## Hints ${ }^{1}$ for exercise sheet 11

## Exercise 1

a. Consider the subspace $c \subseteq \ell^{\infty}(\mathbb{R})$, where $c$ is the set of all convergent sequences, and consider the function $f: c \rightarrow \mathbb{R}$ defined on any convergent sequence $x=\left(x_{k}\right)_{k \in \mathbb{N}}$ by $f(x):=\lim _{k \rightarrow \infty} x_{k}$. The idea is to apply Hahn-Banach, using the function $p$ given in the hint.

The resulting extension $l$ of $f$ is thus an extension of the notion of a limit of a sequence; hence the name Banach limit!
b. This part only need a few short lines, nothing complicated!

## Exercise 2

a. I would suggest using Exercise 5 from Sheet 8, as well as part of the proof of Theorem 2.9.9 from the lecture notes. You don't need to do the technical details again from scratch. b. Without loss of generality one can assume that $I$ is an open interval ( $a, b$ ) (why?). Then estimate

$$
\left|\int_{a}^{b} f_{k}(x)-f(x) d x\right|
$$

using a term of the form

$$
\left|\sum_{m=m_{a}}^{m_{b}} \int_{m / k}^{(m+1) / k} f_{k}(x)-f(x) d x\right|
$$

where $m_{a}, m_{b} \in \mathbb{Z}$ are appropriately chosen.

## Exercise 3

a. The idea is to apply interatively the Lemma of Mazur to the sequences $\left\{x_{n} \mid n \geq k\right\}$, for each $k$. As such: for $k=1$, by the Lemma of Mazur there exists for $\left(x_{n}\right)_{n \geq 1}$ a number $N(1) \in \mathbb{N}$ and numbers $0 \leq \lambda_{i}^{1} \leq 1$ with $\sum_{i=1}^{N(1)} \lambda_{i}^{1}=1$ such that for $y_{1}=\sum_{i=1}^{N(1)} \lambda_{i}^{1} x_{i}$

$$
\left\|y_{1}-x\right\| \leq 1
$$

holds. Then take $k=2$, and construct $y_{2}$ such that $\qquad$

$$
\left\|y_{2}-x\right\| \leq 1 / 2 .
$$

[^0]In general construct $y_{k}$ such that

$$
\left\|y_{k}-x\right\| \leq 1 / 2^{k-1}
$$

b. Note: The notation may seem disconcerting: $F$ is simply an element of $\mathbb{R}^{n}$.

Perhaps start a solution like this:
Consider a subsequence $\left(u_{n_{k}}\right)_{k \in \mathbb{N}}$ of $\left(u_{n}\right)_{n \in \mathbb{N}}$ such that

$$
\lim _{k \rightarrow \infty} \int_{U} f\left(x, \nabla u_{n_{k}}\right) d x=\liminf _{n \rightarrow \infty} \int_{U} f\left(x, \nabla u_{n}\right) d x
$$

Apply part a. to get a sequence $v_{n_{k}}$ with $v_{n_{k}}=\sum_{i=N_{k}}^{N\left(n_{k}\right)} \lambda_{i}^{n_{k}} u_{i} \rightarrow u$ as $k \rightarrow \infty$ in $W^{1, p}(U)$. In particular $\nabla v_{n_{k}} \rightarrow \nabla u$ in $L^{p}(U)$ (these are weak gradients, remember). Then there exists a subsequence such that $\nabla v_{n_{k_{j}}}(x) \rightarrow \nabla u(x)$ pointwise for almost all $x \in U$.

Etimate

$$
\int_{U} f(x, \nabla u(x)) d x
$$

using Fatou's lemma and then the convexity of $f$ in the second argument.
c. The notation $f=f(F)$ just means that we are fixing some $F \in \mathbb{R}^{n}$ (which henceforth will not play any role) and are thinking of $f$ as a function $U \rightarrow[0, \infty)$.

Follow the hint.

## Exercise 4

a. Try looking at $f_{n}(x)=n^{1 / 2} f(n x)$, where $f:[0,1] \rightarrow \mathbb{R}$ is an $L^{2}$ function which is not the zero function.
b . Note: assume that $L^{2}=L^{2}([0,1])$.
One possible strategy is to construct for every subsequence $f_{n_{k}} g_{n_{k}}$ a subsubsequence $f_{n_{k_{j}}} g_{n_{k_{j}}}$ such that for every $\psi \in L^{2}$

$$
\int\left[f_{n_{k_{j}}} g_{n_{k_{j}}}-f(x) g(x)\right] \psi(x) d x \rightarrow 0
$$

as $j \rightarrow \infty$. Then one can conclude by Exercise (4a) of Sheet 3, that $f_{n} g_{n} \rightharpoonup f g$ in $L^{2}$.
For this, suppose $f_{n_{k}} g_{n_{k}}$ is an arbitrary subsequence of $f_{n} g_{n}$. Because the sequence $f_{n_{k}}$ is bounded in $L^{\infty}$, which is (canonically isomorphic to) the dual of $L^{1}$ and $L^{1}$ is seperable, Banach Alaoglu implies that there exists a subsequence $f_{n_{k_{j}}}$ which is weak-* convergent to a $\tilde{f}$, i.e. for all $\varphi \in L^{1}$

$$
\int\left(f_{n_{k_{j}}}(x)-\widetilde{f}(x)\right) \varphi(x) d x \rightarrow 0
$$

as $j \rightarrow \infty$.
The idea then is to show that $\widetilde{f}(x)=f(x)$ almost everywhere (this may be somewhat involved, but go for it). Use the assumptions of the exercise, including that $[0,1]$ is of finite measure.


[^0]:    ${ }^{1}$ Try by yourself first!

