

Introduction to quiver representations and Coxeter functors

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1 INTRODUCTION

This seminar consists of two parts. In the first section we will give some basic definitions and important lemmatas concerning quivers and their representations. Furthermore we will continue with direct sum decomposition of representations and finish the first part with the Krull-Remak-Schmidt theorem. In the second part we will give an introduction to terms like reflection- and coxeterfunctors. The aim of this summary is to give preliminary knowledge which will be important for Gabriel's theorem.

2 QUIVERS AND THEIR REPRESENTATIONS

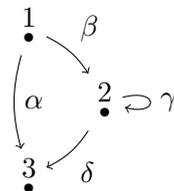
DEFINITION QUIVER:

Generally speaking, a quiver is a directed graph, possibly with multiple arrows and loops, which we assume to be finite. Formally, a quiver is a quadruple $Q=(Q_0, Q_1, s, t)$ which is given by a set Q_0 of vertices and a set Q_1 of arrows where both Q_0 and Q_1 will be finite. $s, t : Q_1 \rightarrow Q_0$ are maps assigning to each arrow its source, resp. target. For an arrow $\alpha \in Q_1$ wich starts at the vertex $s(\alpha)$ and terminates at $t(\alpha)$ we sometimes indicate this as $\alpha: s(\alpha) \rightarrow t(\alpha)$ or also $s(\alpha) \xrightarrow{\alpha} t(\alpha)$.

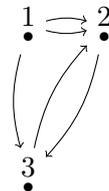
REMARK: Arrows $\alpha \in Q_1$ such that $s(\alpha)=t(\alpha)$ are called loops.

2.1 EXAMPLES OF QUIVERS

EXAMPLE 1.1: The following picture shows the quiver Q , where $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\alpha, \beta, \gamma, \delta\}$, $s(\alpha) = s(\beta) = 1, s(\gamma) = s(\delta) = 2, t(\alpha) = t(\delta) = 3$ and $t(\beta) = t(\gamma) = 2$.



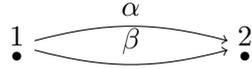
EXAMPLE 1.2: A quiver may have also several arrows in the same or in opposite direction and it may also have loops like in the following example:



EXAMPLE 1.3: The Jordan quiver, which consists of $Q_0 = \{1\}$; $Q_1 = \{\alpha\}$
 $s(\alpha) = t(\alpha)$



EXAMPLE 1.4: The Kronecker quiver, which consists of $Q_0 = \{1, 2\}$; $Q_1 = \{\alpha, \beta\}$;
 $s(\alpha) = s(\beta) = 1$, and $t(\alpha) = t(\beta) = 2$



2.2 PATH ALGEBRAS

A non-trivial path in Q of length $1 \leq m$ is a sequence of arrows $\rho = \rho_m \dots \rho_1$ which satisfies $s(\rho_{i+1}) = t(\rho_i)$ for $1 \leq i \leq m$. We write $i_1 \xrightarrow{\rho_1} i_2 \xrightarrow{\rho_2} i_3 \dots \xrightarrow{\rho_m} i_{m+1}$

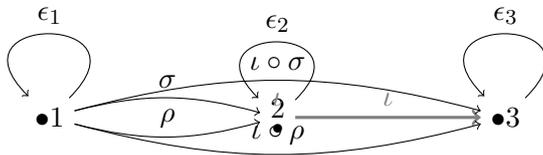
This path ρ starts at $s(\rho_m)$ and terminates at $t(\rho_1)$. Considering the trivial path ϵ_i which has length zero with $s(\epsilon_i) = i = t(\epsilon_i)$, we note that the arrows in a path have the same way of ordering as the composition of functions. Moreover we define the path algebra kQ as the k -algebra with basis the paths in Q and with the product of two paths σ and ρ given by

$$\sigma\rho = \begin{cases} \text{obvious composition} & \text{if } t(\rho) = s(\sigma) \\ 0 & \text{else} \end{cases} \quad (1)$$

where the composition is an associative multiplication. We note also, that the source and target functions of a quiver can be defined on paths, so we write $s(\rho)$ and $t(\rho)$ for the source and target of a path ρ .

