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Summary: Introduction to Category Theory

0. Preliminaries

Category theory occupies a central role in contemporary mathematics and theoretical computer science, while also having applications in mathematical and theoretical physics. It has been characterized as the general mathematical theory of structures and systems of structures. As such, it has become its own field of research. At the very least, it provides a powerful and convenient conceptual language for studying patterns that arise throughout mathematics. It starts with the observation that a lot of properties of mathematical systems can be studied in a unified manner by representation with diagrams of arrows.
Beyond that, category theory provides an alternative for the foundations of mathematics and raises several questions of methodological, epistemological and ontological nature in the philosophy of mathematics.

Remark 0.1: Sets vs. Classes

There is an important distinction between sets and so-called classes. In the context of 'standard' set theory (ZFC), there is no formal definition of what a class is. The axioms determine how sets 'behave' and any object of interest which cannot 'behave' like a set is then informally called (a) "(proper) class". Two of the most widely known examples are the von Neumann universe or the cumu-

lative hierarchy itself (basically the proper class of all pure sets) and the proper class of all ordinals. The latter shares all properties of an ordinal, but cannot be a set because otherwise it would be a member of itself and thus contradict the axiom of regularity.
Other axiomatizations of set theory - such as the von Neumann - Bernays - Gödel set theory - do have a formal definition of classes and are thus sometimes used as a foundation for category theory instead of ZFC in order to provide for a formal distinction between classes and sets.

1. Basic Definitions, Axioms and Examples

Remark 1.1 : Modus Operandi I

We first give some general axioms for categories without using any set theory. To distinguish them from categories in the narrower sense, which we will study later on (and have sets as the collection of objects and arrows), we call them "metacategories". This distinction should, however, become clearer down the line.

Definition 1.1 : Metagraphs

A **metagraph** consists of objects $a, b, c, \ldots$ and arrows $f, g, h, \ldots$ and two operations, as follows:

- *Domain* : Assigning each arrow f an object a ($=: \operatorname{dom} f$)
- *Codomain* : Assigning each arrow f an object b ($=: \operatorname{cod} f$)

In the visual representation, f is thus indicated by an actual arrow in a diagram starting at a (its domain) and going over to b (its codomain): $f: a \to b$ or $a \xrightarrow{f} b$

Definition 1.2 : Metacategories

A **metacategory** C is a metagraph with two additional operations:

Remark: In the context of (meta-)categories, "arrow(s)" and "morphism(s)" are used interchangeably.

- *Identity*: Assigning to each object a an arrow $id_a =: 1_a$, going from a to a.
- *Composition*: Assigning to each pair of arrows $\langle g, f \rangle$ with $\operatorname{dom} g = \operatorname{cod} f$ an arrow $g \circ f$, called (the) "composite", going from $\operatorname{dom} f$ to $\operatorname{cod} g$:

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & c \end{tikzcd}$$

These operations in a metacategory need to satisfy the following axioms:

- *Associativity*: For any given objects and arrows in the configuration $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ one always has the equality: $h \circ (g \circ f) = (h \circ g) \circ f$
  The pictorial representation of this equality is:

$$\begin{tikzcd} a \arrow[r, "h \circ (g \circ f) \,=\, (h \circ g) \circ f"] \arrow[d, "f"'] \arrow[dr, "g \circ f" near end] & d \\ b \arrow[r, "g"'] \arrow[ur, "h \circ g" near start] & c \arrow[u, "h"'] \end{tikzcd}$$

commutes.

In such diagrams, the objects are also called "vertices" and the morphisms "directed edges".
A diagram is commutative / commutes iff for each pair of vertices c, c', any two paths formed from directed edges leading from c to c' yield, by composition of labels, equal arrows from c to c'.

- *Unit Law*: For all arrows $f: a \to b$ and $g: b \to c$ composition with id_b gives:
  - $id_b \circ f = f$
  - $g \circ id_b = g$

  That is,

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "1_b \circ f"'] & b \arrow[d, "1_b"] \arrow[dr, "g \circ 1_b"] & \\
& b \arrow[r, "g"'] & c
\end{tikzcd}$$

  commutes.

  For each object a the corresponding arrow 1_a is uniquely determined by the above two equalities.

Mind again that we are not requiring the objects and/or arrows to be sets. They also do not have to form sets. At times, the collections of all objects of certain categories might for example form proper classes instead of sets. Since in category theory, not every category studied involves sets, it makes sense to first introduce the idea in general fashion, before moving on to more intuitive examples involving sets.

Example 1.1: The Metacategory of all Sets

Consider the metacategory of all sets. Its objects are all sets, while its arrows are the usual functions. Composition is given by the usual functional composition (for $f: X \to Y$ and $g: Y \to Z$ we get $g \circ f: X \to Z: x \mapsto g(fx)$ *omitting unnecessary brackets*) and the identity arrow for each set is just the identity map: $id_X: X \to X: x \mapsto x$.
The axioms are easily verified:

- Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, $id_Y \circ f$ sends an x to y and then said y to itself, i.e. $id_Y \circ f = f$. Similarly: $g \circ id_Y = g$

- Given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, take $x \in X$ and compute:

$$h \circ (g \circ f)(x) = h((g \circ f)(x)) = h(g(f(x)))$$
$$(h \circ g) \circ f(x) = (h \circ g)(f(x)) = h(g(f(x)))$$

Since x was arbitrary: $\forall x \in X \; (h \circ (g \circ f)(x) = (h \circ g) \circ f(x))$ and thus functional composition is associative.

