Real cohomology, Powers of the fundamental ideal of the Witt ring & consequences for homotopy modules



## §0. INTRODUCTION

The main aim of this talk is to sketch proves of the following

**THEOREM** (JACOBSON) : The signature map induces an isomorphism of Zariski sheaves on $Sm/_k$
$$\operatorname{colim}_m \mathcal{I}^m \longrightarrow \operatorname{Sup}_* \underline{\mathbb{Z}}$$

meaning that $(\operatorname{colim}_m \mathcal{I}^m)(X) = \operatorname{colim}\big(W(X) \xrightarrow{\cdot\langle\langle -1\rangle\rangle} \mathcal{I}(X) \xrightarrow{\cdot\langle\langle -1\rangle\rangle} \mathcal{I}^2(X) \to \cdots\big)$
$$(\operatorname{Sup}_* \underline{\mathbb{Z}})(X) = \Gamma(X, \operatorname{Sup}_* \underline{\mathbb{Z}}_{X_r}) = \Gamma(X_r, \underline{\mathbb{Z}}_{X_r})$$

**THEOREM** (BACHMANN) : Let $F_\cdot$ be a homotopy module in which $\rho : F_n \to F_{n+1}$ is invertible $\forall n$. Then $F_k$ is a rét sheaf $\forall k$.

## §1. Jacobson's theorem.

1.1 **Def** :
- $W(X)$ = Groth. ring of quadratic bundles modulo hyperbolic ~~plan~~ spaces
- $\mathcal{I}(X)$ = fundamental ideal = $\ker(rk : W(X) \to \mathbb{Z}/2)$
  $\overset{!}{=}$ ideal generated by pfister forms $\langle\langle a\rangle\rangle := 1 - \langle a\rangle$ for $a \in \mathcal{O}(X)^*$
- ⊛

- There's a "global signature" map $\operatorname{Sign} : W(X) \longrightarrow H^0(X_r, \mathbb{Z})$ defined by mapping $\phi \longmapsto \big(X_r \ni (P, x) \longmapsto$ Signature of $\phi \otimes k(x)$ according to the ordering $P\big)$
  this is well defined (ie $\operatorname{sign}(\phi)$ is a locally constant function) ; because
  $$\operatorname{Sign}(\langle\langle a\rangle\rangle) = 2\chi_{\{a<0\}}$$
  $$\operatorname{Sign}(\langle\langle a_1, \dots, a_m\rangle\rangle) = 2^m \chi_{\{a_1<0, \dots, a_m<0\}}$$
  This gives a comm. diagram (~~after sheafification~~)

$$\begin{tikzcd} W(X) \arrow[r, "\cdot\langle\langle -1\rangle\rangle"] \arrow[d] & \mathcal{I}(X) \arrow[r, "\cdot\langle\langle -1\rangle\rangle"] \arrow[d, "\frac{\operatorname{Signature}(-)}{2}"] & \mathcal{I}^2(X) \arrow[r] \arrow[d, "\frac{\operatorname{Sign}(-)}{4}"] & \cdots & \operatorname{colim}_m \mathcal{I}^m \arrow[d, "\operatorname{Sign}"] \\ H^0(X_r, \mathbb{Z}) \arrow[r] & H^0(X_r, \mathbb{Z}) \arrow[r] & H^0(X_r, \mathbb{Z}) \arrow[r] & \cdots & H^0(X_r, \mathbb{Z}) \end{tikzcd}$$

This is our map

**rk** : When $X = \operatorname{Spec} F$ with $\operatorname{char}(F) \neq 2$ it's a classical result that such signature map is an isomorphism

1.2 **THEOREM** : The signature map we constructed above $\operatorname{colim}_m \mathcal{I}^m \longrightarrow \operatorname{Sup}_* \underline{\mathbb{Z}}_{X_r}$ is an isomorphism of Zariski sheaves on $X$, for every $X$

⊛ $\operatorname{Sup} : X_r \to X$ def by $(x, P) \mapsto x$ (i.e. forgetting the ordering of the residue field)

Sketch of the proof: Immediately reduce to the case $X = \mathrm{Spec}\, A$, $A$ = local ring