EXAMPLE 1.5: Let Q be the quiver $1 \xrightarrow{\rho} 2 \xrightarrow{\sigma} 3$ then kQ has the basis the paths: $\epsilon_1, \epsilon_2, \epsilon_3, \rho$ and $\sigma\rho$, where the product of the paths σ and ρ is the path $\sigma\rho$. On the other hand the product $\rho\sigma$ is zero. Some other products are $\rho\rho=0, \epsilon_1\rho=0, \epsilon_2\rho=\rho, \rho\epsilon_1=\rho, \epsilon_3(\sigma\rho)=\sigma\rho, \epsilon_1\epsilon_1=\epsilon_1, \epsilon_1\epsilon_2=0$, etc.

EXAMPLE 1.6: Let Q be a quiver, where $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\rho, \sigma, \iota\}$, where $s(\rho) = 1 = s(\sigma)$, $t(\rho) = 2 = t(\sigma)$, $s(\iota) = 2$, $t(\iota) = 3$



$$Q(1, 2) = \{\rho, \sigma\}$$

$$Q(1, 3) = \{\iota \circ \rho, \iota \circ \sigma\}$$

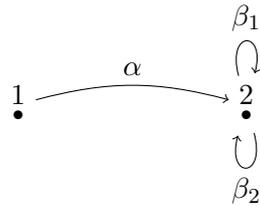
$$Q(2, 3) = \{\iota\}$$

$$Q(2, 1) = \emptyset$$

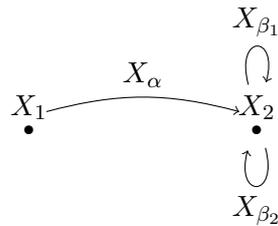
2.3 REPRESENTATIONS OF QUIVERS

A representation X of a quiver Q is a collection $X = (X_i, X_\alpha)_{i \in Q_0, \alpha \in Q_1}$ consisting of a vector space X_i for each vertex i and a linear map $X_\alpha : X_{s(\alpha)} \rightarrow X_{t(\alpha)}$ for each arrow α .

EXAMPLE 1.7: The quiver consisting of the vertices i, j and arrows $\alpha : i \rightarrow j$ and $\beta_1, \beta_2 : j \rightarrow j$ depicted as follows:



For example a representation of the preceding quiver is just a diagram,



where X_1 and X_2 are vector spaces and $X_\alpha, X_{\beta_1}, X_{\beta_2}$ are linear maps.

MORPHISMS OF REPRESENTATIONS:

Given two representations $X = (X_i, X_\alpha)_{i \in Q_0, \alpha \in Q_1}, Y = (Y_i, Y_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q , a morphism ϕ from X to Y is a collection $\phi = (\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : X_i \rightarrow Y_i$ for each vertex i , such that $Y_\alpha \phi_{s(\alpha)} = \phi_{t(\alpha)} X_\alpha$ for each arrow α . For each arrow α we have a commutative diagram

$$\begin{array}{ccc}
X_{s(\alpha)} & \xrightarrow{\phi_{s(\alpha)}} & Y_{s(\alpha)} \\
\downarrow X_\alpha & & \downarrow Y_\alpha \\
X_{t(\alpha)} & \xrightarrow{\phi_{t(\alpha)}} & Y_{t(\alpha)}
\end{array}$$

REMARK:Note that both representations X and Y are from the same quiver, since there is no natural way to define morphisms between representations of different quivers.

EXAMPLE 1.8: For any representation X of Q there is always the **identity morphism** $1_X : X \rightarrow X$ defined by the identity maps $(1_X)_i : X(i) \rightarrow X(i)$, for any vertex i of Q_0 .

COMPOSITION OF MORPHISMS: The composition of ϕ with ψ is given by $(\psi\phi)_i = \psi_i\phi_i$ for each vertex i . For each representation X we have the identity morphism $id_X : X \rightarrow X$ with $(id_X)_i = id_{X_i}$ for all i . For the set of morphisms $X \rightarrow X$ we write $End(X)$.

