Homework 11

Please turn these in on Monday, Feb. 8 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

1. Finish up the valuative criterion for proper morphisms: Suppose $f : X \to Y$ is proper, let K be a field, $\mathcal{O}_v \subset K$ a valuation ring of K, $j : \operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}_v$ the morphism with $j^* : \mathcal{O}_v \to K$ the inclusion, and let

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{g} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} \mathcal{O}_v & \xrightarrow[\bar{g}]{} & Y \end{array}$$

be a commutative diagram. Show that there exists a morphism $h : \operatorname{Spec} \mathcal{O} \to X$ filling in the diagram (we don't need X to be noetherian for this). *Hint*: Consider the fiber product $p_2 : \operatorname{Spec} \mathcal{O}_v \times_Y X \to \operatorname{Spec} \mathcal{O}_v$. Show the pair (j, g) defines a morphism $(j, g) : \operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}_v \times_Y X$. Let C be the closure of $(j, g)(|\operatorname{Spec} K|)$ and show that $p_2(C) = |\operatorname{Spec} \mathcal{O}_v|$. Find a point $c \in C$ with $p_2(c)$ the closed point of $\operatorname{Spec} \mathcal{O}_v$. Show that $k(C) = K$ and that p_2^* induces inclusions $\mathcal{O}_v \subset \mathcal{O}_{,c} \subset K$. Using the $\mathcal{O}_v$ is a valuation ring, conclude that $p_2^* : \mathcal{O}_{C,c} \to \mathcal{O}_v$ is an isomorphism and use this to construct the map h.

2. Let $f : X \to Y$ be a morphism of schemes. Show that f is proper if and only if for all open $U \subset Y$, the restriction of f to $f_U : f^{-1}(U) \to U$ is a proper morphism.

3. This is a refinement of the valuative criterion in the case of schemes of finite type over a field.
a) recall that a discrete valuation ring is a noetherian local domain $(\mathcal{O}, \mathfrak{m})$ with quotient field K such that $\mathcal{O}$ is integrally closed in K and has Krull dimension one. Equivalently: the maximal ideal $\mathfrak{m}$ is principal. $\mathcal{O}$ is also a valuation ring of K. Let k be a field and consider $\mathbf{Sch}/k$, the category of finite type k-schemes.

Theorem 0.1. *Let $f : X \to Y$ be a morphism in $\mathbf{Sch}/k$.*
i) f is separated if and only if, for K a finitely generated field extension of k, for $k \subset \mathcal{O} \subset K$ a discrete valuation ring with quotient field K, and for

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "g"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O} \arrow[r, "\bar{g}"'] \arrow[ur, shift left, "h_1"] \arrow[ur, shift right, "h_2"'] & Y \end{tikzcd}$$

a commutative diagram in $\mathbf{Sch}/k$*, with* $j : \operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}$ *the morphism induced by the inclusion* $\mathcal{O} \subset K$*, we have* $h_1 = h_2$*.*
ii) f *is proper if and only if, for* K *a finitely generated field extension of* k*, for* $k \subset \mathcal{O} \subset K$ *a discrete valuation ring with quotient field* K*, and for*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "g"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O} \arrow[r, "\bar{g}"'] & Y \end{tikzcd}$$

a commutative diagram in $\mathbf{Sch}/k$*, with* $j : \operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}$ *the morphism induced by the inclusion* $\mathcal{O} \subset K$*, there exists a unique* k*-morphism* $h : \operatorname{Spec} \mathcal{O} \to X$ *making*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "g"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O} \arrow[r, "\bar{g}"'] \arrow[ur, "h"'] & Y \end{tikzcd}$$

commute.