**Step 1**: Reduction to the case of $A = R_{\mathfrak p}$ with $R = \mathbb{Z}[T_1, \dots, T_n]/I$

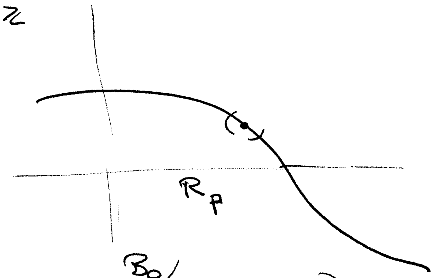
the statement is invariant under filtered colimits $\rightsquigarrow$ we write

$A = \mathrm{colim}_\alpha A_\alpha$ $\quad A_\alpha$ = f.t. per. subring $\quad A = \mathrm{colim}_{\to} (A_\alpha)_{\mathfrak{p}_\alpha}$

$\mathfrak{p}_\alpha = A_\alpha \cap \mathfrak{p} \subset A_\alpha$

**Step 2**: $\mathbb{A}^2_{\mathbb{Z}}$

$$\begin{array}{ccccc} I & \to & \mathbb{Z}[\underline{T}] & \overset{s}{\twoheadrightarrow} & R \\ \downarrow & & \downarrow & & \downarrow \\ I_{s^{-1}(\mathfrak p)} & \to & \mathbb{Z}[\underline{T}]_{s^{-1}(\mathfrak p)} & \to & R_{\mathfrak p} = A \\ \| & & \| & & \\ I_0 & & B_0 & & \end{array}$$




Spec $R$

Take $B :=$ Henselization of $B_0$ along $I_0$

$I := I_0 B$ $\quad$ ( $B$ = colimit of $B_0 \to C$ étale with $B_0/I_0 \xrightarrow{\sim} C/I_0C$ )

and look at

$$\begin{array}{ccc} \mathrm{colim}_{n} I^n(B) & \xrightarrow[\sim]{①} & \mathrm{colim}_{n} I^n(B/I = A) \\ \downarrow & & \downarrow \\ H^0(\mathrm{Spec}\, B, \mathbb{Z}) & \xrightarrow[\sim]{②} & H^0(\mathrm{Spec}\, A, \mathbb{Z}) \end{array}$$

② ~~We've already seen something of~~

- Every point of Spec$(B)$ has a specialization in Spec$(B/I)$
- $\cdot$ $\quad$ $\cdot$ $\quad$ specialize to a unique closed point

$H^*(\mathrm{Spec}\, B, \mathbb{Z}) = H^*(\mathrm{Max\,Spec}\, B, \mathbb{Z}) = H^*(\mathrm{Spec}\, B/I, \mathbb{Z})$

① Needs an ad hoc trick (no time for this)

So we reduced to prove the statement for $B$; moreover using the trick of filtered colimits with Popescu's theorem we further reduce to the case in which $B$ is essentially smooth over $\mathbb{Z}_{(p)}$ $(p \neq 2)$ or $\mathbb{Q}$.

**Step 3** Look at the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathrm{colim}_{n} I^n(B) & \to & \mathrm{colim}_{n} I^n(K) & \to & \bigoplus_{x \in X^{(1)}} \mathrm{colim}_{n \geq -1} I^n(k(x)) \\ & \downarrow & & 2 \downarrow \mathrm{sign} & & \downarrow 2\cdot \mathrm{sign} \\ 0 \to & H^0(\mathrm{Spec}\, B, \mathbb{Z}) & \to & H^0(\mathrm{Spec}\, K, \mathbb{Z}) & \to & \bigoplus_{x \in X^{(1)}} H^0(\mathrm{Spec}\, k(x), \mathbb{Z}) \end{array}$$

where $X = \mathrm{Spec}\, B$ and $\mathrm{colim}_{n \geq -1}$ means $\mathrm{colim}\,(W(\ )\langle\langle -1\rangle\rangle \to W(\ )\langle\langle -1\rangle\rangle \to I\langle\langle -1\rangle\rangle \to I^2 \to \dots)$

- Central vertical map and right hand side vertical map are respectively an isom and injective thanks to classical arguments
- The horizontal sequences are exact: the lower one is a direct check; $\cdot$ upper ~~one~~ needs much work

The exactness of the first line is addressed in two steps

**Step a**: $0 \to \mathrm{colim}_{n} I^n(B) \to \mathrm{colim}_{n} I^n(K)$ is exact

**Step b**: $\mathrm{colim}_{n} I^n(B) \to \mathrm{colim}_{n} I^n(K) \to \bigoplus_{x \in X^{(1)}} \mathrm{colim}_{n \geq -1} I^n(k(x))$

is exact. Here $X = \mathrm{Spec}\, B$
$K = \mathrm{Frac}\, B$.