Example 1.2: The Metacategory of all Groups

The metacategory of all groups has all groups as its objects and group homomorphisms as arrows. From Algebra I, we know that the composition of two group homomorphisms yields another group homomorphism. As such, verification of the axioms is again easy.

Example 1.3: The Metacategory of all Topological Spaces

Similarly, one defines the other metacategories such as the metacategory of all topological spaces with continuous maps as arrows.

Remark 1.2: An Alternative Definition

There is a natural 'correspondence' between an object and its identity arrow. As such, we could also omit the objects from our category talk and only deal with arrows. The basic 'ingredients' of an arrows-only category C are arrows, certain ordered pairs of arrows $\langle g, f \rangle$ (the composable arrows) and an operation assigning to each composable pair its composite, i.e. assigning to each $\langle g, f \rangle$ an arrow $g \circ f$. The sentences "$g \circ f$ is defined" and "$\langle g, f \rangle$ is a composable pair" are treated as equivalent. One defines an identity of C to be an arrow u such that $f \circ u = f$ and $u \circ g = g$ whenever these composites are defined.

The arrows need to satisfy the following axioms:

i) The composite $(h \circ g) \circ f$ is defined iff $h \circ (g \circ f)$ is defined. If one of them is defined, they are equal. We write "$h \circ g \circ f$" for the triple composite.

ii) The triple composite $h \circ g \circ f$ is defined whenever $h \circ g$ and $g \circ f$ are defined.

iii) For each arrow g of C there are identity arrows u and u' of C such that $u' \circ g$ and $g \circ u$ are defined.

The pair u, u' in iii) serves as the pair 'domain, codomain' of g.

Theorem 1.1: Equivalence of Definitions

The definition of a metacategory in Def. 1.2 and the definition of a metacategory in Rmk 1.2 are equivalent.

Proof: Omitted / Exercise

($\square$)

Remark 1.3: Modus Operandi II

Having dealt with the idea of category in an axiomatic way without relying on set theory, we now wish to move on to categories in the narrower sense, i.e. any interpretation of the metacategory axioms within set theory.

Definition 1.3: Directed Graphs

A **directed graph** is a set of objects $\mathcal{O}$ and a set of arrows $\mathcal{A}$ as well as two functions:

- $dom : \mathcal{A} \to \mathcal{O} : f \mapsto dom f$
- $cod : \mathcal{A} \to \mathcal{O} : f \mapsto cod f$

In such a graph, the set of composable arrows is the set $\mathcal{A} \times_{\mathcal{O}} \mathcal{A} := \{ <g, f> : f, g \in A \wedge dom\, g = cod f \}$, called (the) "product over $\mathcal{O}$", hence the notation.

Definition 1.4: Categories

A **category** C is a directed graph with two additional functions:

- $\mathcal{O} \to \mathcal{A} : a \mapsto id_a$ (*identity*)
- $\mathcal{A} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A} : \langle g, f \rangle \mapsto g \circ f$ (*composition*)

such that:

- $\mathrm{dom}\, id_a = \mathrm{cod}\, id_a = a$
- $\mathrm{dom}\, g \circ f = \mathrm{dom}\, f$ and $\mathrm{cod}\, g \circ f = \mathrm{cod}\, g$

for all objects $a \in \mathcal{O}$ and pairs $\langle g, f \rangle \in \mathcal{A} \times_{\mathcal{O}} \mathcal{A}$.
Last but not least, these functions need to satisfy the axioms *Associativity* and *Unit Law*.
By abuse of notation, we usually drop the talk of $\mathcal{O}$ and $\mathcal{A}$ and write "$c \in C$" and "f in C" instead, respectively.
For two objects $a, b \in C$ we write "$\hom(a,b)$" for the set $\{f \in C : \mathrm{dom} f = a \wedge \mathrm{cod} f = b\}$ of arrows between a and b. Such sets are called "hom-sets". Alternatively, we could take composition as a function for each triple of objects a, b and c sending a pair $\langle g, f \rangle \in \hom(b,c) \times \hom(a,b)$ to $g \circ f \in \hom(a,c)$.

Examples 1.4: Some Special Categories

We define the following special categories:

- $\mathbf{0}$ is the empty category: no objects, no arrows.
- $\mathbf{1}$ is the category with one object and one identity arrow.
- $\mathbf{2}$ is the category with two objects a and b and just one arrow $a \to b$ not the identity.
- $\mathbf{3}$ is the category of three objects a, b and c whose non-identity arrows are arranged in a triangle:

$$\begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd}$$

- $\downarrow\downarrow$ is the category of two objects a and b and only two arrows from a to b which are not identity arrows:

$$a \rightrightarrows b$$

Such arrows are called "parallel".

