Homework 2

Please turn these in on Monday, Nov. 2 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

Fix an algebraically closed field k.

1. Just as for k^n, a hypersurface H in $\mathbb{P}^n(k)$ is defined to be a closed algebraic subset with $I_h(H) = (f)$ for some non-constant homogeneous polynomial $f \in k[x_0, \ldots, x_n]$. The polynomial f is called a defining equation for H.
(a) Show that a defining equation for a hypersurface H is uniquely determined up to multiplication by a $\lambda \in k^\times$.
(b) Show that a hypersurface H in $\mathbb{P}^n(k)$ is irreducible if and only if its defining equation is an irreducible polynomial, and that each hypersurface is uniquely the union of finitely many irreducible hypersurfaces.
(c) Let $H_1, \ldots, H_r$ be hypersurfaces in $\mathbb{P}^n(k)$, $r \leq n$. Show that $H_1 \cap \ldots \cap H_r \neq \emptyset$. *Hint*: If $f_1, \ldots, f_r \in k[x_0, \ldots, x_n]$ are homogeneous polynomials of degree > 0, then $(f_1, \ldots, f_r) \subset (x_0, \ldots, x_n)$. Then use Krull's dimension theorem. *Note*: You might view this result as a generalization of the well-known theorem in linear algebra: let $L_1, \ldots, L_r$ be linear homogeneous polynomials in $x_0, \ldots, x_n$ with $r \leq n$. Then the system of equations $L_1 = \ldots = L_r = 0$ has a solution $x \neq 0$.
(d) Let $A \subset \mathbb{P}^n(k)$ be a non-empty closed algebraic subset; $A = V_h(I)$ for some homogeneous ideal $I \subset k[x_0, \ldots, x_n]$. Define the dimension of A by

$$\dim A := \dim C(A) - 1$$

where $C(A)$ is the cone, $C(A) := V(I) \subset k^{n+1}$. Suppose that $\dim A > 0$ and let $H \subset \mathbb{P}^n(k)$ be a hypersurface. Show that $A \cap H \neq \emptyset$ and that $\dim A - 1 \leq \dim(A \cap H) \leq \dim A$. Conclude that $A \cap H_1 \cap \ldots \cap H_r \neq \emptyset$ for hypersurfaces $H_1, \ldots, H_r$ if $r \leq \dim A$. *Hint*: Consider the case of irreducible A.

2. Let $S \subset k^n$ be a closed algebraic subset. A *closed algebraic subset* of S is a subset $S' \subset S$ which as a subset of k^n is a closed algebraic subset of k^n; an irreducible closed algebraic subset of S is a closed algebraic subset S' of S such that, if $S' = S'_1 \cup S'_2$ with each S'_i a closed algebraic subset of S, then $S' = S'_1$ or $S' = S'_2$.
(a) Let T be a closed algebraic subset of k^n. Show that $S \cap T$ is a closed algebraic subset of S.
(b) Show that a closed algebraic subset S' of S is an irreducible closed algebraic subset of S if and only if S' is an irreducible closed algebraic subset of k^n.

(c) Define a bijection

$$\{\text{closed algebraic subsets of } S\} \leftrightarrow \{J \subset k[S] \mid J \text{ is a radical ideal}\}$$

and show that this yields a bijection between the set of irreducible closed algebraic subsets of S and the set of prime ideals in $k[S]$.

3. Let $A \subset k^n$ be a non-empty closed algebraic subset and let $f \in k[x_1, \ldots, x_n]$ be a non-zero polynomial.
(a) Show that there is a closed algebraic subset $A_f \subset k^{n+1}$ and a morphism $f : A_f \to A$ that defines a bijection of A_f with the subset $\{x \in A \mid f(x) \neq 0\}$ of A. Show that $k[A_f]$ is isomorphic to the localization $S_f^{-1}k[A]$, where $S_f = \{f^n \mid n = 0, 1, \ldots\}$. *Hint*: consider the ideal J in $k[x_1, \ldots, x_{n+1}]$ generated by $I(A)$ and $1 - x_{n+1} \cdot t$ and show that $k[x_1, \ldots, x_{n+1}]/J \cong S_f^{-1}k[A]$.
(b) Let $A \subset k^n$ and $B \subset k^m$ be closed algebraic subsets and let $F : A \to B$ be a morphism. Let $f \in k[x_1, \ldots, x_n]$ and $g \in k[y_1, \ldots, y_m]$ be non-zero polynomials and let $U := A \setminus V((f))$, $V := B \setminus V((g))$. From (a), we identify U with a closed algebraic subset of k^{n+1} and V with a closed algebraic subset of k^{m+1}. Suppose that F restricts to a map of sets $G : U \to V$. Show that, under the above identifications, G is a morphism.

4. Let $C \subset k^2$ be the closed algebraic subset $V((x_2^2 - x_1^3))$.
(a) Show that C is irreducible.
(b) Show that the pair (x^2, x^3) represents a morphism $f : k^1 \to C$, which, as a map of sets, is a bijection.
(c) Show that the morphism f in (a) does not admit an inverse morphism $g : C \to k^1$. *Hint*: Consider the image of f^*.
(d) From (3), we may consider the subsets $U := k^1 \setminus \{0\}$ and $V := C \setminus \{(0, 0)\}$ as closed algebraic subsets (of k^2 and k^3, respectively), since $\{0\} = V((x))$ and $\{(0, 0)\} = V((x_1)) \cap C$. Show that the morphism $f : k^1 \to C$ restricts to a morphism $g : U \to V$, and that g is an isomorphism.