

Hints¹ for exercise sheet 9

Exercise 1

Note: as mentioned in exercise class, we are implicitly assuming that f is not the zero function.

Tools you might find useful in this exercise (among other things):

- write out/compute expressions for L^p norms
- check that a sequence is (not) Cauchy
- convergence of a sequence in L^p implies that there is a subsequence which converges point-wise almost everywhere.
- change of variables formula
- consider expressions such as $\|\cdot\|_{L^q(\mathbb{R} \setminus [-R, R])} \rightarrow 0$ as $R \rightarrow \infty$
- Riemann-Lebesgue Lemma
- Don't forget that f has compact support.

Exercise 2

a. For showing weak convergence: assuming that $(x^{(n)})_{n \in \mathbb{N}}$ is bounded and $x_i^{(n)} \rightarrow x_i$ for all $i \in \mathbb{N}$, consider an arbitrary $y \in \ell^q(\mathbb{K})$ and define

$$y_n^{(k)} = \begin{cases} y_n & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$$

Then use that $\|y - y^{(i)}\|_q \rightarrow 0$ as $i \rightarrow \infty$ to estimate

$$\langle x^{(n)} - x, y \rangle = \dots$$

b. Perhaps it's useful to remember that

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 - 2\operatorname{Re}\langle y, x \rangle + \|y\|^2$$

¹Try by yourself first!

Exercise 3

- a. You can do it.
- b. Here is the beginning of one possible solution: Assume that there exists a sequence $\varepsilon_k \rightarrow 0$ such that I_{ε_k} is weak-* convergent in $(L^\infty((0, 1)))^*$. Then there exists a subsequence ε_{k_l} such that $1 \geq \varepsilon_{k_{l+1}}/\varepsilon_{k_l} \rightarrow 0$ as $l \rightarrow \infty$ (why is this true?). Then define

$$f = \sum_{j=1}^{\infty} (-1)^j \chi_{[\varepsilon_{k_{j+1}}, \varepsilon_{k_j}]},$$

and check...

Exercise 4

- a. You can do it.
- b. For showing $\|x_j^* - x^*\|_{\sigma} \rightarrow 0$: split the sum! And use that not only M but also $\|x^*\|$ is a constant.

For showing weak-* convergence, try using a “three term triangle inequality” argument.

- c. A possible approach: consider $X = c_0(\mathbb{K})$, the space of sequences convergent to zero, equipped with the supremum norm $\|\cdot\|_{\infty}$. Inside this space, consider $\sigma = \{e_n\}_{n \in \mathbb{N}}$, where $e_n = (0, \dots, 0, 1, 0, \dots)$ as usual (“1” is the n -th entry), and consider functionals y_j^* defined on $x = (x_k)_{k \in \mathbb{N}} \in X$ by

$$y_j^*(x) = \frac{1}{j} \sum_{k=1}^{j^2} x_k.$$

- d. See Theorem 6.3.6 from the lecture notes.