.
(ii) Show that a morphism of sheaves $f : S' \to S$ is zero if and only if $f_x : S'_x \to S_x$ is zero for all $x \in T$. Conclude that a sheaf S is isomorphic to the 0-sheaf if and only if $S_x = 0$ for all $x \in T$.
(iii) Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence of presheaves. Show that $0 \to P'_x \to P_x \to P''_x \to 0$ is an exact sequence of abelian groups for all $x \in T$.
(iv) If P is a presheaf, $P \to \alpha P$ the sheafification, then $P_x \to (\alpha P)_x$ is an isomorphism for all $x \in T$.
(v) If $0 \to S' \to S \to S'' \to 0$ is an exact sequence of sheaves, then $0 \to S'_x \to S_x \to S''_x \to 0$ is an exact sequence of abelian groups for all

$x \in T$. Conclude that a morphism of sheaves $S' \to S$ is a monomorphism/epimorphism/isomorphism if and only if $S'_x \to S_x$ is a monomorphism/epimorphism/isomorphism for all $x \in T$.
(vi) Show that a sequence of sheaves $P \to Q \to R$ is exact if and only if for all $x \in T$, the sequence of stalks $P_x \to Q_x \to R_x$ is an exact sequence of abelian groups.

6. Give an example of an morphism of sheaves of abelian groups on the circle S^1 for which the presheaf cokernel is not a sheaf. *Hint*: Let $\mathcal{C}_{S^1}$ be the sheaf of continuous real-valued functions on S^1, $i : \mathbb{Z}_{S^1} \to \mathcal{C}_{S^1}$ the subsheaf of $\mathbb{Z}$-valued functions. Show that $\operatorname{coker} i$ is isomorphic to the sheaf of continuous $S^1 :== \mathbb{R}/\mathbb{Z}$-valued maps (with group law induced by the addition in S^1). Then try to lift the identity map $S^1 \to S^1$ to an element of $\mathcal{C}_{S^1}(S^1)$.

7. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called *exact* if F preserves kernels and cokernels.
(i) Give an example of a topological space T such that the inclusion functor $i : \mathbf{Sh}^{\mathbf{Ab}}(T) \to \mathbf{Psh}^{\mathbf{Ab}}(T)$ is not exact. *Hint*: see problem (6).
(ii) Show that the sheafification functor $\alpha : \mathbf{Psh}^{\mathbf{Ab}}(T) \to \mathbf{Sh}^{\mathbf{Ab}}(T)$ is exact. *Hint*: Use (4).