ALGEBRAS
CELINE KUETTEL

Contents

1. Basic properties

1.1. Definitions.

Definition 1.1. An *algebra*, A, is a vector space together with a mapping $A \times A \to A$ such that the conditions (1)and (2) below both hold. The image of two vectors x $\in A$, y $\in A$, under this mapping is called the *product* of x and y and will be denoted by xy.
The mapping $A \times A \to A$ is required to satisfy:

(1) $(\lambda\ x_1 + \mu\ x_2)$y=$\lambda(x_1y)+\ \mu(x_2y)$
(2) x$(\lambda\ y_1 + \mu\ y_2)$=$\lambda(xy_1)+\ \mu(xy_2)$

Definition 1.2. Suppose B is a second algebra. Then a linear mapping φ: $A \to B$ is called a *homomorphism* (of algebras) if φ preserves products; i.e.φ(xy)=φ(x)φ(y).

Remark 1.3. A homomorphism that is injective (resp. surjective, bijective) is called a monomorphism (resp. epimorphism, isomorphism). If A=B, φ is called a endomorphism.

Remark 1.4. Let A be a given algebra and let U, V be two subsets of A. We denote by UV, the set UV:={x$\in A$ | x= $\sum_i u_i\ v_i$, $u_i \in$ U, $v_i \in$ V }

Definition 1.5. Every vector a$\in A$ induces a linear mapping μ(a):A$\to$A defined by μ(a)x=ax. μ(a) is called the *multiplication operator* determined by a.

Definition 1.6. An algebra A is called *associative* if x(yz)=(xy)z, x,y,z $\in A$
and *commutative* if xy=yx, x,y $\in A$.

Definition 1.7. From every algebra A we can optain a second algebra A^{opp} by defining (xy)^{opp}=yx. A^{opp} is called the algebra *opposite* to A.

Remark 1.8. If A is associative then so is A^{opp} and if A is commutative then we have A^{opp} =A

Definition 1.9. If A is an associative algebra, a subset S$\subset$ A is called a *system of generators* of A if each vector x $\in A$ is a linear combination of products of elements in S,
x= $\sum_{(v)} \lambda^{v_1...v_p} x_{v_1}...x_{v_p}$, $x_{v_i} \in S$,$\lambda^{v_1...v_p} \in \Gamma$.

Definition 1.10. A *unit element* (or identity) in an algebra is an element e s.t. $\forall$ x xe=ex=x.

Remark 1.11. If A has a unit element, then it is unique.

Definition 1.12. An algebra with unit element is called a *division algebra*, if to every element $a$$\neq$0 there is an element a^{-1} s.t. aa^{-1}=$a^{-1}a$=e.

Example 1.13. Consider the space $L(E;E)$ of all linear transformations of a vector space E. We define the product of two transformations by $\psi\varphi$=$\psi \circ \varphi$.
The mapping (φ,ψ)$\rightarrow$ $\psi\varphi$ satisfies (1) and (2) (from definition 1.1.) and hence $L(E;E)$ is an algebra. $L(E;E)$ together with this multiplication is called the *algebra of linear transformations of E* and is denoted by $A(E;E)$. Further $A(E;E)$ is associative however it is not commutative if dim$E$ $\geq$2.

Example 1.14. Let $M^{n\times n}$ be the vector space of (n$\times$n)-matrices for a given integer n. Then it follows that the space $M^{n\times n}$ is made into an associative algebra under the matrice multiplication with the unit matrix J as a unit element.
Now consider a vector space E of dimension n with a distinguished basis e_v(v=1...n). Then every linear transformation φ:$E$$\rightarrow$$E$ determines a matrix $M(\varphi)$. The correspondence $\varphi \rightarrow M(\varphi)$ determines a linear isomorphism of $A(E;E)$ onto $M^{n\times n}$. So we have $M(\varphi \circ \psi)$=$M(\varphi)M(\psi)$. This realtion shows that M is an isomorphism of the algebra $A(E;E)$ onto the opposite algebra $(M^{n\times n})^{opp}$.

1.2. **Subalgebras and ideals.**

Definition 1.15. A *subalgebra*, A_1, of an algebra A is a linear subspace which is closed under the multiplication in A; that is, if x and y are arbitrary elements of A_1, then xy$\in A_1$. Thus A_1 inherits the structure of an algebra from A.

Definition 1.16. Let S be a subset of A and suppose that A is associative. Then the subspace B$\subset$ A generated (linearly) by elements of the form $s_1...s_r$, $s_i \in$ S, is clearly a subalgebra of A, called the *subalgebra generated by S*.

Definition 1.17. A *right* (*left*) *ideal* in an algebra A is a subspace I s.t. for every x$\in$$I$, and every y$\in$$A$, xy$\in$$I$ (yx$\in$$I$). A subspace that is both a right and left ideal is called a $two-sided\ ideal$, or simply an *ideal in A*.

Definition 1.18. The *ideal I generated by a set S* is the intersection of all ideals containing S. If A is associative, I is the subspace of A generated (linearly) by elements of the form
s, as, sa s$\in$$S$, $a$$\in$$A$.
In particular every single element a generates an ideal I_a. I_a is called the *principal ideal* generated by a.

Example 1.19. Given an element a of an associative algebra consider the set, N_a, of all elements $x$$\in$$A$ s.t. ax=0. If $x$$\in$$N_a$ then we have
for every $y$$\in$$A$ $a(xy)$$\equiv$$(ax)y = 0$ and so $xy$$\in$$N_a$.
This shows that N_a is a right ideal in A. It is called the *right annihilator of A*. Similarly the left annihilator is defined.