DEFINITION SUBREPRESENTATION: We fix a pair X, Y of representations of Q . We call X a subrepresentation of Y and write $X \subseteq Y$ if X_i is a subspace of Y_i for each vertex i and $X_\alpha(x) = Y_\alpha(x)$ for each arrow α and $x \in X_{s(\alpha)}$.

Given a morphism $\phi : X \rightarrow Y$ its kernel $Ker(\phi)$ is by definition the subrepresentation of X with $(Ker\phi)_i = Ker\phi_i$ for each vertex i . The cokernel $Coker\phi$ and the image $Im\phi$ are defined analogously. The morphism ϕ is an isomorphism if each ϕ_i is an isomorphism. One defines addition and scalar multiplication for morphisms $X \rightarrow Y$ point-wise and that makes $Hom(X, Y)$ into a vector space.

DIMENSION VECTOR: The dimension vector of a finite dimensional representation X is the vector $dim X$ in Z^{Q_0} with $(dim X)_i = dim X_i$ $i \in Q_0$.

2.4 THE CATEGORY OF REPRESENTATIONS

DEFINITION CATEGORY: A category C is an algebraic structure which consists of

- a class of objects $\text{Obj}(C)$
- a set of morphisms $\text{Hom}_C(A, B)$ for every ordered pair (A, B) of objects
- an identity morphisms $\text{id}_A \in \text{Hom}_C(A, A)$ for every object and
- a composition map $\text{Hom}_C(A, B) \times \text{Hom}_C(B, C) \rightarrow \text{Hom}_C(A, C)$ such that the following axioms hold:
- associativity: if $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, then $h \circ (g \circ f) = (h \circ g) \circ f$
- identity: for every object X , there exists a morphism $1_X : X \rightarrow X$ called the identity morphism for X , such that for every morphism $f : A \rightarrow X$ and every morphism $g : X \rightarrow B$, we have $1_X \circ f = f$ and $g \circ 1_X = g$

In our case the representations of a quiver together with the set of morphisms build a representationcategory $\text{Rep}(Q, k)$.

2.5 DUALITY

OPPOSITE QUIVER: We obtain Q^{op} from Q by reversing all arrows. The vector space duality $D = \text{Hom}_k(-, k)$ induces a duality

$$D : \text{Rep}(Q, k) \rightarrow \text{Rep}(Q^{op}, k)$$

Given a representation X of Q we let $(DX)_i = D(X_i)$ and $(DX)_\alpha = D(X_\alpha)$ for $i \in Q_0$ and $\alpha \in Q_1$. For a morphism $\phi : X \rightarrow Y$ we let $(D\phi)_i = D(\phi_i)$.

LEMMA1: Let $X \in \text{Rep}(Q, k)$ and $Y \in \text{Rep}(Q^{op}, k)$. There is a canonical monomorphism $\epsilon_X : X \rightarrow D^2X$ which induces a natural isomorphism

$$\text{Hom}(Y, DX) \cong \text{Hom}(X, DY) \text{ by sending } \phi : Y \rightarrow DX \text{ to } (D\phi)\epsilon_X$$

SIMPLE REPRESENTATIONS: A simple representation is defined to be a non-zero representation with no proper subrepresentations. Now assume that Q has no oriented cycles (that is, non-trivial paths from a vertex to itself). Then for any simple representation S of Q , there exists a unique vertex i such

$$S(i)_j = \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

$$\text{and } S(i)_\alpha = 0$$

for $j \in Q_0$ and $\alpha \in Q_1$. This representation is simple.

2.6 PROJECTIVE AND INJECTIVE REPRESENTATIONS

For each vertex i we define a projective representation $P(i)$ and an injective representation $I(i)$. Given any set X , we denote by $k[X]$ the vector space with basis X , that is, $k[X]$ is the set of linear combinations $\sum_p \lambda_p x_p$ with $x_p \in X$, $\lambda_p \in k$ and almost all $\lambda_p = 0$.