~~Step~~ b) This uses two new ~~step~~ inputs

b1) ~~the following sequence~~ exactness of

$$\begin{tikzcd}
(**) & I^n(B)/I^{n+1}(B) \arrow[r] & I^n(k)/I^{n+1}(k) \arrow[r] & \bigoplus_{x \in X^{(1)}} I^{n-1}(k(x))/I^n(k(x)) \\
& B^* \otimes \cdots \otimes B^* / (\text{Steinberg}, 2) \arrow[u] \arrow[d, two heads] \arrow[r] & K^M_n(k)/2 \arrow[u, "\cong"] \arrow[d, "\cong"', "\leftarrow \text{Voev.'s Milnor conj}"] \arrow[r] & \bigoplus_x K^M_{n-1}(k(x)) \arrow[u, "\cong"] \arrow[d, "\cong"] \\
0 \arrow[r] & H^n_{et}(B, \mathbb{Z}/2) \arrow[r] & H^n_{et}(k, \mathbb{Z}/2) \arrow[r] & \bigoplus_x H^{n-1}_{et}(k(x), \mathbb{Z}/2)
\end{tikzcd}$$

which follows from

~~Old Con~~ Orlov-Vis-Voev. Milnor's conj

- Exactness of bottom row follows from Gillet / Bloch-Ogus
- Surj of the bottom LHS vertical map follows from work of Kerz. on Gersten resolutions

b2) Now we have to use the exact seq. (**) to produce a lift to $\operatorname{colim}_n I^n(B)$ of an element of $\operatorname{colim}_n I^n(k)$ which maps to zero in $\bigoplus_{x \in} \operatorname{colim}_{n \geq -1} I^n(k(x))$. The exact sequence (**) allows us to lift to $I^N(B)/I^{N+1}(B)$ for $N \gg 0$

$\leadsto$ we can find a lift of an element in $I^N(B)$ up to an ~~mistake~~ error term living in $I^{n+1}(k)$

We then proceed by induction, trying to lift the error term from $I^{N+1}(k)$ to $I^{N+2}(B)$.

At this point we need a tool that will tell us that this process finishes in a finite number of steps ~~following~~ and produces an actual lift of our initial element to $I^{N+M}(B)$.

This tool is a hard computation and we don't have time to go into its details.

Step a) We have a commutative square

$$\begin{tikzcd} \operatorname{colim}_m I^m(B) \arrow[r, hook] \arrow[d] & W(B)[\tfrac{1}{2}] \arrow[d, hook] \\ \operatorname{colim}_m \bar{I}^m(k) \arrow[r, hook] & W(k)[\tfrac{1}{2}] \end{tikzcd}$$

and the horizontal maps are injective; the right hand side vertical map is also injective.

The injectivity of the horizontal maps is more or less "classical".
~~because the kernel of 2 Sgn W(B)...~~
The injectivity of the vertical right map is non-trivial, but it's a more or less direct check on the Gersten complex, it doesn't seem to use cannons like Voevodsky's theorems. ■

We give an application of this theorem of Jacobson, which is not needed later:

COROLLARY. If $X$ is a sep. finite type $\mathbb{R}$-scheme of Krull dimension $d$ and $n \geq d+1$, then the signature map induces an isomorphism

$$H^p_{Zar}(X, \mathcal{I}^n) \xrightarrow{\simeq} H^p_{Sing}(X(\mathbb{R}), \mathbb{Z}) \quad \forall p \geq 0$$

Pf: We combine three facts
- $\forall$ abelian gp $M$, $H^*(X_r, M) = H^*(X(\mathbb{R}), M)$
- If $n \geq d+1$ $\mathcal{I}^n \xrightarrow[\simeq]{\cdot 2} \mathcal{I}^{n+1}$ indeed

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^m(U) \arrow[r] \arrow[d, "2"] & I^m(k) \arrow[r] \arrow[d, "2\,\cdot 2"] & \bigoplus_{x \in X^{(1)}} I^{m-1}(k(x)) \arrow[d, "2\,\cdot 2"] \\ 0 \arrow[r] & \mathcal{I}^{m+1}(U) \arrow[r] & I^{m+1}(k) \arrow[r] & \bigoplus_{x \in X^{(1)}} I^{m}(k(x)) \end{tikzcd}$$