The proof is essentially the same as for the theorem we discussed in class, with the help of the following

Proposition 0.2. *Let* $k \subset K$ *be a finitely generated field extension, let* A *be a subring of* K *containing* k *and finitely generated as a* k*-algebra. Let* $P \subset A$ *be a prime ideal. Then there is a discrete valuation ring* $\mathcal{O}$ *of* K *with* $A_P \subset \mathcal{O} \subset K$ *and with* $\mathcal{O}$ *dominating* A_P*, that is, the maximal ideal* $\mathfrak{m}$ *of* $\mathcal{O}$ *satisfies* $\mathfrak{m} \cap A_P = PA_P$*.*

The proof of the proposition goes in steps:

Step 1. Reduce to the case in which K is the quotient field of A. For this, first show there there are elements $t_1, \ldots, t_r \in K$, such that the subring $A[t_1, \ldots, t_r]$ of K is a polynomial ring over A, and there is a subring B of K with $A[t_1, \ldots, t_n] \subset B \subset K$, B finite over $A[t_1, \ldots, t_n]$ and K the quotient field of B. Then show there is a prime ideal $Q \subset B$ with $P = A \cap Q$ and show that it suffices to prove the proposition for $Q \subset B$.

Step 2. We now assume that K is the quotient field of A. Show that $P = (x_1, \ldots, x_r)$ for suitable $x_i \in A$. For each $i = 1, \ldots, r$, we have the subring $A[x_1/x_i, x_2/x_i, \ldots, x_r/x_i]$ of K. Show that there is an i such that $PA[x_1/x_i, \ldots, x_r/_i]$ is not the unit ideal, and in this case, $PA[x_1/x_i, \ldots, x_r/_i] = (x_i)A[x_1/x_i, \ldots, x_r/x_i]$. *Hint*: Choose a valuation ring $\mathcal{O}_v$ of K with $\mathcal{O}_v > A_P$ and take i such that $v(x_i) \leq v(x_j)$ for all $j = 1, \ldots, r$.

Step 3. Choosing i as in Step 2, let $A' = A[x_1/x_i, \ldots, x_r/x_i]$. Let $B \subset K$ be the normalization of A' in K. Show that there is a prime ideal $Q \subset B$

with $Q \cap A' \supset (x_i)A'$ and that $B_Q > A_P$.
Step 4. Use Krull's principal ideal theorem to show that Q has height 1. Conclude that B_Q is a discrete valuation ring.

4. a. Show that a finite morphism is proper.
b. Show that an open immersion $j : U \to X$ is proper if and only if j is also a closed immersion.
c. Let R be a ring. Show that the projection $p_1 : \mathbb{A}^2_R \to \mathbb{A}^1_R$ is not proper; here $\mathbb{A}^n_R := \operatorname{Spec} R[T_1, \ldots, T_n]$ and p_1 is the morphism induced by the inclusion $R[T_1] \subset R[T_1, T_2]$.

5. Let k be a field, let $k[t]$ denote a polynomial ring in one variable over k, let $X = \operatorname{Spec} k[t, t^{-1}]$, $Y = \operatorname{Spec} k[t^2 - 1, t^3 - t^2]$, where $k[t^2 - 1, t^3 - t^2]$ is the sub-k-algebra of $k[t]$ generated by $t^2 - 1, t^3 - t$. The inclusions $k[t^2 - 1, t^3 - t^2] \subset k[t] \subset k[t, t^{-1}]$ define morphisms $f : X \to Y$, $g : \mathbb{A}^1_k \to Y$ and $h : X \to \mathbb{A}^1_k$.
a) Show that h is an open immersion, giving an isomorphism of X with the open subscheme $\mathbb{A}^1_k \setminus \{0\}$, where 0 is the point of $\mathbb{A}^1_k$ corresponding to the prime ideal $(t) \subset k[t]$.
b) Show that the map f is a separated closed morphism of finite type.
c) Show that f is not proper. *Hint*: We have the diagram

$$\begin{tikzcd} X \times_Y \mathbb{A}^1_k \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ \mathbb{A}^1_k \arrow[r, "g"'] & Y \end{tikzcd}$$

Show that the maps $h : X \to \mathbb{A}^1$, $\mathrm{Id}_X : X \to X$ define a map $(\mathrm{Id},_X h) : X \to X \times_Y \mathbb{A}^1_k$ with $p_1 \circ (\mathrm{Id}, h) = \mathrm{Id}_X$. Show that (Id, h) is a closed immersion, but $p_2 \circ (\mathrm{Id}, h)$ does not have closed image.