1.3. **Factor algebras.**

Theorem 1.20. *Let A be an algebra and B an arbitary subalgebra of A. Consider the canonical projection* π*:* $A \rightarrow A/B$*. Then* A/B *admits a multiplication s.t.* π *is a homomorphism iff B is an ideal in A.*

Proof.

- ⇒: Assume there exists such a multiplication A/B. Then for every x∈A, y∈B, we have π(xy)=π(x)π(y)=π(x)0=0, hence xy∈B.
  Similarly it follows that yx∈B ans so B must be an ideal.
- ⇐: Assume B is an ideal. Then we define the multiplication in A/B by
  $\tilde{x}\tilde{y}$=π(xy) $\tilde{x}$, $\tilde{y}$ ∈ A/B
  where x and y are any representatives of $\tilde{x}$ and $\tilde{y}$ respectively.
  WTS: the product does not depend on the choice of x and y.
  Let x' and y' be two other elements s.t. πx'=$\tilde{x}$ and πy=$\tilde{y}$. Then x' - x ∈ B and y' - y ∈ B.
  Hence we can write x'=x+b, b∈B and y=y'+c, c∈B.
  It follows that x'y'-xy=by+xc+bc ∈ B
  and so π(x'y')=π(xy).
  The multiplication in A/B clearly satisfies (1) and (2) from the definition of an algebra as follows from the linearity of π. Finally we have π(xy)=πxπy and we see that π is a homomorphism and that the multiplication in A/B is uniquely determined by the requirement that π is a homomorphism.

□

Remark 1.21. The vector space A/B together with the multiplication is called the *factor algebra of* A with respect to the ideal B. It is clear that if A is associative (commutative) then so is A/B. If A has a unit element e then $\tilde{e}$=πe is the unit element of the algebra A/B.

1.4. **Homomorphisms.**

Theorem 1.22. *Suppose* A *and* B *are algebras and* φ:$A \to B$ *is a homomorphism. Then the kernel of* φ *is an ideal in* A.

Theorem 1.23. *Now let* $\bar{\varphi}$:$A/ker\varphi \to B$ *be the induced injective linear mapping. Then we have the commutative diagram* φ:$A \to B$, $\bar{\varphi}$:$A/ker\varphi \to B$ *and* π:$A \to A/ker\varphi$ *and since* π *is a homomorphism, it follows that*
$\bar{\varphi}(\pi x \cdot \pi y) = \bar{\varphi}\pi(xy) = \varphi(xy) = \varphi(x)\cdot\varphi(y) = \bar{\varphi}(\pi x)\cdot\bar{\varphi}(\pi y)$
This relation shows that $\bar{\varphi}$:$A/ker\varphi \to Im(\varphi)$ *is an isomorphism.*

Proposition 1.24. *Let* φ_0:$S \to B$ *be an arbitrary set map. Then* φ_0 *can be extended to a homomorphism* φ:$A \to B$ *if and only if* $\sum_{(v)} \lambda^{v_1 \dots v_p} \varphi_0 x_{v_1} \dots \varphi_0 x_{v_p}=0$ *whenever* $\sum_{(v)} \lambda^{v_1 \dots v_p} x_{v_1} \dots x_{v_p}=0$

Definition 1.25. Let A be an arbitray algebra. $C(A)$ is the set of elements $a \in A$ that commute with every element in A and is called the *centre* of A.

2. Ideals

2.1. **The lattice of ideals.**
Let A be an algebra, and consider the set $\mathscr{I}$ of ideals in A. We order this set by inclusion; i.e. if I_1 amd I_2 are ideals in A, then we write $I_1 \leqq I_1$ iff $I_1 \subset I_2$. The relation $\leqq$ is clearly a partial order in $\mathscr{I}$. Now let I_1 and I_2 be ideals in A. Then it is easily checked that I_1+I_2 and $I_1 \bigcap I_2$ are again ideals and are in fact the least upper bound and the greatest lower bound in I_1 and I_2. Hence, the realtion $\leqq$ induces in $\mathscr{I}$ the structure of a lattice.

2.2. **Nilpotent ideals.**

Definition 2.1. Let A be an associative algebra. Then an element $a \in A$ will be called *nilpotent* if for some k, a^k=0.
The least k for which this holds is called the *degree of niloptency of* a.

Definition 2.2. An ideal I will be called nilpotent if for some k, I^k=0.
The least k for which this holds is called *degree of niloptency of* I and will be denoted by deg$I$.

2.3. **Radicals.**

Remark 2.3. Let A be an associative commutative algebra. Then the nilpotent elements in A form an ideal. In fact, if x and y are nilpotent of degree p and q respectively we have that
$(\lambda x + \lambda y)^{p+q}=0$ and $(xy)^p = x^p y^p = 0$

Definition 2.4. The ideal consisting of the nilpotent elements is called the *radical* of A and will be denoted by rad$A$.

Remark 2.5. rad(rad A)=rad$A$.

2.4. **Simple algebras.**

Definition 2.6. An algebra A is called *simple* if it has no proper non-trivial ideals and if $A^2 \neq 0$.

Theorem 2.7. *Let A be a simple commutative associative algebra. Then A is a division algebra.*

2.5. **Totally reducible algebras.**

Definition 2.8. An algebra A is called *totally reducible* if to every ideal I there is a complementary ideal I' s.t $A= I \oplus I'$.

Remark 2.9. Every ideal I in a totally reducible algebra is itself a totally reducible algebra.