- $H^*_{Zar}(X, \operatorname{colim} \mathcal{I}^m) \underset{Sgn}{\simeq} H^*(X_r, \mathbb{Z})$

REMARK: we can upgrade the isom of the theorem to a isom $\operatorname{colim}_m \mathcal{K}^{MW}_m \to \mathrm{Sgn}_* \mathbb{Z}$

$$\begin{tikzcd} K_0^{MW} \arrow[r, "\cdot\rho = -[-1]"] \arrow[d, two heads] & K_1^{MW} \arrow[r, "\cdot\rho = -[-1]"] \arrow[d, two heads] & K_2^{MW} \arrow[r] \arrow[d, two heads] & \cdots \\ K_0^{MW}/h = W \arrow[r, "\cdot 2"'] & K_1^{MW}/h = \mathcal{I} \arrow[r, "\cdot 2"'] & K_2^{MW}/h = \mathcal{I}^2 \arrow[r] & \cdots \end{tikzcd}$$

$\cdot 2 = \langle\langle -1 \rangle\rangle = 1 - \langle -1 \rangle$

$$\begin{tikzcd} K_0^{MW} \arrow[r, "\rho = -[-1]"] \arrow[d] & \cdot \arrow[d, "\eta"] \\ K_0^{W} \arrow[r, "\sim"] & W \end{tikzcd} \quad \eta\rho = -\eta[-1] = -\langle -1 \rangle + 1$$

$\rightsquigarrow$ the squares commute

Surjectivity is by default; injectivity: if $a \in K_m^{MW}$ is mapped to zero $\Rightarrow$ $a$ is a multiple of $h$; but $\rho h = 0$ $\Rightarrow$ $a = 0$ ■

Moreover such isomorphism is compatible with transfers

PROPOSITION: The isomorphism $\operatorname{colim}_m \tilde{\mathcal{I}}^m \longrightarrow \mathrm{Sg}_* \mathbb{Z}$ is compatible with transfers if we endow the target with the transfer given by summing over preimages.

Pf: Thanks to the projection formula, the trace is well defined on $\operatorname{colim}_m \mathcal{I}^m$ and in order to check the compatibility with transfers in $H^0((-)_r, \mathbb{Z})$ we just have to look at finite étale ext covers of fields $L - k$. In this setting there's a classical formula of Scharlau (I guess) stating that

$$\mathrm{Sgn}_P(\mathrm{tr}_f(\phi)) = \sum_{R \supset P} \mathrm{Sgn}_R(\phi)$$ ■

§ Surjectivity of Rét transfers

2.1 PROPOSITION Let $k$ be a field of char 0 and $L \mid k$ a field extension inducing a rét cover $\mathrm{Sper}\, L \to \mathrm{Sper}\, k$. Then the transfer map $tr: H^0_{\text{rét}}(L, \mathbb{Z}) \to H^0_{\text{rét}}(k, \mathbb{Z})$ "summing over preimages" is onto

2.2 proof: Step 1: If $f: X \to Y$ is a map of Stone (= compact, Hausdorff, totally disconnected) spaces then the above transfer map is surjective.

Assume we have a finite covering $\coprod Y_i = Y$ on which $f^{-1}(Y_i) = \coprod_j X_{ij}$ and for every $i$ and $j$ $f|_{X_{ij}}: X_{ij} \to Y_i$ is a homeomorphism. Then one writes $1 = \chi_Y = \sum_i \chi_{Y_i}$, so if we set $U = \coprod_i X_{i1}$, the loc. function $\chi_U: X \to \mathbb{Z}$ will map to $\chi_Y$ under the transfer map. For this to be useful we need $\chi_U$ to be locally constant, i.e. $Y_i$ must be clopen.

NOTE: If $y \in V \underset{op}{\subseteq} Y$ then I can find a clopen nbd $V_y$ of $y$ in $Y$ which is contained in $V$. Indeed, since every $z \in Y$ has a clopen nbd I have $Y \setminus V \subseteq \bigcup_{z \notin V} V_z$ and only a finite number of $V_z$ suffice (by compactness)

$\leadsto Y \setminus V \subseteq \bigcup_{i=1}^{n} V_i$ and $\underbrace{Y \setminus \left(\bigcup_{i=1}^{n} V_i\right)}_{\text{clopen nbd of } y \text{ contained in } V} \subseteq V$

Now, since $f$ is local homeo, I can always find an open cover $Y_i$ of $Y$ on which $f^{-1}(Y_i) = \coprod X_{ij}$ and I can assume $Y_i$'s are clopen and that there are just a finite number of those, $Y_1, \dots, Y_m$. They might not be pairwise disjoint, so I change them inductively setting

$Y_1 = Y_1, \quad Y_k = Y_k \setminus \bigcup_{i<k} Y_i$.