For a map $\phi : X \rightarrow Y$, let $k[\phi] : k[X] \rightarrow k[Y]$ be the linear map sending $\sum_p \lambda_p x_p$ to $\sum_p \lambda_p \phi(x_p)$. Now we define $P(i)_j = k[Q(i, j)]$ and $P(i)_\alpha = k[Q(i, \alpha)]$ $j \in Q_0$ and $\alpha \in Q_1$. Dually, we define

$$I(i)_j = Dk[Q(j, i)] \text{ and } I(i)_\alpha = Dk[Q(\alpha, i)].$$

LEMMA 2: Let X be a representation of Q . Then there are natural isomorphisms

$$\text{Hom}(P(i), X) \cong X_i \text{ and } \text{Hom}(X, I(i)) \cong DX_i$$

LEMMA 3: (1) The representations $P(i)$ are pairwise non-isomorphic. (2) The representations $I(i)$ are pairwise non-isomorphic.

LEMMA 4: Suppose Q has no oriented cycles. The $P(i)$ and $I(i)$ are finite dimensional with $\text{End}(P(i)) \cong k \cong \text{End}(I(i))$.

2.7 DIRECT SUM DECOMPOSITIONS

In this section we consider finite dimensional representations. We show that each representation decomposes essentially uniquely into indecomposable representations.

DIRECT SUMS: Let X_1, \dots, X_r be a finite number of representations. A direct sum

$$X = X_1 \oplus \dots \oplus X_r$$

is a representation X together with morphisms $\iota_i : X_i \rightarrow X$ and $\pi_i : X \rightarrow X_i$ for $1 \leq i \leq r$ such that $\sum_{i=1}^r \iota_i \pi_i = id_X$ and $\pi_i \iota_i = id_{X_i}$ for all i . Note that we can identify each X_i via ι_i with a subrepresentation of X . Then we obtain

$$X = \sum_{i=1}^r X_i \text{ and } X_i \cap \sum_{i' \neq i} X_{i'} = 0 \text{ for } 1 \leq i \leq r \text{ (2.1.1)}$$

Here $\sum_{i \in I} X_i$ refers to the smallest subrepresentation of X containing X_i for all $i \in I$. Conversely, if X_1, \dots, X_r is a family of subrepresentations of a representation X satisfying (2.1.1) then $X = X_1 \oplus \dots \oplus X_r$. In that case we take for $\iota_i : X_i \rightarrow X$ the inclusion morphism and let $\pi_i = (\rho_i \iota_i)^{-1} \rho_i$, where $\rho_i : X \rightarrow X / \sum_{i' \neq i} X_{i'}$ denotes the canonical morphism.

A family of subrepresentations X_1, \dots, X_r of X satisfying (2.1.1) is called a direct sum decomposition of X . We have a decomposition $X = X_1 \oplus \dots \oplus X_r$ if and only if we have a vector space decomposition $X_j = (X_1)_j \oplus \dots \oplus (X_r)_j$ for each vertex j .

LEMMA 5: Let $X = X_1 \oplus \dots \oplus X_r$ and $Y = Y_1 \oplus \dots \oplus Y_s$. Then we have induced vector space decompositions

$$\bigoplus_{i=1 \dots r} \text{Hom}(X_i, Y) = \text{Hom}(X, Y) = \bigoplus_{j=1 \dots s} \text{Hom}(X, Y_j).$$

It follows from **LEMMA 5** that a direct sum of $X_1 \dots X_r$ is unique up to an isomorphism. Thus we may speak of **THE** direct sum and the notation $X_1 \oplus \dots \oplus X_r$ is well defined. We write $X^r = X \oplus \dots \oplus X$ for the direct sum of r copies of a representation X . **INDECOMPOSABLE REPRESENTATIONS:** A representation X is called indecomposable if $X \neq 0$ and $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

PROPOSITION 1: A representation X is indecomposable if and only if $\text{End}(X)$ is local.