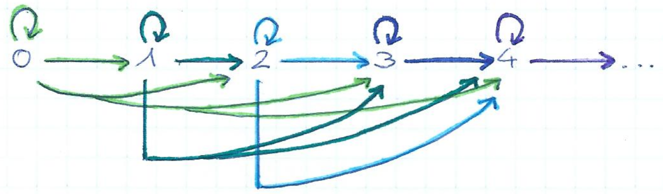
Definitions 1.5 : Some Vocabulary on Categories

- A **discrete category** is a category in which all arrows are identity arrows. Every set X can function as the set of objects of a discrete category. The arrows of such a category are simply arrows of the sort $a \to a$ for every $a \in X$. On the other hand, every discrete category is determined solely by its objects in exactly this manner. Thus, discrete categories are sets.
- A **monoid** is a category with only one object. Each monoid is thus determined solely by its arrows, the identity arrow and by the operation of composition of arrows. As such, every two arrows in a monoid have a composite and the monoid can be described as a set M endowed with a semigroup structure and an identity element (in virtue of being a category, it satisfies *Associativity* and *Unit Law*). Thus, a monoid in the context of category theory is exactly a semigroup with an identity element, i.e. a monoid in the context of algebra.

  In any case, for any category C and an object $a \in C$ the set $\hom(a, a)$ is a monoid.
- A **group** is a category with one object in which every arrow has a two-sided inverse under composition.
- For each commutative ring K, the set **Mat(K)** of rectangular matrices with entries in K forms a category: the objects are positive integers n, m, etc. and each

$m \times n$ matrix A is regarded as an arrow $A: n \to m$, with composition being the usual matrix product.

- For any set of sets V, **Ens(V)** is the category with objects elements in V and arrows all functions between members of V. Composition is given by the usual functional composition.
- A preorder is a category P in which, given two objects $p, p' \in P$, there is at most one arrow $p \to p'$. In any preorder P we can define a binary relation as follows: for $p, p' \in P$ we have $p \leq p' :\Leftrightarrow p \to p'$ is in P. Due to composition and the existence of identity arrows, $\leq$ is transitive and reflexive. Conversely, every set P with such an order defines a preorder.
  Preorders include partial orders (antisymmetry added) and linear orders (antisymmetry + totality added).
- Since every ordinal is linearly ordered by $\in$, it constitutes a preorder. For example, **1**, **2** and **3** are the preorders belonging to the ordinals 1, 2 and 3. ω is the first infinite ordinal defining the category:



- $\mathbf{\Delta}$ is the category with objects all finite ordinals and arrows all order-preserving maps ($x \in y$ in $m \in \mathbf{\Delta}$ implies $f(x) \in f(y)$ in $n \in \mathbf{\Delta}$ for $f: m \to n$ in $\mathbf{\Delta}$). This category is also called (the) "simplicial category".
- **Finord** has the same objects as $\mathbf{\Delta}$ but the arrows are all maps between finite ordinals.

Definitions 1.6: Small, Locally Small and Large Categories

Remark: I had to piece these things together myself, since none of the references were very clear about this. Enjoy at your own risk.

In addition to the metacategory of all sets, which cannot be a set itself (it would have to be a member of itself, contradicting the axiom of regularity), we would like to have an actual category **Set**, the category of all small sets. We assume that there is a big enough 'set' $\mathcal{U}$, the **universe**, and call a 'member' $u \in \mathcal{U}$ a **small set**. Similarly, a small group is a small set endowed with a group structure, a small monoid is a small set endowed with a monoid structure, etc. Accordingly, a function $f: X \to Y$ is small iff it is a small set, i.e. a function between small sets. We now form the category of the 'elements' of $\mathcal{U}$, the small sets. As such, the arrows of **Set** are all functions between the small sets. Similarly to this, we construct other so-called **large categories**, i.e. (meta-)categories whose collection of objects and/or arrows are no longer small sets, but for example proper classes, such as:

Name	Objects	Arrows
Cat	small categories	functors (cf. section 2)
Mon	small monoids	morphisms of monoids
Grp	small groups	morphisms of groups
Ab	small abelian groups	morphisms of (abelian) groups
Rng	small rings	morphisms of rings
CRng	small commutative rings	morphisms of (commutative) rings
R-**Mod**	small left modules over ring R	linear maps
Mod-R	small right modules over ring R	linear maps
K-**Mod**	small modules over comm. ring K	linear maps
Top	small topological spaces	continuous maps

A (meta-)category is **small** iff both the collection of objects and arrows are small sets.

A (meta-) category is **locally small** iff all its hom-sets are small sets.

Cf. section 4 for a slightly more formal treatment in ZFC.

2. Functors

Definition 2.1: Functors

A **functor** is basically a morphism of categories. Let B and C be two categories. A functor $T: C \to B$ with domain C and codomain B consists of two functions:

- the object function T assigning each object of C an object of B: $c \mapsto Tc$ and
- the arrow function (also written) T assigning each morphism $f: c \to c'$ in C a morphism $Tf: Tc \to Tc'$ in B,

such that

- $T(id_c) = id_{Tc}$ for all objects $c \in C$
- $T(g \circ f) = Tg \circ Tf$, whenever the composite is defined in C

A functor can also be characterized using only arrows. In this case, it carries every arrow f in C to an arrow Tf in B such that each identity of C is carried to an identity of B and each composable pair $\langle g, f \rangle$ is carried to a composable pair $\langle Tg, Tf \rangle$ satisfying $Tg \circ Tf = T(g \circ f)$