2.3 Step 2: [If $L/k$ is a rét cover then $\mathrm{Sper}(L) \to \mathrm{Sper}(k)$ is a local homeo of Stone spaces. We can do this in greater generality for later use]
If $X \to Y$ is étale, then $f_r: X_r \to Y_r$ is a local homeo.
i.e. $\exists$ open cover $W_i$ of $X_r$ st $f_r|_{W_i}$ is an open embedding.

$\boxed{f_r \text{ is open}}$ Recall that since $f$ is fin. pres. $\Rightarrow f_r$ maps constr. to constr. so we need to check that $\mathrm{Im}(f_r)$ is stable under generization. i.e. $\forall\, \xi \in X_r$ and $f_r(\xi) = \eta < \eta'$ $\exists\, \xi' \in X_r$ s.t. $f_r(\xi') = \eta'$. From Marco's talk the dotted

$$\begin{tikzcd} v \arrow[r, "\eta"] \arrow[d] & X \arrow[d, "f"] \\ V \arrow[r] \arrow[ur, dotted, "\exists"] & Y_0 \end{tikzcd}$$
$\eta < \eta'$

corresponds to $V$ = real closed valuation ring $\leadsto$ henselian

$\boxed{f_r \text{ is local homeo}}$ We have to show that both $f_r$ and $X_r \to Y_r \times_{X_r} Y_r$ are open

Now: $f_r$ is open; $(-)_r$ commutes to products

$(X \xrightarrow{\Delta} Y \times_Y Y)_r$

$\Delta$ is open bec $f$ is étale

$\Rightarrow \Delta$ is et $\Rightarrow \Delta_r$ is open ✓

Assume

**LEMMA**: $(\mathcal{C}, t)$ is a site, $F$ is a separated presheaf on $(\mathcal{C}, t)$, $U_\bullet$, $V_\bullet$ are $t$-coverings and $V_\bullet$ refines $U_\bullet$. Then, if $F$ satisfies the sheaf condition w.r.t. $V_\bullet$ it does as well w.r.t. $U_\bullet$. ☐

We use this in the following form:

2.4 **PROPOSITION**: Let $F$ be a sheaf on $Sm_k$ w.r.t. the Nis topology. Then $F$ is a rét sheaf iff it satisfies the sheaf condition with resp. to rét covers of the following form:

$X$ is local Henselian, essentially smooth over $k$

$f : U \to X$ is a finite étale map and a rét-cover. (=: fret cover)

Proof: $\boxed{\Longrightarrow}$ is clear so we prove the other direction $\boxed{\Longleftarrow}$.

Take $F$ a Nis sheaf on $Sm_k$, $X \in Sm_k$ and a rét cover $U_\bullet \to X$. Let $F_X$ the restriction to the small Nis site of $X$ and $\underline{Hom}$ the internal hom sheaf.

$$(*) \qquad F(X) \to F(U_\bullet) \rightrightarrows F(U_\bullet \times_X U_\bullet)$$

is obtained as global sections on $Nis_X$ of the diagram

$$F_X \to \underline{Hom}(U_\bullet, F_X) \rightrightarrows \underline{Hom}(U_\bullet \times_X U_\bullet, F_X)$$

of sheaves on $Nis_X$

$$\underline{Hom}(U_\bullet, F_X) = F_X(U_\bullet \times_X -), \qquad \underline{Hom}(U_\bullet \times_X U_\bullet, F_X) = F_X(U_\bullet \times_X U_\bullet \times_X -)$$

The exactness of such can be checked on stalks $\mathcal{O}^h_{X,x}$ so we are reduced to check exactness of $(*)$ when $X$ is ess smooth over $k$ and local henselian.

CLAIM: Every rét cover $U_\bullet \to X$ when $X$ is ess smooth$/k$ and local henselian can be refined by a fret cover.