THE KRULL-REMAK-SCHMIDT THEOREM: Let X be a finite dimensional representation. Then there exists a decomposition $X = X_1^{a_1} \oplus \dots \oplus X_r^{a_r}$ with the X_i pairwise non-isomorphic indecomposable representations and each $1 \leq a_i$. If $X = Y_1^{b_1} \oplus \dots \oplus Y_s^{b_s}$ is another such decomposition, then $r=s$ and after reordering, $X_i \cong Y_i$ and $a_i = b_i$ for $1 \leq i \leq r$.

The Krull-Remak-Schmidt theorem says that the classification of finite dimensional representation reduces to the classification of indecomposable representations. There is also a similar statement about morphisms between representations. Let $X = X_1 \oplus \dots \oplus X_r$ and $Y = Y_1 \oplus \dots \oplus Y_s$ be two representations with decompositions into indecomposable representations. Then we have

$$\text{Hom}(X, Y) = \bigoplus_{i,j} \text{Hom}(X_i, Y_j) \text{ by Lemma 5.}$$

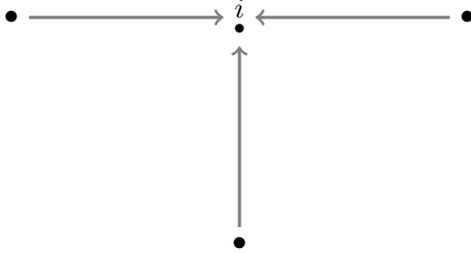
Thus each morphism $\phi : X \rightarrow Y$ can be written uniquely as a matrix $\phi = (\phi_{ij})$ where each entry $\phi_{ij} : X_i \rightarrow Y_j$ is a morphism between indecomposable representations.

3 REFLECTION FUNCTORS

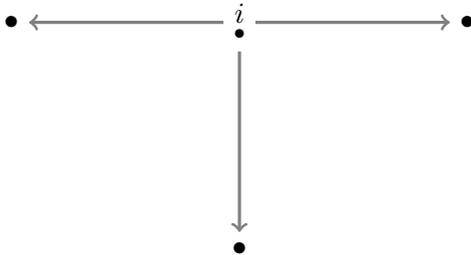
3.1 ORIENTATION

DEFINITION SINK AND SOURCE: Let Q be a quiver. A vertex $i \in Q$ is called
 i) a sink if there is no arrow in Q starting at i ii) a source, if there is no arrow ending at i .

Sink:



Source:



DEFINITION 3.1.2: For a vertex i , $\sigma_i Q$ is the quiver obtained from Q by reversing all arrows which start or end at i .

DEFINITION 3.1.3: Let Q be a quiver.

An ordering

$$i_1 \dots i_n$$

is called admissable if for each p the vertex i_p is a sink for $\sigma_{i_{p-1}} \dots \sigma_{i_1} Q$. Then $Q = \sigma_{i_n} \dots \sigma_{i_1} Q$

LEMMA 3.1.1: There exists an admissable ordering of the vertices of Q if and only if there is no oriented cycle in Q .

3.2 EULER FORM

DEFINITION 3.2.1: The Euler form is the bilinear form:

$$\langle \cdot, \cdot \rangle: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z} \text{ defined by } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)} \quad \forall \mathbf{x}, \mathbf{y}$$

A symmetric bilinear form is defined by $(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle$

DEFINITION 3.2.2: Let Q be a quiver without loops. A reflection with respect to $i \in Q_o$ is a map $\sigma_i: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ with $\sigma_i(x) = x - \frac{(x, e_i)}{(e_i, e_i)} \cdot e_i$, where e_i is the i -th coordinate vector. σ_i is an automorphism of order two preserving the bilinear form.

3.2 REFLECTION FUNCTORS

DEFINITION 3.3.1: Let Q be a quiver and i a vertex of Q . We define a pair of reflection functors S_i^+ and S_i^- . We fix representations X, X' of Q and a morphism $\phi: X \rightarrow X'$.