Example 2.1: Powerset-functor

Consider the functor $\mathcal{P}: \mathbf{Set} \to \mathbf{Set}$. Its object function sends each set X to its powerset $\mathcal{P}(X)$. Its arrow function maps each function $f: X \to Y$ to a function $\mathcal{P}f: \mathcal{P}(X) \to \mathcal{P}(Y)$, which sends each $A \subseteq X$ to $f(A) \subseteq Y$. One verifies the two axioms of a functor:

- In the case of the map id_X, the resulting morphism $\mathcal{P} id_X : \mathcal{P}(X) \to \mathcal{P}(X)$ sends $A \subseteq X$ to $id_X(A) = A$, i.e. $A \mapsto A$. Thus: $\mathcal{P} id_X = id_{\mathcal{P}(X)}$
- Assume we have two functions g, f, which are composable: $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then $\mathcal{P}(g \circ f)$ sends each $A \subseteq X$ to $g(f(A)) \subseteq Z$. This is the same as sending the subset $f(A) \subseteq Y$ to the subset $g(f(A))$ after sending A to $f(A)$, ergo: $\mathcal{P}(g \circ f) = \mathcal{P}g \circ \mathcal{P}f$

Examples 2.2: Functors in Algebra

Functors arise naturally in algebra.
Consider for some commutative ring K the set of all non-singular square matices of length n ($\in \mathbb{N}$) with entries in K. It constitutes the general linear group $GL_n(K)$. Additionally, each homomorphism of rings $f: K \to K'$ produces a homomorphism of groups $GL_n f: GL_n(K) \to GL_n(K') : (a_{ij})_{i,j} \mapsto (f(a_{ij}))_{i,j}$. These data define for each natural number n a functor $GL_n : \mathbf{CRng} \to \mathbf{Grp}$, sending each commutative ring K to $GL_n(K)$ and each morphism $f: K \to K'$ to $GL_n f: GL_n(K) \to GL_n(K')$.
Consider the derived subgroup $D(G)$ of a group G: $D(G) = \{ [x,y] : x, y \in G \}$. Since any morphism of groups $G \to H$ carries commutators to commutators, the assignment $G \mapsto D(G)$ defines a functor $\mathbf{Grp} \to \mathbf{Grp}$
Similarly, the assignment $G \mapsto G/D(G)$ induces a functor $\mathbf{Grp} \to \mathbf{Ab}$, the factor-commutator functor.

Definition 2.2: Forgetful Functors

A functor which 'forgets' some or all of the structure between two categories is called a "**forgetful**" or "underlying".

Examples 2.3: Forgetful Functors

Consider the functor $U : \mathbf{Grp} \to \mathbf{Set}$ sending as its object function each group G to the set UG, which is basically just the underlying set G of the group G without any of the algebraic structures. As such, morphisms are mapped to themselves, regarded then as simple functions, however: $f : G \to H \mapsto Uf : UG \to UH$.
Another example is the functor $S : \mathbf{Rng} \to \mathbf{Ab}$, sending each ring R to the abelian group of R and, again, sending each morphism to itself, regarded then as a morphism of groups instead of a morphism of rings.

Definition 2.3: Composite Functors

Functors can be composed. Let $T : C \to B$ and $S : B \to A$ be two functors between categories A, B and C. The composite functions:

- $c \mapsto S(Tc)$ (*composite object function*)
- $f \mapsto S(Tf)$ (*composite arrow function*)

form the **composite functor** of S with T (mind the order) $S \circ T : C \to A$. This composition is associative and for each category B we have an identity functor $I_B : B \to B$, which acts as an identity for this composition.
Thus, we can form the metacategory of all categories with objects all categories and morphisms all functors. Additionally, we can form the category **Cat** of all small categories, but not the category of all categories, whose collection of objects is most certainly not a small set.

Definition 2.4: Isomorphisms of Categories

An **isomorphism** $T : C \to B$ between two categories B

and C is a functor from C to B, which is a bijection both on objects and morphisms. Alternatively, yet equivalently, a functor $T: C \to B$ is an isomorphism iff there is a functor $S: B \to C$ such that both $S \circ T$ and $T \circ S$ are identity functors, on C and B respectively. S is called (the) "two-sided inverse of T" and we write "$S = T^{-1}$".

Definitions 2.5: Full and Faithful Functors

A functor $T: C \to B$ of two categories C, B is **full** iff for every pair of objects c and c' in C and every arrow $g: Tc \to Tc'$ in B There is an arrow in C $f: c \to c'$ such that $g = Tf$. The composite of two full functors is again full.

A functor $T: C \to B$ of two categories C, B is **faithful** iff to every pair of objects $c, c' \in C$ and to every pair of parallel arrows $f_1, f_2: c \to c'$ in C we have: $Tf_1 = Tf_2 : Tc \to Tc' \Rightarrow f_1 = f_2$. The composite of two faithful functors is again faithful. Faithful functors are also called "embeddings".

A functor that is both full and faithful is called "**fully faithful**".

Example 2.4: Isomorphism of Groups

Let $\varphi: G \to G'$ be an isomorphism of groups. As such, φ also functions as an isomorphism of categories, namely between the two group categories formed from G and G'.

Example 2.5

The forgetful functor U: **Grp** $\to$ **Set** is faithful but not full and not a bijection on objects.

