Homework 10

Please turn these in on Monday, Jan. 25 (in class or in the problem session). You should be prepared to present these problems on the board during the problem session.

1. Let B be a scheme, X and Y B-schemes and $f : X \to Y$ a morphism over B.
a) Let $p_X : X \times_B Y \to X$, $p_Y : X \times_B Y \to Y$ be the projections. Show there is unique morphism $\gamma_f : X \to X \times_B Y$ with $p_X \circ \gamma_f = \mathrm{Id}_X$, $p_Y \circ \gamma_f = f$.
b) Show that (X, γ_f) is the equalizer of $f \circ p_X, p_Y : X \times_B Y \to Y$.
c) Suppose that Y is separated over B. Show that γ_f is a closed immersion.
d) In case Y is separated over B, let Γ_f be the closed subscheme of $X \times_B Y$ determined by γ_f; Γ_f is called the *graph* of f. Show that the restriction of p_X to Γ_f defines an isomorphism $p : \Gamma_f \to X$.

2. a) Let X be a separated scheme, and let $U, V \subset X$ be affine open subschemes. Show that $U \cap V$ is affine. *Hint*: Letting $\Delta_X : X \to X \times_{\mathrm{Spec}\,\mathbb{Z}} X$ be the diagonal, show that $U \cap V$ is isomorphic to $\Delta_X^{-1}(U \times_{\mathrm{Spec}\,\mathbb{Z}} V)$. Conclude there is a closed immersion $U \cap V \to U \times_{\mathrm{Spec}\,\mathbb{Z}} V$.
b) Let k be a field, and let $\mathbb{A}^2_k = \mathrm{Spec}\, k[T_1, T_2]$. Let X be two copies of $\mathbb{A}^2_k$, glued along the open subscheme $\mathbb{A}^2_k \setminus \{(0,0)\}$ by the identity map. Show that X has affine open subschemes U, V with $U \cap V$ *not* an affine scheme. *Hint* Show that $Y := \mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine by computing $\mathcal{O}_Y(Y)$.
c) Let $U \to B$, $V \to B$ be affine B-schemes, with B a separated scheme. Show that $U \times_B V$ is an affine scheme. Give an an example of $i : U \to B$, $j : V \to B$ affine B-schemes, with $U \times_B V$ *not* an affine scheme. *Hint* Show that if the morphisms i and j are inclusions of open subschemes, then $U \times_B V$ is isomorphic to $U \cap V$.

3. Recall that a scheme X is quasi-compact if the underlying space $|X|$ is a quasi-compact topological space. I made a mistake in class in the definition of a quasi-compact morphism of schemes. The correct definition is: A morphism of schemes $f : Y \to X$ is quasi-compact if for each *affine* open subscheme U of X, $f^{-1}(U)$ is quasi-compact (I had omitted the condition that U is affine).
a) Show that a scheme X is quasi-compact if and only if X is a union of finitely many affine open subschemes.
b) Show that $f : Y \to X$ is quasi-compact if and only if there is an affine open cover of X, $X = \cup_\alpha U_\alpha$ with $f^{-1}(U_\alpha)$ quasi-compact.
c) Show that $f : Y \to X$ is quasi-compact if and only if for all quasi-compact open subschemes $U \subset X$, $f^{-1}(U)$ is quasi-compact.
d) Suppose X is noetherian. Show that $f : Y \to X$ is quasi-compact if and only if for all open subschemes $U \subset X$, $f^{-1}(U)$ is quasi-compact.
e) Let X be quasi-compact and let $f : X \to \mathrm{Spec}\, A$ be a morphism. Show

that f is a quasi-compact morphism.

4. Show that the classes of morphisms locally of finite type, morphisms of finite type and finite morphisms are all closed under base-change, that is: Let $f : Y \to X$ be a morphism of schemes, satisfying property P, where P is one of: the morphism is locally of finite type, the morphism is of finite type, the morphism is finite. Let $g : Z \to X$ be a morphism of schemes. Show that $p_Z : X \times_Y Z \to Z$ also satisfies P.

5. a) Let $f : Y \to X$ be a morphism of schemes. Show that f is a morphism of finite type if and only if f is locally of finite type and quasi-compact.
b) Let $f : Y \to X$ be a morphism of finite type. Show that for all affine open subschemes $U = \operatorname{Spec} A$ of X, $f^{-1}(U)$ has a finite affine open cover, $f^{-1}(U) = \cup_{i=1}^n V_i$, $V_i = \operatorname{Spec} B_i$, with each B_i a finitely generated A-algebra.
c) Let $f : Y \to X$ be a morphism of finite type and suppose that X is noetherian. Show that Y is noetherian. *Hint*: Hilbert basis theorem.
d) Let $f : Y \to X$ be a finite morphism, and let $U \subset X$ be an affine open subscheme, $U = \operatorname{Spec} A$. Show that $V := f^{-1}(U)$ is affine, $V = \operatorname{Spec} B$, and B is a finite A-module. *Hint*: First show that you can cover U by finitely many principal open subschemes $U_i = \operatorname{Spec} A_{f_i}$ such that $f^{-1}(U_i) = \operatorname{Spec} B_i$ with B_i a finitely generated A_{f_i}-module for each i. Next, factor the morphism $f_{|V} : V \to U$ as

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[dr, "f_{|V}"'] & \operatorname{Spec} \mathcal{O}_X(V) \arrow[d] \\ & U \end{tikzcd}$$

and show that p is an isomorphism. Then use the fact: Let M be an A-module, $f_1, \ldots, f_n \in A$ generating the unit ideal. Suppose that M_{f_i} is a finitely generated A_{f_i}-module for all i. Then M is a finitely generated A-module.