GABRIEL'S THEOREM

Seminar: Advanced topics in linear algebra

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Content

1. Basic definitions

1.1. Quivers

Definition 1.1. A quiver is a directed finite graph $Q = (Q_0, Q_1, s, t)$.
Q_0: finite set of vertices
Q_1: finite set of arrows
s, t: two maps $Q_1 \to Q_0$ such that an arrow α starts at $s(\alpha)$ and ends at $t(\alpha)$

Definition 1.2. A representation of Q is a collection $X = (X_i, X_\alpha)$, $i \in Q_0$, $\alpha \in Q_1$.

Definition 1.3. A representation X is called indecomposable if
$X \neq 0$ and for $X = X_1 \oplus X_2 \Rightarrow X_1 = 0$ or $X_2 = 0$

Definition 1.4. Let X be an indecomposable representation of Q. We define
X is preprojective if $X \cong C^r P(i)$ for some vertex i and some $r \leq 0$
X is preinjective if $X \cong C^r I(i)$ for some vertex i and some $r \geq 0$
X is regular if $C^r X \neq 0$ for all $r \in \mathbb{Z}$

Remark For the definitions of C^r, see [1] pages 8-10.

1.2. Finite graphs

Definition 1.5. Let Γ be a finite graph with n vertices. Denote the edges between two vertices i and j with $d_{ij} = d_{ji}$.
Γ induces a symmetric bilinear form $(-, -)$ and a quadratic form q:

$$(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad (e_i, e_j) = \begin{cases} -d_{ij} & \text{if } i \neq j \\ 2 - 2d_{ii} & \text{if } i = j \end{cases}$$

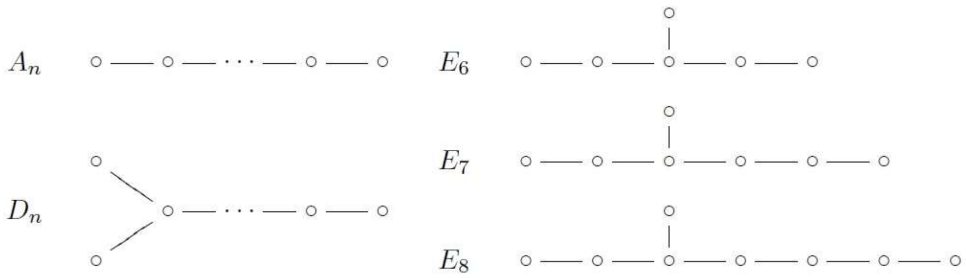
$$q\colon \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j.$$

Definition 1.6. $\mathrm{rad}q = \{x \in \mathbb{Z}^n \mid (x, -) = 0\}$ is called the radical of q

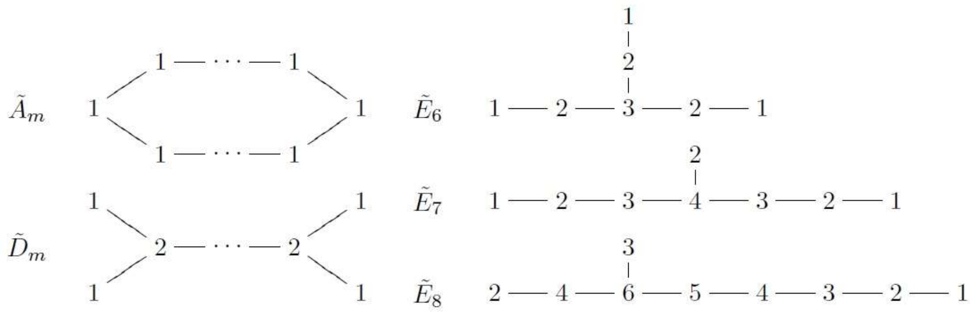
Definition 1.7. For a quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ we call
q positive definite if $q(x) > 0 \ \forall\ x \in \mathbb{Z}^n$ where $x \neq 0$
q positive semi-definite if $q(x) \geq 0 \ \forall\ x \in \mathbb{Z}^n$

1.3. Dynkin an Euclidean diagrams

Definition 1.8. Let $n \in \mathbb{N}$ be the number of vertices.
The following diagrams are called Dynkin diagrams:



Definition 1.9. Let $n \in \mathbb{N}$ be the number of vertices.
Let $m = n - 1$, with $m \geq 0$ for $\tilde{A}_m$ or $m \geq 4$ for $\tilde{D}_m$
The following diagrams are called Euclidean diagrams:



Each vertex is labeled with δ_i for $\delta \in \mathbb{Z}^n$.