Remark 2.1: Alternative Characterization

The two properties in Def. 2.5 can also be visualized in terms of hom-sets. Given a pair of objects $c, c' \in C$ in a category C, the arrow function of a functor $T: C \to B$ to some category B assigns each arrow $f: c \to c'$ an arrow $Tf: Tc \to Tc'$ and thus defines a function: $T_{c,c'}: \hom(c, c') \to \hom(Tc, Tc'): f \mapsto Tf$
T is full iff every such function is surjective and faithful iff every such function is injective. In the case of a fully faithful functor every such function is a bijection. This, however, does not imply that that the functor is an isomorphism of categories, since there may be objects in B not in the image of T.

Definitions 2.6: Subcategories and Inclusion Functors

A **subcategory** S of a category C is a collection of objects and arrows of C such that S is a category on its own:
- With each arrow f of S both its domain and codomain belong to S.
- For each object $s \in S$ the arrow id_s is in S.
- For each pair of composable arrows $s \to s' \to s''$ in S their composite is in S.

The inclusion map $S \to C$ is a functor, called (the) "**inclusion functor**". It is automatically faithful.
S is a **full subcategory** iff the inclusion functor is full.
A full subcategory S of a category C is thus determined by the set of its objects alone, since the arrows between any two objects s, s' in S are all morphisms $s \to s'$ in C.

Example 2.5: $\mathbf{Set}_f$

Consider the category $\mathbf{Set}_f$ of all finite sets and functions.

It is a full subcategory of **Set**.

Example 2.6 : Functors with Domain $\mathbf{1}$

Fix a category C. Let $T: \mathbf{1} \to C$ be functor. Since $\mathbf{1}$ has only one object and one arrow (identity) and needs to satisfy $T 1_c = 1_{Tc}$, the object function of T essentially chooses one object c^* in C and the arrow function maps the identity in $\mathbf{1}$ to id_{c^*}. Thus, a functor $T: \mathbf{1} \to C$ corresponds to an object (with its identity arrow) in C.

Theorem 2.1 : Functors between Preorders

A functor T between two preorders P and P' 'is' a function that is monotonic, i.e. $p \leq p'$ in P implies $Tp \leq Tp'$ in P'

Proof: Recall that a preorder $(P, \leq)$ in the context of set theory can be seen as a preorder in the context of category theory and vice versa, via the equivalence : $p_1 \leq p_2 \iff$ there is an arrow $p_1 \to p_2$.
Let $T: P \to P'$ be a functor between two preorders $\Rightarrow$ every pair p, p' with $p \to p'$ in P is sent to $Tp \to Tp'$ in P'. Thus $p \leq p'$ implies $Tp \leq Tp'$, concerning the object function of T.
Let $T: P \to P'$ be a function between two preorders $(P, \leq)$, $(P', \leq)$ that is monotonic. We immediately consider P, P' as categories and fix T as the object function. Since T is order-preserving, it must 'implicitly' send each arrow $p \to p'$ in P to $Tp \to Tp'$ in P'. We simply make this explicit by additionally defining a function which deals with those arrows. It follows that these two functions form a functor. ▨

Theorem 2.2: Functors between Groups

A functor T between two groups G, G' in the context of category theory 'is' a morphism of groups between G, G' in the context of group theory.

Proof: We use the same idea as in Thm 2.1: Every category that is a group (one object, every arrow has a two-sided inverse under composition) can also be considered a group in the context of group theory, by relating the identity arrow to the neutral element and the composition between arrows to the group operation.
As such, a functor $T: G \to G'$ between groups sends the object to the object and sends the identity arrow 1_G to $1_{G'}$ while also satisfying $T(g \circ f) = Tg \circ Tf$. By the above relation, T exactly satisfies the group homomorphism conditions. Conversely, a group homomorphism $\varphi: G \to G'$ can function as the arrow function of a functor. The object function is additionally defined as the function $G \mapsto G'$. The functor axioms are immediately verified. ▨

'Theorem' 2.3: Functors **Grp** $\to$ **Ab**

There is no functor T: **Grp** $\to$ **Ab** sending as its object function G to its center $Z(G)$.

Proof: Assume the converse, i.e. there is such a functor. Consider the diagram:

$$S_2 \xrightarrow{f} S_3 \xrightarrow{g} S_2$$

with S_2, S_3 being the usual symmetric groups.

The image of this diagram under T is:

$$\begin{array}{ccccc} \mathcal{Z}(S_2) & \xrightarrow{Tf} & \mathcal{Z}(S_3) & \xrightarrow{Tg} & \mathcal{Z}(S_2) \\ \| & & \| & & \| \\ S_2 & & \{id\} & & S_2 \end{array}$$

S_2 is abelian

Recall: $S_2 = \{id, \tau\}$ with τ being the transposition of two elements. Let us make the following choice for f, g:

$f: id_{S_2} \mapsto id_{S_3}$; $g: id_{S_3} \mapsto id_{S_2}$

$\tau \mapsto (1,2)$ $\quad (1,2) \mapsto \tau$

In the case of Tg and Tf, we only have one option each:

$Tf: id_{S_2} \mapsto id_{S_3}$; $Tg: id_{S_3} \mapsto id_{S_2}$

$\tau \mapsto id_{S_3}$

$\Rightarrow Tg \circ Tf(id_{S_2}) = id_{S_2}$, $Tg \circ Tf(\tau) = id_{S_2}$

Yet by our choice of f and g, we have $g \circ f = 1_{S_2}$

T is a functor $\Rightarrow T(g \circ f) = T(1_{S_2}) = 1_{S_2} = Tg \circ Tf$. But $Tg \circ Tf$ cannot be 1_{S_2} ↯

∎

3. Natural Transformations

Definition 3.1: Natural Transformations

Given two functors $S, T: C \to B$ between two categories, a **natural transformation** is a function $\tau: S \xrightarrow{\cdot} T$, which assigns to each object $c \in C$ an arrow $\tau c: Sc \to Tc$ in B such that every arrow $f: c \to c'$ yields a diagram

$$\begin{tikzcd} c \arrow[d, "f"] & Sc \arrow[r, "\tau c"] \arrow[d, "Sf"'] & Tc \arrow[d, "Tf"] \\ c' & Sc' \arrow[r, "\tau c'"'] & Tc' \end{tikzcd}$$

, which commutes.

Iff this holds, we call τc "natural" in c.

By thinking of the functor S as giving an image of C in B, we can think of the function τ as translating said image into the image of T with all squares and parallelograms such as

$$
\begin{tikzcd}
a \arrow[dr, "f"] \arrow[dd, "h"'] & \\
& b \arrow[dl, "g"] \\
c &
\end{tikzcd}
\qquad
\begin{tikzcd}
Sa \arrow[rrr, "\tau a"] \arrow[dr, "Sf"] \arrow[dd, "Sh"'] & & & Ta \arrow[dr, "Tf"] \arrow[dd, "Th"] & \\
& Sb \arrow[rrr, "\tau b"] \arrow[dl, "Sg"] & & & Tb \arrow[dl, "Tg"] \\
Sc \arrow[rrr, "\tau c"] & & & Tc &
\end{tikzcd},
$$

commuting.

We call τa, τb, etc. (the) "components" of the natural transformation. A natural transformation is sometimes also called (a) "morphism of functors"

A natural transformation τ with every element $\tau c \in B$ invertible is called (a) "**natural equivalence**" or "natural isomorphism": $\tau: S \cong T$. In such a case, the inverses $(\tau c)^{-1}$ are the components of a natural isomorphism $\tau^{-1}: T \dot{\to} S$.

Example 3.1: The Determinant

The determinant of matrices is a natural transformation. Let K be a commutative ring and $\det_K(M)$ be the determinant of the $n \times n$-matrix M with entries in K. M is non-singular iff $\det_K(M)$ is a unit, i.e. $\det_K(M) \in K^*$, $K^* = \{k \in K : k \text{ is invertible}\}$. Now, $\det_K : GL_n(K) \to K^*$ is a morphism of groups and as such an arrow in **Grp**. Because the determinant is defined by the same formula for all commutative rings, each morphism $f: K \to K'$ of commutative rings produces a commutative diagram:

$$
\begin{tikzcd}
GL_n K \arrow[r, "\det_K"] \arrow[d, "GL_n f"'] & K^* \arrow[d, "f^*"] \\
GL_n K' \arrow[r, "\det_{K'}"] & K'^*
\end{tikzcd}
$$

This states that the transformation $\det : GL_n \xrightarrow{\cdot} (\,)^*$ is natural between two functors **CRng** $\to$ **Grp**, the first one being GL_n from Ex 2.2. The second one is a functor which sends each ring K to the group K^* and each morphism of (commutative) rings $f : K \to K'$ to the corresponding morphism of groups $f^* : K^* \to K'^*$.

Example 3.2: The projection $p_G : G \to G/D(G)$

For each group G the projection $p_G : G \to G/D(G)$ defines a transformation π from the identity functor on **Grp** to the factor-commutator functor from Ex. 2.2. π is natural, because each group homomorphism $f : G \to H$ defines a homomorphism f' such that

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d, "f"'] & G/D(G) \arrow[d, "f'"] \\ H \arrow[r, "\pi_H"] & H/D(H) \end{tikzcd}$$

commutes.

Example 3.3: **Finord** and **Set$_f$**

Another example arises when we compare **Finord** with **Set$_f$** (is some universe U). Every ordinal in ω is a finite set, so the inclusion S: **Finord** $\to$ **Set$_f$** is a functor. On the other hand, each finite set X has an ordinal $n \in \omega$ corresponding to it via the equation $n = |X|$. We choose for each X a bijection $\theta_X : X \to |X|$. If X itself is an ordinal, we choose $\theta_X = id_X$. For any function $f : X \to Y$ between finite sets we have a corresponding function $|f| : |X| \to |Y|$ between ordinals by $|f| = \theta_Y \circ f \circ \theta_X^{-1}$. As such,

$$\begin{tikzcd} X \arrow[r, "\theta_X"] \arrow[d, "f"'] & {|X|} \arrow[d, "|f|"] \\ Y \arrow[r, "\theta_Y"] & {|Y|} \end{tikzcd}$$

commutes. Thus, θ is a natural transformation and 1.1 : **Set**$_f \to$ **Finord** is a functor. By choosing $\theta_X = id_X$ if $X = |X|$, we ensure that the composite functor $1.1 \circ S$ is the identity functor I' on **Finord**. On the other hand, the composite $S \circ 1.1$ is not the identity functor on **Set**$_f$, because it sends each X in **Set**$_f$ to a special finite set: its cardinal (which in the case of finite sets is the same as an ordinal). However, the diagram above does show that $\theta : I \xrightarrow{\cdot} S \circ 1.1$ is a natural isomorphism. Thus, we get: $I \cong S \circ 1.1$ and $I' = 1.1 \circ S$. This gives rise to

Definition 3.2: Equivalence between Categories

An **equivalence** between two categories C and D is a pair of functors $S : C \to D$ and $T : D \to C$ together with natural isomorphisms $I_C \cong T \circ S$ and $I_D \cong S \circ T$. This notion allows us to compare categories which are 'alike' but of different 'sizes.'

