HODGE THEORY

Abstract. Program WS 2014/1015 for the "Baby Seminar" on Hodge Theory.

1. Basic complex differential geometry

Review complex manifolds, almost complex structure and integrability criterion, actions of the differential operators $d, \partial, \bar{\partial}$ on vector bundles, Hermitian and Käler metrics. References for this can be found in [1] chaper 2 and 3. The point is to explain the three main theorems of Hodge theory (for a compact Käler manifold) that are proved in chapter 6 of [1]. The proof is technical and strongly analytic: we don't really want a proof, but we want to understand how it works, what kind of tools are needed and where the assumptions become necessary. Voisin does a great job hiding most of the technical part in chapter 5 of [1]. Feel free to add more background if it is necessary to understand the direction of the proof. A concise introduction is also given in Green's lecuter in [8].

2. Homological algebra of filtered objects

We have to develop the theory of filtered objects in an abelian category, see [3] I.(1-2), culminating in the theorem that characterizes those filtration that make filtered objects an abelian category (1.2.10 of [3]). In [3] 1.(3-4) it is explained how filtered objects induce filtrations on pages of spectral sequences. The main result of this part is theorem [3] 1.3.16: we will use this in the main proves of the next secions. You are supposed to cover the whole [3] 1: this is very long and great piece of homological algebra, but the accent should be on the meaning of thins rather then on the details. This means don't loose time in homological caveats, because you will not have time.

3. Hodge structures

In [3] 2.1 *pure Hodge structures* are introduced, related to filtered complex vector spaces and also related to actions of the Deligne torus, see [3] Proposition 2.1.9. After this switch to *Integral Hodge structures* and polarization: see the rest of 2.1 in [3]. In 2.2 of [3] Deligne recalls that the main example of Hodge structure comes from that induced on the cohomology of a compact Käler manifold (thanks to the classical Hodge theorems). more or less the same material is covered in 7.1 of [1]. Another good surce for this material is chapter 3 section 1 of [5]. Conclude with 2.3 of [3] where *mixed hodge structures* are introduced and give some hints on the proof of theorem 2.3.5 of [3].

4. Examples

[1] 7.(2-3) and [5]3.(2-4) and Hodge structure for curves, Jacobian varieties and Torelli theorem: a good reference might be Forster's book on Riemann Surfaces.

5. Mixed Hodge theory

Cover chapter 3 of [3]: the main point of this is to induce a mixed Hodge structure on the cohomology of a smooth scheme of finite type over $\mathbb{C}$ reducing to the smooth projective case. Essential tools are log deRham complexes and their role in the computation of the cohomology of an open variety (this is 3.1) together with an application of the homological algebra developed in the second lecture (this is 3.2). Stress theorem 3.2.5 giving a sketch of the proof and on its corollaries: 3.2.(13-18). Another reference for this part is chapter 8 of [1]. Alternative reference, if you feel less foundational, [6].

6. Applications and more examples

Cover some of the four applications that Deligne gives in chapter 4 of [3]. To get some good feeling on how to use mixed Hodge Structures to prove things, a good idea might be to cover the paper of D.Arapura "On the Abel Jacobi map for non-compact varieties". This is elementary, but contains and uses all the theory presented so far. Do as much as you can.

7. Deformations of complex manifolds

[1] chapter 9. This is the first section devoted to variations of Hodge structures. After having established the basic properties of (Mixed) Hodge Structures, we wish to describe how they vary with the complex structure. This talk is devoted to the some deformation theory, from the analytic point of view. Things that you cannot forget to mention: Kodaira-Spencer map, Gauß-Manin connections. As usual, the focus has to be on the ideas, not on the technicalities. If Voisin's book is too much for your tastes, feel free to use [6] instead.

8. Variations of Hodge structures

[1] chapter 10. See the comments on the previous talk. Give as much details as possible on 10.3.1, that is applications to curves. A possible strategy is to start directly from that section and then collect all the background material from the rest of the chapter that is needed. It's a good idea to discuss with the person giving the talk on deformation of complex manifolds to check exactly what is needed here.

