UNIVERSITY OF ZURICH HS 2019

SEMINAR ON PRISMATIC COHOMOLOGY

THURSDAY, 10:15-11:45, ROOM Y32 E06

INTRODUCTION

In this seminar we will study the first part of the recent preprint "Prisms and Prismatic Cohomology" by Bhatt-Scholze [BS19].

Let p be a fixed prime number. Prismatic cohomology is a new p-adic cohomology theory which specialises to old theories such as cristalline cohomology, de Rham cohomology, or étale cohomology. To any so-called bounded *prism* (A, I) (a ring with a Frobenius lift ϕ and a locally invertible ideal subject to certain conditions) and any smooth p-adic formal scheme X over A/I there is assigned a so-called *prismatic site* $(X/A)_{\Delta}$ together with a structure sheaf $\mathcal{O}_{\Delta}$. The derived global sections

$$R\Gamma_{\Delta}(X/A) := R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta})$$

fulfil several comparison isomorphisms whereof we will see the following ones in the seminar:

- *Crystalline comparison.* If $I = (p)$, then there is a canocical ϕ-equivariant isomorphism

$$R\Gamma_{\mathrm{crys}}(X/A) \cong \phi^* R\Gamma_{\Delta}(X/A)$$

- *Hodge-Tate comparison.* If $X = \mathrm{Spf}(R)$ is affine, then there is a canonical R-linear isomorphism

$$\Omega^i_{R/(A/I)}\{i\} \cong \mathrm{H}^i(R\Gamma_{\Delta}(X/A) \otimes^{\mathrm{L}}_A A/I$$

  where $M\{i\} = M \otimes_{A/I} (I/I^2)^{\otimes i}$ for an A/I-module M.
- *de Rham comparison.* There is a canonical isomorphism

$$R\Gamma_{\mathrm{dR}}(X/(A/I)) \cong R\Gamma_{\Delta}(X/A) \hat{\otimes}^{\mathrm{L}}_{A,\phi} A/I$$

  of commutative algebras in $D(A)$.

Remark. This programme goes along the lines of lecture notes by Bhargav Bhatt [Bha] and of a seminar by Moritz Kerz and Georg Tamme at Universität Regensburg.

TALKS

Talk 0: Panorama and distribution of talks (03.10.)
We will watch a lecture by Bhatt at the IHES from June 2018 (Link). Afterwards we will distribute the talks.

Talk 1: Crystalline Cohomology I (10.10.)
Introduce the basic notions needed for Crystalline Cohomology. Witt vectors (and truncations) of a perfect field with their operations (F,V) and with examples. Introduce DP (=Divided-Powers)-structures on ideals, and the crystalline site. Introduce the DP-envelope of a ring resp. scheme and state the Crystalline-de Rham comparison. *References:* [BO78], [Ill75], [CL98].

Talk 2: Crystalline Cohomology II (17.10.)
This talk is intended as a tour through the uses and motivation for Crystalline Cohomology. Sketch the ingredients of the Crystalline-de Rham comparison. Compare with the situation in characteristic zero, where a variation of Crystalline Cohomology also computes the de Rham Cohomology.

In addition it would be nice to discuss similarities, analogies and differences with the chracteristic zero setting. For instance one could focus on the following topics:

(i) the Betti-de Rham comparison;
(ii) the Hodge-de Rham spectral sequence and the de Rham-Witt complex;
(iii) the Hodge-Witt decomposition;
(iv) crystals;
(v) (a)symmetry of the Hodge numbers;
(vi) ....

Refences: [BO78], [Ill75], [CL98].

Talk 3: δ-rings (24.10.)
Define δ-rings and provide examples. Present the category of δ-rings, its adjunctions with the category of rings, (co)limits of δ-rings, and free objects. Finally, prove the equivalence between the category of perfect $\mathbb{F}_p$-algebras and the category of perfect p-adically complete δ-rings. *References:* [Bha, II.], [BS19, §2.1, §2.4].

Talk 4: Distinguished elements and derived completions (31.10.)
Discuss distinguished elements in δ-rings and examples and prove some characterisations of these. Give a brief account of derived completions as needed for the next talk. *References:* [Bha, III. §1-§2], [BS19, §2.3].

Talk 5: Perfectoid rings (07.11.)
Introduce integral perfectoid rings and characterise those in characteristic p. Define the tilt of an inegral perfectoid ring and construct Fontaine's map θ. Prove the tilting correspondence for integral perfectoid rings. *References:* [Mor, §1], [BMS18, §3].

Talk 6: Prisms (14.11.)
Introduce the notion of a prism. Show that the category of perfect

prisms is equivalent to the category of perfectoid rings (in the sense of [BMS18, Def. 3.5] which is equivalent to the notion in Bhatt's lecture [Bha, IV. Def. 2.1]). *References:* [Bha, IV.], [BS19, §3]

Talk 7: The prismatic site (21.11.)
Define the prismatic site and explain why one can avoid working with Grothendieck topologies by restricting to the affine case. Recall the algebraic de Rham complex and construct the Hodge-Tate comparison map. Finally state the Hodge-Tate comparison theorem [Bha, V. Thm. 3.8] which will be proved in the next talk. *Reference:* [Bha, V.]

Talk 8: Crystalline comparison (28.11.)
Explain how to realise divided power structures via δ-structures and recall how to compute prismatic cohomology. Then prove the cristalline comparison isomorphism and deduce the de Rham comparison. *References:* [Bha, VI. §2.3], [BS19, §5, Thm. 6.4].

Talk 9: Hodge-Tate comparison isomorphism (05.12.)
Prove the Hodge-Tate comparison isomorphism. *Rferences:* [Bha, VI. §4], [BS19, §6].

Talk 10: Buffer slot (12.12.)

Talk 11: Buffer slot (19.12.)

References

[Bha] Bhargav Bhatt, *Lectures on prismatic cohomology*, (Link).

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397.

[BO78] Pierre Berthelot and Arthur Ogus, *Notes on crystalline cohomology*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[BS19] Bhargav Bhatt and Peter Scholze, *Prisms and Prismatic Cohomology*, arXiv:1905.08229, 2019.

[CL98] Antoine Chambert-Loir, *Cohomologie cristalline: un survol*, Exposition. Math. **16** (1998), no. 4, 333–382.

[Ill75] Luc Illusie, *Report on crystalline cohomology*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), 1975, pp. 459–478.

[Mor] Matthew Morrow, *Foundations of perfectoid spaces*, lecture notes (PDF).