Example 3.4: Final Example

- If H is a fixed group, then the assignment $G \mapsto H \times G$ induces a functor $H \times - :$ **Grp** $\to$ **Grp**, which additionally sends each morphism $G \xrightarrow{f} G'$ of groups to the morphism $1_H \times f : H \times G \to H \times G' : \langle h, g \rangle \mapsto \langle h, f(g) \rangle$
- Observe that each morphism of groups $\varphi : H \to K$ defines a natural transformation as follows:
  Let $\tau : H \times - \xrightarrow{\cdot} K \times -$ be defined by sending G to the function $\varphi \times 1_G : H \times G \to K \times G : \langle h, g \rangle \mapsto \langle \varphi(h), g \rangle$. As such,

$$\begin{tikzcd} G \arrow[d, "f"'] & H \times G \arrow[r, "\varphi \times 1_G"] \arrow[d, "1_H \times f"'] & K \times G \arrow[d, "1_K \times f"] \\ G' & H \times G' \arrow[r, "\varphi \times 1_{G'}"] & K \times G' \end{tikzcd}$$

commutes.

4. A Foundational Remark

We wish to study totalities of mathematical objects in category theory, such as the category of all groups or of all topological spaces. Singling out these objects/categories within the framework of axiomatic set theory would mean that we apply a comprehension principle of the following form:

Let φ be a first-order formula with $\mathrm{free}(\varphi) = \{x\}$. Then we can form the set $\{x : \varphi(x)\}$ of all sets satisfying φ.

This principle, however, is unacceptable, since it leads to contradictions such as Russell's paradox: Take $\varphi \equiv x \notin x$ and form $A := \{x : x \notin x\}$. Now, being a member of A is equivalent to satisfying $x \notin x$. Thus: $A \in A \Leftrightarrow A \notin A$, a contradiction.
As a consequence, in axiomatic set theory, we restrict the comprehension principle to already existing sets:

Axiom Schema of Restricted Comprehension: Let φ be a first-order formula with $\mathrm{free}(\varphi) = \{x\}$. Then the following formula is an axiom: $\forall z \exists y \forall x \, (x \in y \leftrightarrow (x \in z \wedge \varphi(x)))$

This is the formulation in ZFC. There is an analoguous axiom schema in NBG.
This is one of the ten axioms of ZFC. The others are:

Axiom of the Empty Set: $\exists x \forall y \, (y \notin x)$
Axiom of Extensionality: $\forall x \forall y \, (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.
Axiom of Pairing: $\forall x \forall y \exists u \forall z \, (z \in u \leftrightarrow (z = x \vee z = y))$
Axiom of Union: $\forall x \exists u \forall z \, (z \in u \leftrightarrow \exists w \in x \, (z \in w))$ *)
Axiom of Infinity: $\exists I \, (\emptyset \in I \wedge \forall x (x \in I \rightarrow x \cup \{x\} \in I))$
Axiom of Powerset: $\forall x \exists y \forall z \, (z \in y \leftrightarrow z \subseteq x)$

*) *This u is usually denoted by* "$\bigcup x$"

Axiom Schema of Replacement: A formula $\varphi(x,y)$ with $\mathrm{free}(\varphi) = \{x,y\}$ is a class function iff $\forall x \, \exists! y \, \varphi(x,y)$. Now, is $\varphi(x,y)$ a class function, then the following formula is an axiom: $\forall A \, (\forall x \in A \, \exists! y \, \varphi(x,y) \rightarrow \exists B \, \forall x \in A \, \exists y \in B \, \varphi(x,y))$

Remark: "For any class function φ, there is a set B which is a superset of the image of φ" would be a more intutive formulation of the above axiom schema.

Axiom of Foundation: $\forall x \, (x \neq \emptyset \rightarrow \exists y \in x \, (y \cap x = \emptyset))$

Axiom of Choice: $\forall \mathcal{F} \, (\emptyset \notin \mathcal{F} \rightarrow \exists f : \mathcal{F} \rightarrow \bigcup \mathcal{F} \, (\forall x \in \mathcal{F} \, (f(x) \in x)))$

Let us quickly go back at Def. 1.6. In it, we assumed the existence of a set $\mathcal{U}$, called (the) "universe", in which we made the distinction between small, locally small and large in relation to $\mathcal{U}$. In general, this set $\mathcal{U}$ needs to satisfy the following conditions:

i) $\mathcal{U}$ is pure *(this is omitted in some parts of the literature)*

ii) $x \in \mathcal{U}$ and $v \in \mathcal{U}$ implies $\{u,v\}$, $\langle u,v \rangle$ and $u \times v \in \mathcal{U}$

iii) $x \in \mathcal{U}$ implies $\mathcal{P}(x) \in \mathcal{U}$ and $\bigcup x \in \mathcal{U}$

iv) $\omega \in \mathcal{U}$; $\omega = \{0, 1, 2, 3, \dots\}$ the set of all finite ordinals / all natural numbers / all finite cardinals.

v) If $f: a \rightarrow b$ is a surjective function with $a \in \mathcal{U}$ and $b \subseteq \mathcal{U}$, then $b \in \mathcal{U}$

vi) $x \in u \in \mathcal{U}$ implies $x \in \mathcal{U}$.

