Hints[1] for exercise sheet 12

Exercise 1

This exercise is solved in the book of Lieb and Loss; if you get stuck, look there! In their proof, there are still plenty of details to fill in.

Exercise 2

a. Let $\omega \in S^{n-1}$. Argue why

$$f(x) = -\int_0^\infty \frac{d}{dt} f(x+t\omega)dt = -\int_0^\infty \omega \cdot (\nabla f)(x+t\omega)dt.$$

Integrate both sides of this equation. Bring one of the resulting integrals into the desired form by using coordinate transformations.

b. Using that $(L^q(\mathbb{R}^n))^* = L^{\tilde{q}}(\mathbb{R}^n)$ where $1/p + 1/\tilde{q} = 1$, write

$$\|f\|_q = \sup_{g \in L^{\tilde{q}}(\mathbb{R}^n), \|g\|_{\tilde{q}}=1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|. \tag{1}$$

Approximate f in $H_0^{1,p}$ with a sequence $(\varphi_j) \subseteq C_0^\infty$ and estimate

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \quad ... \quad + \left| \int_{\mathbb{R}^n} \varphi_j(x)g(x)dx \right|$$

Then use part a) and exercise 1.

Exercise 3

For calculating the norms, it is useful to choose coordinates in a smart way (e.g. cylindrical coordinates or spherical coordinate, etc.).

Exercise 4

This is very much worth the effort!

[1] Try by yourself first!