## Homework 1

These are problems for the first meeting of the problem session on Monday. You should be prepared to present these problems on the board during the problem session.

Fix an algebraically closed field $k$.

1. We have the subset $U_{0}$ of $\mathbb{P}^{n}(k)$ defined as

$$
U_{0}:=\left\{\left[x_{0}: \ldots: x_{n}\right] \in \mathbb{P}^{n}(k) \mid x_{0} \neq 0\right\}
$$

and have mentioned in class that sending $\left[x_{0}: \ldots: x_{n}\right]$ to $\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)$ defines a bijection of $U_{0}$ with $k^{n}$, with inverse the map

$$
\left(y_{1}, \ldots, y_{n}\right) \mapsto\left[1: y_{1}: \ldots: y_{n}\right]
$$

Let $i: k^{n} \rightarrow \mathbb{P}^{n}(k)$ be the resulting inclusion.
(a) Let $A$ be a closed algebraic subset of $\mathbb{P}^{n}(k)$. Show that $i^{-1}(A)$ is a closed algebraic subset of $k^{n}$ and that

$$
I\left(i^{-1}(A)\right)=\left\{g\left(1, x_{1}, \ldots, x_{n}\right) \mid g \in I_{h}(A)\right\}
$$

(b) For $f \in k\left[x_{1}, \ldots, x_{n}\right]$ of degree $n$, define the homogeneous polynomial of degree $n, f^{h} \in k\left[x_{0}, \ldots, x_{n}\right]$, as follows: write $f$ as a sum of homogeneous terms

$$
f=\sum_{d=0}^{n} f_{d}, f_{n} \neq 0
$$

and set $f^{h}:=\sum_{d=0}^{n} x_{0}^{n-d} f_{d}$; we set $0^{h}=0$. Let $B$ be a closed algebraic subset of $k^{n}$ and let $A=V_{h}\left(I_{h}(i(B))\right)$. Show that $A$ is the smallest closed algebraic subset of $\mathbb{P}^{n}(k)$ containing $i(B)$ and that for $g$ a homogeneous element of $k\left[x_{0}, \ldots, x_{n}\right], g$ is in $I_{h}(A)$ if and only if there is a $f$ in $I(B)$ and an $i \geq 0$ such that $g=x_{0}^{i} f^{h}$.
(c) Consider the subset $C$ of $k^{3}$ defined by

$$
C:=\left\{\left(t, t^{2}, t^{3}\right) \mid t \in k\right\}
$$

Show that $C$ is an algebraic subset of $k^{3}$ with $I(C)=\left(x_{2}-x_{1}^{2}, x_{3}-x_{1}^{3}\right)$, but $I_{h}(i(C)) \neq\left(\left(x_{2}-x_{1}^{2}\right)^{h},\left(x_{3}-x_{1}^{3}\right)^{h}\right)$. Find generators for $I_{h}(i(C))$ (you will need three).
2. Let $f \in k\left[x_{0}, x_{1}, x_{2}\right]$ be a homogeneous polynomial of degree $d>0$ and let $L \in k\left[x_{0}, x_{1}, x_{2}\right]$ be a non-zero linear polynomial, that is $L$ is homogeneous of degree one. Let $A=V_{h}((f))$ and $B=V_{h}((L))$.
(a) Show that $A \cap B \neq \emptyset$.
(b) Suppose that $L$ is not a factor of $f$. Show that $A \cap B$ is a finite set with at most $d$ elements.
(c) Given degrees $e, d>0$, give an example of polynomials $f, g \in k\left[x_{1}, x_{2}\right]$ with $\operatorname{deg}(f)=d$, $\operatorname{deg}(g)=e$, such that $V((f)) \cap V((g))=\emptyset$ and show that
$V_{h}\left(\left(f^{h}\right)\right) \cap V_{h}\left(\left(g^{h}\right)\right) \subset V_{h}\left(\left(x_{0}\right)\right)$.
(d) In case $e=1$ in (c), show that $V_{h}\left(\left(f^{h}\right)\right) \cap V_{h}\left(\left(g^{h}\right)\right) \cap V_{h}\left(\left(x_{0}\right)\right) \neq \emptyset$.
3. Let $A$ be a closed algebraic subset of $k^{n}$. The coordinate $\operatorname{ring}$ of $A$ is the $k$-algebra $k[A]:=k\left[x_{1}, \ldots, x_{n}\right] /(I(A))$. Define the dimension of $A$ as the dimension of the ring $k[A]$. We say that $A$ has pure dimenion $d$ if each irreducible component of $A$ has dimension $d$.
(a) Let $H$ be a closed algebraic subset of $k^{n}$ such that each irreducible component of $H$ has dimension $n-1$. Show that $I(H)=(f)$, with $f \in$ $k\left[x_{1}, \ldots, x_{n}\right]$ non-zero. Such an $H$ is called a hypersurface in $k^{n}$ and $f$ is a defining equation for $H$.
(b) Let $H \subset k^{n}$ be a hypersurface. Show that a defining equation $f$ for $H$ is unique up to multiplication by an element $a \in k^{\times}$, and thus there is a unique monic defining equation for $H$. Show that a monic defining equation $f$ is uniquely (up to order) a product $f=\prod_{i=1}^{r} f_{i}$ with the $f_{i}$ monic and irreducible, and that $H$ is irreducible if and only if $f$ is irreducible.
(c) Let $A \subset k^{n}$ be a closed algebraic subset of pure dimension $d$ and let $H \subset k^{n}$ be a hypersurface such that $H$ contains no component of $A$. Then $A \cap H$ has pure dimension $d-1$, or is empty. Hint first reduce to the case of irreducible $A$.