(1) If the vertex i is a sink of Q , then we construct

$$S_i^+ : \text{Rep}(Q, k) \rightarrow \text{Rep}(\sigma_i Q, k)$$

as follows. We define $S_i^+ X = Y$ by letting $Y_j = X_j$ for a vertex $j \neq i$, and letting Y_i be the kernel of the map $\xi = (X_\alpha)$ in the following sequence

$$Y_i \xrightarrow{\xi'} \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X_{s(\alpha)} \xrightarrow{\xi} X_i$$

where ξ' denotes the inclusion map of the kernel. For an arrow $\alpha \in Q$, let $Y_\alpha = X_\alpha$ if $t(\alpha) \neq i$ and $Y_\alpha: Y_i \rightarrow X_{s(\alpha)} = Y_{s(\alpha)}$ be the map ξ' followed by the canonical projection onto $X_{s(\alpha)}$ if $t(\alpha) = i$. For the morphism $S_i^+ \phi = \psi$ let $\psi_j = \phi_j$ if $j \neq i$ and let $\psi_i: Y_i \rightarrow Y_i'$ be the restriction of the map

$$(\phi_{s(\alpha)}): \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X_{s(\alpha)} \rightarrow \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X'_{s(\alpha)}$$

(2) If the vertex i is a source of Q , then we construct dually

$$S_i^- : \text{Rep}(Q, k) \rightarrow \text{Rep}(\sigma_i Q, k)$$

as follows. We define $S_i^- X = Y$ by letting $Y_j = X_j$ for a vertex $j \neq i$, and letting Y_i be the cokernel of the map $\xi = (X_\alpha)$ in the following sequence

$$X_i \xrightarrow{\xi} \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X_{t(\alpha)} \xrightarrow{\xi'} Y_i$$

where ξ' denotes the canonical map onto the cokernel.

For an arrow $\alpha \in Q$, let $Y_\alpha = X_\alpha$ if $s(\alpha) \neq i$ and $Y_\alpha: Y_{t(\alpha)} = X_{t(\alpha)} \rightarrow Y_i$ be the restriction ξ' to $X_{t(\alpha)}$ if $s(\alpha) = i$. For the morphism $S_i^- \phi = \psi$ let $\psi_j = \phi_j$ if $j \neq i$ and let $\psi_i: Y_i \rightarrow Y_i'$ be map which is induced by

$$(\phi_{t(\alpha)}): \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X_{t(\alpha)} \rightarrow \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X'_{t(\alpha)}$$

DEFINITION 3.3.2:(1) Let i be a sink of Q . Then we define a natural monomorphism

$$\iota_i X : S_i^- S_i^+ X \rightarrow X$$

by letting $(\iota_i X)_j = id_{X_j}$ for a vertex $j \neq i$ and letting $(\iota_i X)_i$ be the canonical map

$$(S_i^- S_i^+ X)_i = Coker \xi' \cong Im \xi \rightarrow X_i$$

(2) Let i be a source of Q . Then we define a natural epimorphism

$$\pi_i X : X \rightarrow S_i^+ S_i^- X$$

by letting $(\pi_i X)_j = id_{X_j}$ for a vertex $j \neq i$ and letting $(\pi_i X)_i$ be the canonical map

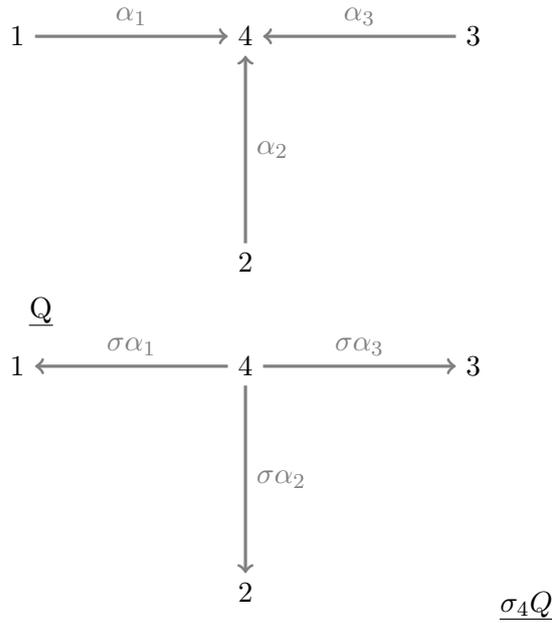
$$X_i \rightarrow Im \xi \cong Ker \xi' = (S_i^+ S_i^- X)_i$$