9. Cycle, Chern and Hodge classes

[1] chapter 11. Considering the determinant rôle of these objects also in the Motive Seminar of this term, great care is needed for this talk. Voisin's book gives a very comprehensive introduction to the topic, but another reference is [6]. A more motivic oriented presentation is in [7]. Don't forget to state the Hodge Conjecture. Another beautiful reference is Murre's lecture in [8].

10. Jacobians, Albanese varieties and Abel-Jacobi maps

Cover [1] chapter 12, from 12.1 to 12.3.1. You can avoid 12.3.2 and 12.3.3. Define intermediate Jacobians and explain how to use Hodge theory to construct the Abel-Jacobi map. Again you can use D.Arapura's paper to get some feeling of what's going on. If that paper was not entirely presented in the talk "Applications", this is a good place to finish explaining it.

11. Topics 1: limits of hodge structures

[**?**]. More details to come.

12. Topics 2: Griffiths computation for Hypersurfaces

If time permits. Other possibilities will be discussed on the way. A suggestion is Van Geemen's introduction to the Hodge conjecture for Abelian Varieties in [8].

13. Time and place

We meet on the scheduled days (usually tuesday) in the Tea room at 10 a.m. (sharp).

14. Schedule

Lecture number	Title	Speaker	Date
0	Introduction and distribution of talks	Federico	21.10
1	Basic complex differential geometry	Wei	28.10
2	Homological algebra of filtered objects	Gabriela	4.11
3	Hodge structures	Rin	11.11
4	Examples	Niels	18.11
5	Mixed Hodge theory	Chiara	25.11
6	Applications and more examples	Adeel	2.12
7	Deformations of complex manifolds	Toan	9.12
8	Variations of Hodge structures	Jin	16.12
9	Cycle, Chern and Hodge classes	Thi	13.12
10	Jacobians, Albanese varieties and Abel-Jacobi maps	Federico/Lorenzo	20.1
11	Topics1	?	27.1
12	Topics2	?	3.2
13	topics3	?	10.2

15. General Remarks that apply to all the talks

The talks are supposed to last 2 whole hours: do not count on some additional time; you should not exceed the given time for many reasons, one of those is that we would like to discuss on the talk after it (and before lunch).

We all know that every single talk is quite dense. You are supposed to present a lot of material which, at the moment, you are not familiar to. In addition the quantity of mathematics that you are supposed to tell in just two hours is would probably require way more time. These two issues make the preparation of the seminar more difficult and suppose a great responsibility from every single speaker. It would be very nice if you could discuss in advance with Federico or Lorenzo about the skeleton of your talk, the perspective you want to give or just the points that need to be stressed. As a result the chance of losing time in irrelevant points would be hopefully very low. This doesn't mean we are forcing you to follow our oligarchical will, but it does mean that we would like the seminar to be as helpful as possible for all of us.

Finally, it would be really great if you could produce some notes to accompany your talk. This would help a lot if, for instance, you can not give some proof you consider important for timing reasons. Ideally the notes should contain at least what you are going to write on the board: clearly additional material is always welcome.

References

[1] C.Voisin, *Hodge Theory and Complex algebraic geometry. I*
[2] C.Voisin, *Hodge Theory and Complex algebraic geometry. II*
[3] P.Deligne, *Thorie de Hodge, II*
[4] P.Deligne, *Thorie de Hodge, III*
[5] D.Huybrechts, *Lectures on K3 surfaces.*
[6] C. Peters, J. Steenbrink, *Mixed Hodge Structures.*Springer (2008)
[7] E. Esnault, E. Viehweg, *Deligne-Beilinson cohomology.*
[8] M.Green, J. Murre and C. Voisin eds. *Algebraic Cycles and Hodge Theory*, Springer LNM 1594.