For our purposes, we fix some $\mathcal{U}$ once, which is large enough so that we can do our category theory in it.

To give this issue a more formal treatment, assume that ZFC is consistent, i.e. no contradiction can be deduced from the axioms. Then, in virtue of Gödel's completeness theorem, ZFC has a model, denoted $\mathbb{V}$ and called (the) "von Neumann universe" or (the) "cumulative hierarchy". $\mathbb{V}$ is the collection of all pure sets and all of ordinary mathematics can be understood to take place in $\mathbb{V}$.

$\mathbb{V}$ satisfies all properties that $\mathcal{U}$ in Def. 1.6 needs to have (v) corresponds to *Replacement*, for example), with the exception of i): While all its 'elements' are pure sets, making $\mathbb{V}$ 'pure', $\mathbb{V}$ cannot be a set. Indeed, if it were, $\mathbb{V} \in \mathbb{V}$. But *Foundations* says the each non-empty set needs to have an $\in$-minimal element. But $\mathbb{V} \in \mathbb{V}$ would mean that $\{\mathbb{V}\}$ (constructable thanks to *Pairing*) would contradict *Foundation*. This is what I meant in Rmk 0.1 about the behaviour of sets and (proper) classes. We are forced to conclude that $\mathbb{V}$ is a proper class. Still, taking it as $\mathcal{U}$ in Def. 1.6 allows us to define small, locally small and large accordingly: We have a precise idea of sets with which "small set" becomes equivalent (rather: coextensional) with just "set", and, for example, large categories are those whose collection of objects and/or arrows form(s) proper classes. The downside to this is that **Set** is no category anymore, but stays a meta-category, because the collection of objects is $\mathbb{V}$, a proper class.
Even worse, in higher category theory, one also studies collections of proper classes, which can fail to be proper classes themselves, such as **Cls**, the category of all (proper) classes. Since these objects cannot be dealt with in either ZFC or NBG, mathematicians and logicians seek foundations of category theory (and by extension the whole of mathematics) without set theory.

5. A Closing Philosophical Remark

Concerning the latter endeavor above, one observes that doing mathematics in categorial framework is almost always radically different from doing it in the framework of

axiomatic set theory, such as ZFC. As such, discussions arise concerning the precise nature of the entities studied, the nature of the knowledge involved and the nature of the methods employed in mathematics.
As has become clear, three characteristics of mathematical objects in a categorial framework are especially different from their characteristics in a set-theoretical framework:

- Objects are always given together with an ambient category; there no objects existing without being part of some category.
- Objects are characterized solely by the morphisms coming to and going from the objects themselves.
- Objects are always characterized up to an isomorphism.

For example, there is no such thing as *the* natural numbers. It can be argued, however, that there is still one *concept* of the natural numbers. Still, it is hard to resist the conclusion that founding mathematics on category theory embodies some sort of structuralism. Structuralism is the position in philosophy of mathematics that maintains that every mathematical theory describes structures, i.e. 'places' having some structural relations to each other. As such, a mathematical theory describes positions or places in structures, but not objects in the sense of uniquely determined things. For example, the natural number 3 is not an object (in the sense above) anymore, but a place in the structure of the natural numbers. This view arises from the identity criterion at work in a categorial framework. Contrast it with the one in ZFC: Sets are equal iff they have the same elements (*Extensionality*). Here, elements (which are always sets themselves) are given by direct reference via logical constants ("$\emptyset$" for the unique set which has no members), which is why in ZFC sets

are determined uniquely.
Alternatively, one could also interpret the situation in a categorial framework by thinking of mathematical objects as *types*, for which there are different tokens in different contexts, similarly to 'redness' and red (concrete) objects. In such an interpretation, one refers to a token of a type, yet the mathematical theory in question directly characterizes the type, not the tokens.

Observe that this discussion makes no claim concerning the ontology (i.e. the existence) of mathematical objects, but only concerning identification (the nature) of mathematical objects. In the case of structuralism, three different ontological positions have been developed:

- *Ante rem* structuralism: Structures are thought to exist without any concrete instantiations and are thus treated as abstract entities in a platonic fashion (counterpart to classical platonism).
- *In rem* structuralism: Structures are real insofar they are instantiated concretely in the physical world (counterpart to Aristotelian realism).
- *Post rem* structuralism: Structures are treated as 'human inventions'. While mathematical systems exist, talk of their structural 'identities' or 'similarities' is nothing but a convenient and instrumental talk - they have no independent existence (counterpart to nominalism).

This demonstrates that category theory and its related foundational issues reach beyond the discipline of mathematics to questions regarding the nature, existence and knowledge of mathematical entities themselves.

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