DEFINITION 3.3.3: For the morphism $S_i^+ \phi = \psi$ let $\psi_j = \phi_j$ for $j \neq i$ and for $j = i$ let $\psi_i : Y_i \rightarrow Y'_i$ the restriction of the map: $(\phi_s(\alpha)) : \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X_{s(\alpha)} \rightarrow \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X'_{s(\alpha)}$

DEFINITION 3.3.4: For the morphism $S_i^- \phi = \psi$ let $\psi_j = \phi_j$ for $j \neq i$ and for $j = i$ let

let $\psi_i : Y_i \rightarrow Y'_i$ the restriction of the map: $(\phi_t(\alpha)) : \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X_{t(\alpha)} \rightarrow \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X'_{t(\alpha)}$

3.3 EXAMPLES REFLECTION FUNCTOR



$S_4^+ : Rep(Q, k) \rightarrow Rep(\sigma Q), (X)_{j \in Q_0} \rightarrow (S_4^+ X)_{j \in \sigma Q_0}$

$$(S_4^+ X)_j = \begin{cases} X_j & \text{if } j \neq 4 \\ \ker(\bigoplus_{i=1,2,3} X_i) \xrightarrow{\xi} X_4 & \text{if } j = 4 \end{cases} \quad (2)$$

$$\begin{aligned}
S_4^+ \alpha_1: (S_4^+ X)_4 &\hookrightarrow \bigoplus_{i=1,2,3} X_i \twoheadrightarrow X_1 \\
S_4^+ \alpha_2: (S_4^+ X)_4 &\hookrightarrow \bigoplus_{i=1,2,3} X_i \twoheadrightarrow X_2 \\
S_4^+ \alpha_3: (S_4^+ X)_4 &\hookrightarrow \bigoplus_{i=1,2,3} X_i \twoheadrightarrow X_3
\end{aligned}$$

DEFINITION: For the morphism $S_i^+ \phi = \psi$ let $\psi_j = \phi_j$ for $j \neq i$
 $\psi_i: Y_i \rightarrow Y_i'$ the restriction of the map: $(\phi_{s(\alpha)}): \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X_{s(\alpha)} \rightarrow \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X'_{s(\alpha)}$

DEFINITION: For the morphism $S_i^- \phi = \psi$ let $\psi_j = \phi_j$ for $j \neq i$
 $\psi_i: Y_i \rightarrow Y_i'$ the restriction of the map: $(\phi_{t(\alpha)}): \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X_{t(\alpha)} \rightarrow \bigoplus_{\alpha \in Q_1, s(\alpha)=i} X'_{t(\alpha)}$

LEMMA 3.3.1: S_i^- and S_i^+ are functors with properties

- i) $S_i^\pm id_x = id_{S_i^\pm X}$ for every representation X.
- ii) $S_i^\pm(\Psi\Phi) = (S_i^\pm\Psi)(S_i^\pm\Phi)$ for every pair of morphism $\Phi: X \rightarrow Y$ and $\Psi: Y \rightarrow Z$

LEMMA 3.3.2: Let X and X' be two representations of Q and i a vertex. Then

- i) $S_i^\pm(X \oplus X') = S_i^\pm X \oplus S_i^\pm X'$
- ii) $X = (S_j^- S_i^+ X) \oplus Coker \iota_j X$ and $X = (S_j^+ S_i^- X) \oplus Ker \Pi_j X$
- iii) If $Coker \iota_i X = 0$ then $dim S_i^+ X = \sigma_i(dim X)$
- iv) If $Ker \Pi_i X = 0$ then $dim S_i^- X = \sigma_i(dim X)$

REMARK: The representations $Coker \iota_i X$ and $Ker \Pi_i X$ are concentrated at the vertex i. They are direct sums of copies of S(i).