Pf: Up to refinement we assume $U_\bullet$ is affine too and since $X$ is local henselian, $U_\bullet = U_1^\bullet \sqcup U_2^\bullet$ with $U_2^\bullet$ mapping outside the closed point. → so that $U_1^\bullet \to X$ is finite étale. Now, since $(U_1^\bullet)_r \xrightarrow{f_r} X_r$ has as image an open subset of $X_r$ containing the closed point and the only open in $X_r$ with this property is the whole $X_r$, we deduce that $U_1^\bullet \to X$ is a. Moreover since $X_r$ is quasi-compact we just need a finite number of $U_1^\bullet$ to cover $X$ and set $U$ to be their disjoint union.

$F$ is rét separated: Take a rét cover $U_\bullet \to X$. $X$ can be assumed ess. smooth henzel local $\Rightarrow$ $\exists$ fret refinement $V \to U_\bullet \to X$ and now $F(X) \to F(U_\bullet) \to F(V)$ is inj $\Rightarrow$ $F(X) \hookrightarrow F(U_\bullet)$

$F$ is a rét sheaf: use the previous lemma. ∎

## ~~THEOREM~~ §3. Applications to homotopy modules



THEOREM : Let $F$ be a homotopy module over $K$ (= field of char 0) on which $\rho : F_n \to F_{n+1}$ is an isomorphism $\forall n$. Then $F_\cdot$ is a rét sheaf.

Proof : Thanks to proposition 2.4 we just need to check the sheaf condition on fét covers $U \xrightarrow{f} X$ with $X$ local Hensel and ess. smooth $/K$.
Last time we have seen that $F_\cdot$ is a module over $K_*^{TW}$, they both have transfers and they are related by the proj formulas

$$tr_f(a \cdot f^* b) = tr_f(a) \cdot b \qquad a \in K^{TW}_\cdot$$
$$tr_f(f^* a \cdot b) = a \cdot tr_f(b) \qquad b \in F_\cdot$$

- By the assumption on $\rho$, $F_*$ is naturally a module over $\operatorname{colim}_n K^{TW}_m \underset{/h}{\xrightarrow{\sim}} \operatorname{colim}_n I^n \overset{sgn}{\xrightarrow{\sim}} Sp_* \mathbb{Z}$
and these isomorphisms all respect transfers.
- We have a com. square

$$\begin{tikzcd} H^0_{ret}(u, \mathbb{Z}) \arrow[d, "tr"', two heads] & H^0_{ret}(U, \mathbb{Z}) \arrow[l, "="'] \arrow[d, "tr"] \\ H^0_{ret}(x, \mathbb{Z}) & H^0_{ret}(X, \mathbb{Z}) \arrow[l, "="] \end{tikzcd}$$

- Since $X$ is Hensel and $U/X$ finite ét $\Rightarrow$ $U$ is a finite disjoint union of local Hensel ess smooth schemes $/K$. Then equalities of horiz. arrows follow from Hensel condition as seen above
- Surjectivity of LHS transfer follows by PROPOSITION 2.1
$\Rightarrow \exists\, \alpha \in H^0_{ret}(U; \mathbb{Z})$ s.t. $tr(\alpha) = 1$.

$F_*$ is ret-separated :
If $s \in F_\cdot(X)$ s.t. $s_{|U} = 0$, then we have that
$$s = 1 \cdot s = tr_f(\alpha) \cdot s = tr_f(\alpha \cdot s_{|U}) = 0$$

$$\begin{tikzcd} U \times_X U \arrow[r, "p_1"] \arrow[d, "p_2"'] & U \arrow[d, "f"] \\ U \arrow[r, "f"'] & X \end{tikzcd}$$

$F_*$ is ret-sheaf :
If $s \in F_\cdot(U)$ s.t. $p_1^* s = p_2^* s$ in $F_\cdot(U \times_X U)$
$\rightsquigarrow tr(\alpha s) \in F_\cdot(X)$ will do the job

$$tr_f(\alpha s)_{|U} = f^*(tr_f(\alpha s)) = tr_{p_2} p_1^*(\alpha s) = tr_{p_2}(p_1^*(\alpha) \cdot p_1^*(s)) =$$
$$= tr_{p_2}(p_1^*(\alpha)\, p_2^*(s)) = tr_{p_2}(p_1^* \alpha) \cdot s = f^*(tr_f(\alpha)) \cdot s = 1 \cdot s = s$$ ∎