Hints[1] for exercise sheet 4

Exercise 1

a. Define $x_0 := T(0)$ and $L(x) := T(x) - x_0$, and show that L is linear. For this, proceed in a similar way as in in Exercise 4 of Sheet 2.

b. Suggestion: show that the vectors $T(y) - T(z)$ and $T(z) - T(x)$ are scalar multiples of one another, because for these vectors the triangle inequality is an equality (use Cauchy-Schwarz and the parallelogram and polarization identities). This implies that the three points $T(x)$, $T(y)$, and $T(z)$ lie on one line. Then show that $T(z)$ is the same convex linear combination of $T(x)$ and $T(y)$ as z is of x and y.

Exercise 2

a. Approximate $|u|(x)$ using $K_\varepsilon(x) = G_\varepsilon(u(x))$ with $G_\varepsilon(s) = \sqrt{\varepsilon^2 + s^2} - \varepsilon$.

For the part about min and max functions, write these functions with the help of absolute values...

b. Argue first why it is enough to consider a connected open subset $V \subseteq U$ of finite measure. Then approximate u using functions $j_\varepsilon \in C_c^\infty(V)$ and $u_\varepsilon = u * j_\varepsilon \in C^\infty(V)$ such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ in $L^p(V)$.

Exercise 3

Warning! The hint in the original version of the exercise sheet was a bit misleading. Look at the current version.

To show that

$$\int_\Omega \phi(x)\chi_O(f(x))\nabla f(x)dx = -\int_\Omega \nabla\phi(x)G_O(f(x))dx \quad \forall \phi \in C_0^\infty(\Omega),\ \forall \text{ open sets } O \subset \mathbb{R}. \tag{1}$$

implies the solution of the exercise, use an approximation of A from above with a decreasing sequence O_j of open sets (recall from Analysis III that such sequences exist by the regularity properties of the Lebesgue measure). Use (and show) that $G_{O_j}(g) \to 0$ as $j \to \infty$ for such a sequence of open sets.

To show that (1) is true, reduce first to the case of an open interval I (use Lebesgue dominated convergence and sigma-additivity), and then approximate χ_I using an increasing sequence of continuous functions; this also gives an approximation of G_I....

[1] Try by yourself first!

Exercise 4

a. Try a proof by contradiction using exercises 3 and 2.
b. One needs to show that

$$\left\| \partial_i f(\cdot) - \frac{f(\cdot + te_i) - f(\cdot)}{t} \right\|_2$$

converges to zero. Use an approximating sequence $f_\varepsilon = f * j_\varepsilon \in C^\infty(\mathbb{R}^n)$ (where $j_\varepsilon \in C_0^\infty(\mathbb{R}^n)$) such that $f_\varepsilon \to f$ in $L^2(\mathbb{R}^n)$ in order to expand the above expression and estimate it from above.