LEMMA 3.3.3: Let i be a sink and X an indecomposable representation of Q. The the following are equivalent:

- (1) X is not isomorphic to S(i).
- (2) $S_i^+ X$ is indecomposable.
- (3) $S_i^+ X \neq 0$.
- (4) $S_i^- S_i^+ X \cong X$
- (5) The map $(X_\alpha): \bigoplus_{\alpha \in Q_1, t(\alpha)=i} X_{s(\alpha)} \rightarrow X_i$ is an epimorphism
- (6) $\sigma_i(dim X) > 0$
- (7) $dim S_i^+ X = \sigma_i(dim X)$

THEOREM 3.3.1: The functors S_i^+ and S_i^- induce mutually inverse bijections between the isomorphism classes of indecomposable representations of Q and the isomorphism classes of indecomposable representations of $\sigma_i Q$ except the simple representation S(i). Moreover, $dim S_i^\pm X = \sigma_i(dim X)$ for every indecomposable representation X not isomorphic to S(i).

3.4 COXETER FUNCTORS

DEFINITION 3.4.1: Let Q be a quiver with no oriented cycles and

$$i_1 \dots i_n$$

an admissible ordering of the vertices of Q .
The Coxeter functors is defined as :

$$\begin{aligned} C^+ &: S_{i_n}^+ \dots S_{i_1}^+ : \text{Rep}(Q, k) \rightarrow \text{Rep}(Q, k) \\ C^- &: S_{i_1}^- \dots S_{i_n}^- : \text{Rep}(Q, k) \rightarrow \text{Rep}(Q, k) \end{aligned}$$

For $r \in \mathbb{Z}$ we write:

$$C^r = \begin{cases} (C^+)^r & \text{for } r > 0 \\ Id & \text{for } r = 0 \\ (C^-)^{-r} & \text{for } r < 0 \end{cases}$$

LEMMA 3.4.1: The functors C^+ and C^- do not depend on the choice of ordering of the vertices of Q .

LEMMA 3.4.2: Let i be a vertex.

- i) $\dim P(i) = \sigma_1 \dots \sigma_{i-1}(e_i)$ and $\dim I(i) = \sigma_n \dots \sigma_{i+1}(e_i)$
- ii) $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$ and $I(i) \cong S_n^+ \dots S_{i+1}^+ S(i)$

PROPOSITION 3.4.3: Let X be an indecomposable representation of Q .

- (1) $C^+ X = 0$ if $X \cong P(i)$ for some vertex i , and $C^- C^+ X \cong X$ otherwise.
- (2) $C^- X = 0$ if $X \cong I(i)$ for some vertex i , and $C^+ C^- X \cong X$ otherwise.

3.5 PREPROJECTIVE AND PREINJECTIVE REPRESENTATIONS

DEFINITION 3.5.1: Let X be an indecomposable representation of Q .

- (1) X is preprojective if $X \cong C^r P(i)$ for some vertex i and some $r \leq 0$.
- (2) X is preinjective if $X \cong C^r I(i)$ for some vertex i and some $r \geq 0$.
- (3) X is regular if $C^r X \neq 0$ for all $r \in \mathbb{Z}$.

Note that X is preprojective if and only if $C^r X = 0$ for some $r > 0$ and X is preinjective if and only if $C^r X = 0$ for some $r < 0$.

PROPOSITION 3.5.1: An indecomposable representation is preprojective, preinjective or regular. Given indecomposable representations X, Y with X preprojective or preinjective, we have $X \cong Y$ if and only if $\dim X = \dim Y$. Moreover,

- (1) $C^r P(i) \cong C^s P(j) \neq 0$ implies $i = j$ and $r = s$;
- (2) $C^r I(i) \cong C^s I(j) \neq 0$ implies $i = j$ and $r = s$;

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