Theorem 1.10. Let Γ be a connected graph with quadratic form q.
(i) Γ is Dynkin $\Leftrightarrow$ q is positive definite
(ii) Γ is Euclidean $\Leftrightarrow$ q is positive semi-definite, but not positive definite
(iii) If Γ is Euclidean, then there is a unique positive vector $\delta \in \mathbb{Z}^n$ such that $\text{rad}q = \mathbb{Z}\delta$.

Proof [1] page 14

2. Roots and Coxeter transformation

2.1. Roots

Definition 2.1. $\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}$
$\{x \in \Delta \mid x \neq 0\}$ is the set of all roots.

Proposition 2.2. Let Γ be a Dynkin or Euclidean diagram.
(i) each standard basis vector e_i is a root
(ii) $x \in \Delta$ and $y \in \text{rad}q \Rightarrow -x \in \Delta$ and $x + y \in \Delta$
(iii) every root is positive or negative
(iv) Γ Euclidean $\Rightarrow \Delta/\text{rad}q$ finite
(v) Γ Dynkin $\Rightarrow \Delta$ finite

Proof [1] page 14

Lemma 2.3. Let Q be a quiver with a Dynkin or Euclidean graph.
x a positive root and $\sigma_i(x)$ not positive $\Rightarrow x = e_i$

Proof by Prop. 2.3.

2.2. Coxeter transformation

Definition 2.4. Let Q be a quiver without oriented cycles. Fix $i_1, \ldots, i_n$ as an admissible ordering of its vertices.
The automorphism $c: \mathbb{Z}^n \to \mathbb{Z}^n$ such that $c(x) = \sigma_{i_n} \ldots \sigma_{i_1}(x)$
is called Coxeter transformation.

Lemma 2.5. (i) $c(\text{dim}P(i)) = -\text{dim}I(i)\ \forall i$
(ii) $\{\text{dim}P(i) \mid i \in Q_0\}$, $\{\text{dim}I(i) \mid i \in Q_0\}$ are two bases of $\mathbb{Z}^n$

Lemma 2.6. Let $x, y \in \mathbb{Z}^n$.
(i) $\langle \text{dim}P(i), x\rangle = x_i = \langle x, \text{dim}I(i)\rangle\ \forall$ vertices i
(ii) $\langle x, y\rangle = -\langle y, c(x)\rangle = \langle c(x), c(y)\rangle$

Lemma 2.7. Let $x \in \mathbb{Z}^n$.
$c(x) = x \Leftrightarrow x \in \text{rad}q$

Notation In the following part, assume that the graph of Q is Dynkin or Euclidean.

Remark (i) The map c induces a permutation of the finite set $\Delta/\text{rad}q$.
(ii) $e_i \in \Delta\ \forall i \Rightarrow c^h$ is the identity on $\mathbb{Z}^n/\text{rad}q$

Lemma 2.8. Let Q be Dynkin type and $x \in \mathbb{Z}^n$.
$\exists\ r \geq 0$ such that $c^r(x) \leq 0$.

Lemma 2.9. Let Q be Euclidean type and $x \in \mathbb{Z}^n$.

(i) $c^r(x) > 0 \ \forall\ r \in \mathbb{Z} \Rightarrow c^h(x) = x$

(ii) $c^h(x) = x \Rightarrow \langle \delta, x \rangle = 0$

Proofs [1] page 15

3. Gabriel's theorem

3.1. Dynkin case

Theorem 3.1. [Gabriel] Let Q be a quiver of Dynkin type.
The function between X and dim$X$ induces a bijection from the isomorphism classes of indecomposable representations of Q to the positive roots corresponding to the diagram of Q.
There are only finitely man isomorphism classes of indecomposable representations.

Proof [1] page 16

3.2. Defect

Definition 3.2. Let Q be a quiver of Euclidean type.
For $x \in \mathbb{Z}^n$, $\partial x = \langle \delta, x \rangle = - \langle x, \delta \rangle$ is called defect of x.
For a representation X, $\partial X = \partial \dim X$ is called defect of X.

Proposition 3.3. Let X be an indecomposable representation.
(i) X is preprojective $\Leftrightarrow \partial X < 0$
(ii) X is preinjective $\Leftrightarrow \partial X > 0$
(iii) X is regular $\Leftrightarrow \partial X = 0$

Proof [1] page 17

3.3. Euclidean case

Theorem 3.4. Let Q be a quiver of Euclidean type with a diagram of n vertices and without oriented cycles.
The function between X and dim$X$ induces a bijection from the isomorphism classes of indecomposable preprojective or preinjective representations of Q to the positive roots with non-zero defect corresponding to the diagram of Q.
The preprojective and preinjective indecomposables form $2n$ countably infinite series $C^{-r}P(i)$ and $C^{r}I(i)$ of pairwise non-isomorphic representations, where $r \in \mathbb{N}_0$ and $i \in Q_0$.

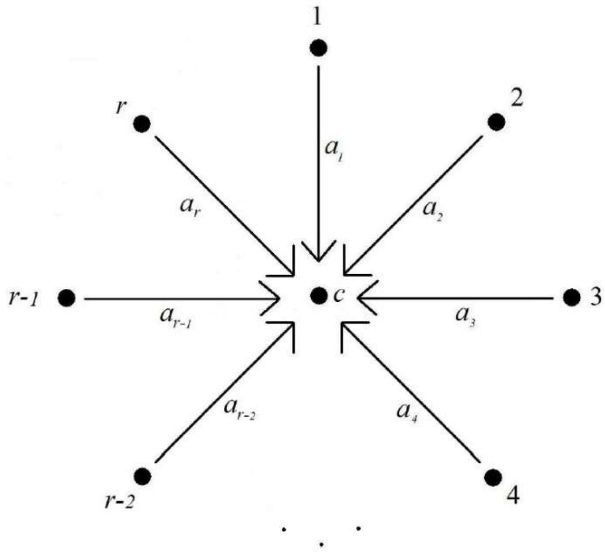
Proof [1] pages 17 and 18

Proposition 3.5. [Gabriel] Let Q be a connected quiver.
There are only finitely many isomorphism classes of indecomposable representations $\Leftrightarrow$ the underlying graph is a Dynkin diagram

Proof [1] page 18

4. Example

Let S_r be the quiver with
vertices $Q_0 = \{1, \ldots, r, c\}$ and
arrows $Q_1 = \{a_1, \ldots, a_r\}$
with $t(a_i) = i$ and $h(a_i) = c \ \forall \ i \in \{1, \ldots, r\}$.



Gabriel's theorem shows, that S_r has finitely many isomorphism classes of representations for $r < 4$.
$r = 1$: Dynkin diagram A_2
$r = 2$: Dynkin diagram A_3
$r = 3$: Dynkin diagram D_4

Gabriel's theorem says, that S_r has infinitely many isomorphism classes of representations for $r \geq 4$.
The underlaying graph is not one of the Dynkin diagrams.

For more details see [2], pages 18-21.

5. References

[1] Krause, H., *Representations of quivers via reflection functors*, Universität Paderborn 2007.
Online available: https://www2.math.uni-paderborn.de/fileadmin/Mathematik/AG-Krause/publications_krause/quiver.pdf

[2] Cummin, E., *Representations of Quivers & Gabriel's Theorem*, 2011.
Online available: http://people.bath.ac.uk/ac886/students/emmaCummin.pdf